Confining Strings in the Abelian-Projected SU(3)-Gluodynamics

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Abstract. – String representation of the Wilson loop in 3D Abelian-projected SU(3)-gluodynamics is constructed in the approximation that Abelian-projected monopoles form a gas. Such an assumption is much weaker than the standard one, demanding the monopole condensation. It is demonstrated that the summation over world sheets, bounded by the contour of the Wilson loop, is realized by the summation over branches of a certain effective multivalued potential of the monopole densities. Finally, by virtue of the so-constructed representation of the Wilson loop in terms of the monopole densities, this quantity is evaluated in the approximation of a dilute monopole gas, which makes confinement in the model under study manifest.

On the way of constructing the string representation of SU(3)-gluodynamics by means of the method of Abelian projections \cite{1} (see Ref. \cite{2} for recent reviews), the main results have up to now been obtained under the assumption of the monopole condensation. Indeed, this assumption leads to an effective Ginzburg-Landau type theory \cite{3}, whose string representation can further be investigated \cite{4} analogously to that of the usual dual superconductor \cite{5}. On the other hand, recently string representation of the Abelian-projected SU(2)-gluodynamics has been derived \cite{6} under a weaker assumption, which states that Abelian-projected monopoles form a gas, rather than condense into the dual Higgs field. Such a way of treating Abelian-projected monopoles in the SU(2)-gluodynamics makes the string representation of the Wilson loop in this theory (which describes an external test particle, electrically charged \textit{w.r.t.} the U(1) Cartan subgroup of SU(2)) similar to that of the Wilson loop in compact QED \cite{7}. The aim of the present Letter is to extend the results of Ref. \cite{6} to the case of Abelian-projected SU(3)-gluodynamics in 2+1 dimensions and finally to emphasize confinement in this theory in the sense of the Wilson area law \cite{8}.

Let us start our analysis with considering the pure monopole contribution to the action of this theory, keeping for a while aside the noncompact part of diagonal fields. (The off-diagonal fields are as usual disregarded on the basis of the so-called Abelian dominance hypothesis \cite{9}. That is because they are argued to become very massive (and thus short-ranged) and therefore...
irrelevant to the IR region, where confinement holds.) The partition function describing the grand canonical ensemble of monopoles has the form

\[
Z = 1 + \sum_{N=1}^{\infty} \frac{\zeta^N}{N!} \left( \prod_{a=1}^{N} \int d^3z_a \sum_{\alpha_a = \pm 1, \pm 2, \pm 3} \right) \exp \left[ -\frac{g_m^2}{4\pi} \sum_{a < b} \frac{g_{\alpha_a} g_{\alpha_b}}{|z_a - z_b|} \right].
\] (1)

Here, \(g_m\) is the magnetic coupling constant, related to the QCD coupling constant \(g\) according to the equation \(g g_m = 4\pi\), \(\zeta \propto \exp (-\text{const.}/g^2)\) is the fugacity (Boltzmann factor) of a single monopole, and \(g_{\alpha_a}\)'s are the nonzero weights of the zero triality adjoint representation of \(\ast SU(3)\). These weights are defined as \(\tilde{q}_1 = (1/2, \sqrt{3}/2), \tilde{q}_2 = (-1, 0)\), \(\tilde{q}_3 = (1/2, -\sqrt{3}/2), \tilde{q}_{-\alpha} = -\tilde{q}_\alpha\). It is worth noting that for \(\lambda = (\lambda_3, \lambda_8)\), where in the Gell-Mann basis \(\lambda_3 = \text{diag}(1, -1, 0)\), \(\lambda_8 = \text{diag}(1/\sqrt{3}, 1/\sqrt{3}, -2/\sqrt{3})\), the following relations hold: \(\tilde{q}_1 \lambda = \text{diag}(1, 0, -1)\), \(\tilde{q}_2 \lambda = \text{diag}(-1, 1, 0)\), \(\tilde{q}_3 \lambda = \text{diag}(0, -1, 1)\). Therefore for every \(\alpha = \pm 1, \pm 2, \pm 3\), one has \(\tilde{q}_\alpha \lambda = \tilde{n}\), where \(\tilde{n}\) is a certain traceless diagonal matrix with the elements 0, ±1. This matrix can thus be written as \(\tilde{n} = w \lambda_3 w^{-1}\), where \(w\) is any of the six elements of the permutation group \(S_3\).

Equation (1) can be represented as

\[
Z = 1 + \sum_{N=1}^{\infty} \frac{\zeta^N}{N!} \left( \prod_{a=1}^{N} \int d^3z_a \sum_{\alpha_a = \pm 1, \pm 2, \pm 3} \right) \exp \left[ -\frac{2\pi}{g^2} \int d^3x d^3\rho \frac{1}{|x-y|} \rho_{\text{gas}}(y) \right],
\] (2)

or further

\[
Z = \int D\chi \exp \left[ -\frac{1}{2} \int d^3x (\nabla \chi)^2 \right] \times \left[ 1 + \sum_{N=1}^{\infty} \frac{\zeta^N}{N!} \left( \prod_{a=1}^{N} \int d^3z_a \sum_{\alpha_a = \pm 1, \pm 2, \pm 3} \right) \exp \left( ig_m \int d^3x \chi \rho_{\text{gas}} \right) \right].
\] (3)

Here, \(\rho_{\text{gas}}(x) = \sum_{\alpha_a} \tilde{q}_\alpha \delta (x - \tilde{z}_\alpha)\) stands for the density of the monopole gas, and the measure \(D\chi\) is normalized by the condition

\[
\int D\chi \exp \left[ -\frac{1}{2} \int d^3x (\nabla \chi)^2 \right] = 1.
\]

Equation (3) thus yields the following representation for the partition function (1):

\[
Z = \int D\chi \exp \left\{ - \int d^3x \left[ \frac{1}{2} (\nabla \chi)^2 - \zeta \sum_{\alpha = \pm 1, \pm 2, \pm 3} \exp (ig_m \tilde{q}_\alpha \chi) \right] \right\},
\] (4)

or, in the form analogous to the one of compact QED,

\[
Z = \int D\chi \exp \left\{ - \int d^3x \left[ \frac{1}{2} (\nabla \chi)^2 - 2 \zeta \sum_{\alpha = 1}^{3} \cos (g_m \tilde{q}_\alpha \chi) \right] \right\}.
\] (5)

Denoting \(\tilde{q}_\alpha \chi, \alpha = 1, 2, 3\), by \(\chi_\alpha\) and performing the rescaling \(\chi_\alpha = \sqrt{\frac{\zeta}{2}} \chi_\alpha\), we can represent the partition function (3) as
\[ Z = \int \left( \prod_{\alpha=1}^{3} D\chi_{\alpha} \right) \delta \left( \sum_{\alpha=1}^{3} \chi_{\alpha} \right) \exp \left\{ -\int d^{3}x \left[ \frac{1}{2} (\nabla \chi_{\alpha})^{2} - 2\zeta \sum_{\alpha=1}^{3} \cos \left( g_{m} \sqrt{\frac{3}{2}} \chi_{\alpha} \right) \right] \right\} . \quad (6) \]

Integrating out one of the fields \( \chi_{\alpha} \)'s, e.g. for concreteness \( \chi_{3} \), and denoting \( \xi_{1} = \sqrt{\frac{3}{2}}(\chi_{1} + \chi_{2}) \), \( \xi_{2} = \frac{1}{\sqrt{2}}(\chi_{1} - \chi_{2}) \), we get for the partition function (6) the following expression:

\[ Z = \int D\xi_{1}D\xi_{2} \exp \left\{ -\int d^{3}x \left[ \frac{1}{2} (\nabla \xi_{1})^{2} + \frac{1}{2} (\nabla \xi_{2})^{2} - 2\zeta \left[ \cos(g_{m}\xi_{1}) + \cos \left( g_{m} \sqrt{\frac{3}{2}} \xi_{2} \right) \right] \right] \right\} . \quad (7) \]

However, for constructing the string representation of the Wilson loop in the theory (2) (or (4)), the representation (8) will not be necessary. To construct this representation, it is first convenient to derive an expression for the partition function (8) in terms of the integral over monopole densities. This procedure is analogous to the one employed in Ref. [12] for the case of compact QED. Firstly, let us multiply Eq. (2) by the following unity: \( 1 = \int D\bar{\rho} \delta(\bar{\rho}(\vec{x}) - \bar{\rho}_{\text{gas}}(\vec{x})) \). After that, this equation reads

\[ Z = \int D\vec{\lambda}D\bar{\rho} \left\{ 1 + \sum_{N=1}^{\infty} \frac{\zeta^{N}}{N!} \left( \prod_{\alpha=1}^{N} \int d^{3}z_{\alpha} \sum_{\alpha_{a} = \pm 1, \pm 2, \pm 3} \right) \times \exp \left[ -\frac{2\pi}{g^{2}} \int d^{3}x d^{3}y \frac{1}{|\vec{x} - \vec{y}|} \bar{\rho}(\vec{y}) \right] \right\} \exp \left[ -i g_{m} \int d^{3}x \vec{\lambda}(\bar{\rho} - \bar{\rho}_{\text{gas}}) \right] , \quad (9) \]

where \( \vec{\lambda} \) stands for the Lagrange multiplier. Next, upon the normalization of the measure \( D\vec{\lambda} \) by the condition (10)

\[ \int D\vec{\lambda} \exp \left[ -\frac{2\pi}{g^{2}} \int d^{3}x d^{3}y \frac{1}{|\vec{x} - \vec{y}|} \bar{\rho}(\vec{y}) - i g_{m} \int d^{3}x \vec{\lambda} \bar{\rho} \right] = 1, \]

Eq. (9) can be written as follows:

\[ \int \frac{1}{2} \int d^{3}x \left( \nabla \vec{\lambda} \right)^{2} = 1. \]

(1) Clearly, this condition can be written as \( \int D\vec{\lambda} \exp \left[ -\frac{1}{2} \int d^{3}x \left( \nabla \vec{\lambda} \right)^{2} \right] = 1. \)
The integration over $\vec{\lambda}$ reduces now to the problem of finding a solution to the following saddle-point equation:

$$\sum_{\alpha=1}^{3} \bar{q}_{\alpha} \sin \left(g_{m} \bar{q}_{\alpha} \vec{\lambda}\right) = -\frac{i\bar{\rho}}{2\zeta}. \quad (11)$$

This equation can be solved w.r.t. $\vec{q}_{\alpha} \vec{\lambda}$ by noting that an arbitrary vector $\vec{\rho} \equiv (\rho_{1}, \rho_{2})$ can always be represented as $\sum_{\alpha=1}^{3} \bar{q}_{\alpha} \rho_{\alpha}$, where $\rho_{1} = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \rho_{1}^{3} + \rho_{2}^{3}\right)$, $\rho_{2} = -\frac{2}{3} \rho_{1}$, $\rho_{3} = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \rho_{1}^{3} - \rho_{2}^{3}\right)$. Then, inserting the so-obtained expression for $\vec{q}_{\alpha} \vec{\lambda}$ back into the action standing in the argument of the exponent on the R.H.S. of Eq. (10), we eventually arrive at the following representation for the partition function (2) in terms of the monopole densities:

$$Z = \int D\vec{\rho} D\vec{\lambda} \exp \left\{ -\frac{2\pi}{g^{2}} \int d^{3} x d^{3} y \vec{\rho}(\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \vec{\rho}(\vec{y}) + \int d^{3} x \left[ 2\zeta \sum_{\alpha=1}^{3} \cos \left( g_{m} \bar{q}_{\alpha} \vec{\lambda}\right) - i g_{m} \vec{\rho} \right] \right\}. \quad (12)$$

Here, the effective multivalued monopole potential $V[\vec{\rho}]$ has the form

$$V[\vec{\rho}] = \sum_{n=-\infty}^{+\infty} \sum_{\alpha=1}^{3} \int d^{3} x \left\{ \rho_{\alpha} \left[ \ln \left( \frac{\rho_{\alpha}}{2\zeta} \right) + \sqrt{1 + \left( \frac{\rho_{\alpha}}{2\zeta} \right)^{2}} \right] + 2\pi in \right\} - 2\zeta \sqrt{1 + \left( \frac{\rho_{\alpha}}{2\zeta} \right)^{2}}. \quad (13)$$

We are now in a position to discuss the string representation of the Wilson loop. The contribution of diagonal gluons $A_{\mu}^{3} \equiv (A_{\mu}^{3}, A_{\mu}^{8})$ into this quantity, which we are interested with, reads

$$\langle W(C) \rangle \equiv \frac{1}{3} \left\langle \text{tr} P \exp \left( \frac{i}{2} \oint_{C} d\lambda \vec{A}_{\mu} \vec{\lambda} \right) \right\rangle. \quad (14)$$

Clearly, since both $\lambda_{3}$ and $\lambda_{8}$ are diagonal, the path ordering can be omitted (which becomes obvious from the definition of the path-ordering prescription). Owing to the Stokes theorem, one then obtains for the desired monopole contribution to the Wilson loop the following expression (cf. Ref. [12] for the case of compact QED):

$$(^2)$$Here and in the next equation, $\vec{\lambda}$ again denotes the vector $(\lambda_{3}, \lambda_{8})$. 


\[ \langle W(C) \rangle_m = \frac{1}{3} \left\langle \text{tr} \exp \left( \frac{i}{2} \int d^3x \tilde{\rho} \tilde{\eta} \lambda_3 \right) \right\rangle. \]

Here,

\[ \eta \left[ \vec{x}, \Sigma \right] = \frac{1}{2} \varepsilon_{\mu\nu\lambda} \frac{\partial}{\partial x_{\mu}} \int d\sigma_{\nu\lambda}(\vec{y}) \frac{1}{|\vec{x} - \vec{y}|} \]

(15)
denotes the solid angle, under which a certain surface \( \Sigma \), bounded by the contour \( C \), shows up to an observer located at the point \( \vec{x} \). (Clearly, \( \eta \) is a function of the point \( \vec{x} \) and a functional of the surface \( \Sigma \), i.e. of the vector \( \vec{y}(\xi^1, \xi^2) \), which parametrizes this surface.) Owing to the explicit form of the matrices \( \lambda_3 \) and \( \lambda_8 \), one finally obtains (cf. Ref. [10])

\[ \langle W(C) \rangle_m = \frac{1}{3} \left\langle \sum_{\alpha=1}^{3} \exp \left( i \int d^3x \tilde{\rho} \tilde{\eta}_\alpha \eta \right) \right\rangle. \]

(16)

Noting now that the average in Eq. (16) is taken w.r.t. the monopole partition function (2), we can apply to this equation the same procedure, which led to the representation (12)-(13). In this way, we conclude that the monopole contribution to the Wilson loop (14) is given by the following expression:

\[ \langle W(C) \rangle_m = \frac{1}{3} \left\langle \sum_{\alpha=1}^{3} \exp \left( i \int d^3x \tilde{\rho} \tilde{\eta}_\alpha \eta \right) \right\rangle. \]

(17)

Similarly to compact QED [7, 12], a seeming \( \Sigma \)-dependence of the R.H.S. of this equation, brought about by the solid angle, actually disappears due to the summation over all the complex-valued branches of the effective potential (13) at every point \( \vec{x} \). This observation is the essence of the string representation of the Wilson loop in the monopole gas both in compact QED and in the \( SU(3) \)-case under study. The \( SU(3) \)-analogue of the so-called confining string theory, proposed for the case of compact QED in Ref. [7], can be obtained by the following change of variables in the functional integral standing on the R.H.S. of Eq. (17):

\[ \tilde{\rho} \rightarrow \tilde{F}_{\mu\nu}. \] Here, the monopole field strength tensor \( \tilde{F}_{\mu\nu}(\vec{x}) = \frac{1}{2} \varepsilon_{\mu\nu\lambda} \frac{\partial}{\partial x_{\lambda}} \int d^3y \tilde{\rho}(\vec{y}) \) obeys Bianchi identities, modified by monopoles, \( \frac{1}{2} \varepsilon_{\mu\nu\lambda} \partial_{\lambda} \tilde{F}_{\mu\nu} = 4\pi \tilde{\rho} \). Such a substitution is obvious, and we will not discuss it here, referring the reader to Ref. [12] for a detailed comparison of our approach with the theory of confining strings in the case of compact QED. Note only that such a reformulation of the functional integral allows one to account automatically also for the noncompact part of the \( \tilde{A}_\mu \)-fields. Clearly, that is because \( \tilde{F}_{\mu\nu} \) is defined up to an addendum \( \partial_{\nu} \tilde{A}_\mu - \partial_{\mu} \tilde{A}_\nu \) with single-valued \( \tilde{A}_\nu \)'s.

The obtained string representation (17) can now be applied to the evaluation of the Wilson loop in the approximation when the monopole gas is dilute, i.e. \( |\tilde{\rho}| \ll \zeta \). In this way, we can restrict ourselves to the real branch of the monopole potential (13), provided that in Eq. (15) the replacement \( \Sigma \rightarrow \Sigma_{\text{min}} \), with \( \Sigma_{\text{min}} \) standing the surface of the minimal area for a given contour \( C \), has been performed. Then, the Wilson loop in the dilute monopole gas reads

\[ \langle W(C) \rangle_m = \frac{1}{3 \langle W(0) \rangle_m} \sum_{\alpha=1}^{3} \int D\tilde{\rho} \exp \left\{- \frac{2\pi}{g^2} \int d^3x d^3y \tilde{\rho}(\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \tilde{\rho}(\vec{y}) + V[\tilde{\rho}] - i \int d^3x \tilde{\rho} \tilde{\eta}_\alpha \right\}. \]
+ \int d^3x \left( -6\zeta + \frac{1}{6\zeta} \hat{\rho}^2 - i\hat{\rho}\eta(x) \right) \right) = \exp \left\{ -\frac{\zeta}{8\pi} \int d^3x d^3y \frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \partial^\mu \eta(\vec{x}) \partial_\mu \eta(\vec{y}) \right\}. \tag{18}

Here, we have for brevity denoted \( \eta(\vec{x}) \equiv \eta[\vec{x}, \Sigma_{\text{min}}] \) and used the fact that for every \( \alpha \), \( \hat{\eta}_\alpha = \frac{1}{3} \). It is further worth employing the following formula for the derivative of the solid angle (see e.g. [12]):

\[
\partial^\mu \eta[\vec{x}, \Sigma_{\text{min}}] = \varepsilon_{\mu\nu\lambda} \int_C dy \frac{1}{|\vec{x}-\vec{y}|} - 2\pi \int_{\Sigma_{\text{min}}} d\sigma_{\nu\lambda}(\vec{y}) \delta(\vec{x}-\vec{y})
\]

(which is actually valid for an arbitrary surface \( \Sigma \), bounded by \( C \)), after which the derivation of \( \langle W(C) \rangle_m \) becomes straightforward. Combining the so-obtained result with the contribution to the Wilson loop, stemming from the noncompact ("photon") part of the \( \vec{A}_\mu \)-fields, which according to Eq. (14) has the form

\[
\langle W(C) \rangle_{\text{ph}} = \exp \left\{ -\frac{g^2}{24\pi} \int_C dx_\mu \int_C dy_\mu \frac{1}{|\vec{x}-\vec{y}|} \right\},
\]

we finally obtain the following result for the full Wilson loop:

\[
\langle W(C) \rangle = \langle W(C) \rangle_m \langle W(C) \rangle_{\text{ph}} = \exp \left\{ -\frac{\zeta}{8\pi} \int_{\Sigma_{\text{min}}} d\sigma_{\mu\nu}(\vec{x}) \int_{\Sigma_{\text{min}}} d\sigma_{\mu\nu}(\vec{y}) \frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} + \frac{g^2}{24\pi} \int_C dx_\mu \int_C dy_\mu \frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \right\}. \tag{19}
\]

By virtue of the results of Ref. [13], it is easy to get the string tension of the Nambu-Goto term, which is the leading term in the gradient (or \( 1/m \)-) expansion of the nonlocal string effective action standing as the first argument of the exponent on the R.H.S. of Eq. (19) \( [14] \). \( [3] \)

The string tension reads \( \pi g \sqrt{\frac{\zeta}{3}} \) and is therefore nonanalytic in \( g \) (owing to the nonanalyticity in the \( g \)-dependence of the fugacity), similarly to what happens in the real QCD. However, it remains unclear within the model under study how to derive the fugacity itself from the QCD Lagrangian. Owing to this, the approach to the problem of string representation of QCD, elaborated in the present Letter, does not answer the question: what is the proportionality coefficient between the string tension in QCD and \( \Lambda_{\text{QCD}}^2 \)? This question, the answer to which is very important for understanding the connection between the perturbative and nonperturbative phenomena in QCD, will be addressed in future publications.

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\( [3] \) Note that at the surface of the minimal area, the next-to-leading term of this expansion (the so-called rigidity term) vanishes \( [14] \).
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