Averaging inhomogeneities in scalar-tensor cosmology

Vincenzo Vitagliano\textsuperscript{1}, Stefano Liberati\textsuperscript{1} and Valerio Faraoni\textsuperscript{2}

\textsuperscript{1} SISSA - International School for Advanced Studies, Via Beirut 2-4, 34151, Trieste, Italy and INFN, Sezione di Trieste
\textsuperscript{2}Physics Department, Bishop’s University, 2600 College St., Sherbrooke, Québec, Canada J1M 1Z7

E-mail: vitaglia@sissa.it, liberati@sissa.it, vfaraoni@ubishops.ca

Abstract. The backreaction of inhomogeneities on the cosmic dynamics is studied in the context of scalar-tensor gravity. Due to terms of indefinite sign in the non-canonical effective energy tensor of the Brans–Dicke-like scalar field, extra contributions to the cosmic acceleration can arise. Brans–Dicke and metric $f(R)$ gravity are presented as specific examples. Certain representation problems of the formalism peculiar to these theories are pointed out.

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1. Introduction

The latest cosmological data sets and the increasing number of ongoing satellite missions dedicated to cosmology are poised to raise a radically new theoretical scenario as opposed to the description proper of the classical General Relativity (GR) schemes. The cosmic acceleration detected by supernova surveys\textsuperscript{[1]} provides the starting point for a New Deal in cosmology, since dark energy and dark matter components seem to be needed in order to reproduce the observed phenomenology. Over the last decade, there have been many attempts to build models of effective fluids playing the role of dark energy: the taxonomy of possible explanations includes the resurrection of Einstein’s cosmological constant (\textsuperscript{2} and reference therein), as well as the introduction of large-scale modifications of gravity \textsuperscript{3, 4}. Recently, a new proposal about the nature of the current cosmic acceleration has been advanced, involving the backreaction of inhomogeneities \textsuperscript{[5]} \textsuperscript{[6]} \textsuperscript{[7]} \textsuperscript{[8]} as a possible source.

Even if the assumptions of spatial homogeneity and isotropy of the matter distribution inspired by the Cosmological Principle appear to give an adequate, although approximate, description of the universe on large scales, the lumpiness of structures and the existence of huge voids are well-known observable properties in smaller regions and at late epochs. The fitting problem, \textit{i.e.}, the problem of matching a coarse-grained matter distribution with a spacetime metric obtained with an independent smoothing operator,
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2 has been pointed out in Refs. [9, 10]. The development of an averaging procedure — smoothing out inhomogeneities of scalar quantities — allows us to implement a new set of averaged contracted Einstein equations. The lack of commutativity between time evolution and the averaging procedure enables the encoding of the kinematics of the universe in terms of new quantities with recognizable backreaction features.

While the averaging formalism is interesting in itself, and the idea of explaining the cosmological data through backreaction in the context of pure Einstein gravity with no dark energy is very appealing, it has not been demonstrated yet that this idea works in practice. It is undeniable that matter inhomogeneities have a backreaction effect but it is not clear that over/under-densities such as those observed around us are sufficiently large to significantly affect the cosmic dynamics, and are not limited to small perturbative effects. While the jury is still out on whether backreaction explains the observed cosmic acceleration or not, one realizes that virtually all high energy theories attempting to quantize gravity or unifying it with the other interactions predict deviations from GR. In string theories and supergravity the gravitational field includes, in addition to the massless spin two graviton, a dilaton whose presence is unavoidable and that couples non-minimally to the curvature of spacetime [11]. Such a behaviour is mimicked by scalar-tensor gravity [12, 13] (for example, an early representative of string theories, the bosonic string theory reduces to an $\omega = -1$ Brans–Dicke theory in the low-energy limit [14]).

While scalar-tensor theories are constrained on Solar System scales and by the binary pulsar [15], we do not have many constraints on larger scales (except, possibly, those due to the variation of the effective gravitational coupling during Big Bang nucleosynthesis). It is possible, therefore, that the backreaction idea may have to be implemented in alternative theories of gravity. In fact, it could even be that, if backreaction doesn’t quite work in GR, it is “helped” by a non-Einsteinian component of gravity. In [16] a formalism that implements Buchert’s scheme into models with variable Newton “constant” was already developed, motivated by the non-perturbative renormalization group improvement of the action functional [17]. Here, instead, we restrict our attention to scalar-tensor gravity as the prototypical generalization of GR.

The following observation can be made a priori: the Brans–Dicke-like field that necessarily permeates all of spacetime can be described as an effective form of matter by writing the scalar-tensor field equations in the form of effective Einstein equations. The effective energy-momentum tensor characterizing this form of $\phi$-matter easily violates all the energy conditions and, therefore, is more likely to produce the cosmic acceleration.

Another aspect is worth pointing out: it is widely believed that quantum corrections to the Einstein–Hilbert action introduce quadratic deviations from the usual Lagrangian density $R$, which may well have propelled the inflationary epoch in the early universe, \footnote{We do not refer here specifically to $f(R)$ theories based on large-scale modifications of gravity [3,4]. It would be rather pointless to study the backreaction effect in those $f(R)$ theories since it is already known that, in their metric version, they may provide viable models to explain the cosmic acceleration [20].} as in Starobinsky’s inflation [18]. For a spatially homogeneous and isotropic universe,
quadratic corrections die off quickly as the universe expands and $R$ decreases. However, in an inhomogeneous universe, they might help the backreaction mechanism. Now, it is well-known [19, 20] that a theory described by a non-linear Lagrangian density $f(R)$ in the metric formalism is equivalent to an $\omega = 0$ Brans–Dicke theory with a scalar field degree of freedom given by $\phi = f'(R)$ with a suitable scalar field potential. Therefore, by studying scalar-tensor theory, we also catch the effect of the simplest quadratic corrections to GR.

The scalar-tensor action expressed in the Jordan frame is

$$S_{ST} = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} \left[ \phi R - \frac{\omega(\phi)}{\phi} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) \right] + \alpha_m \mathcal{L}^m \right\},$$  \hspace{1cm} (1)$$

where $\phi$ is the Brans–Dicke-like scalar field with potential $V(\phi)$ and coupling function $\omega(\phi)$, $g$ is the determinant of the metric tensor $g_{\mu\nu}$, $R$ is the Ricci curvature, $\mathcal{L}^m$ is the Lagrangian density describing the ordinary matter sector with coupling costant $\alpha_m$, and we adopt the notations of Ref. [21].

The conformal transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega = \sqrt{G\phi}$$  \hspace{1cm} (2)$$

and the scalar field redefinition

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi) + 3}{16\pi G}} \frac{d\phi}{\phi}$$  \hspace{1cm} (3)$$

turn the action (1) into its Einstein frame form

$$S_{ST} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - U\left(\tilde{\phi}\right) + \tilde{\alpha}_m(\tilde{\phi}) \tilde{\mathcal{L}}^m \right],$$  \hspace{1cm} (4)$$

where a tilde denotes quantities in the rescaled world, and

$$U\left(\tilde{\phi}\right) = \frac{V[\phi(\tilde{\phi})]}{[G\phi(\tilde{\phi})]^2}, \quad \tilde{\alpha}_m(\tilde{\phi}) = \frac{\alpha_m}{[G\phi(\tilde{\phi})]^2}. $$  \hspace{1cm} (5)$$

The “new” scalar field $\tilde{\phi}$ couples minimally to the curvature but non-minimally to the matter fields.

2. Averaging procedure for GR cosmology

Our goal is studying the backreaction mechanism of spatial inhomogeneities on the cosmic dynamics in the context of scalar-tensor gravity. Before doing this, we briefly review the Buchert formalism in GR for a universe filled with an irrotational dust. In this case it is possible to choose a foliation of spacetime with spacelike hypersurfaces orthogonal to the flow at any event. We will then apply the averaging procedure with respect to a family of observers comoving with the dust and characterized by a four-velocity field $u^\mu$, thus avoiding gauge complications related to the choice of an arbitrary set of observers tilted with respect to the cosmological matter fluid [22]. In actual fact, in an inhomogeneous universe the four-velocity of these observers is not simply $u^\mu = \delta^{0\mu}$.
but there are also local fluctuations $\delta u^\mu$, so that $u^\mu = \delta^0 u^\mu + \delta u^\mu$ corresponding to the possible choices of time on the inhomogeneous hypersurfaces. Therefore, the procedure adopted here of projecting the Einstein equations onto $u^\mu$ and then averaging is not free of ambiguities and gauge-dependence issues. This projection and the spatial average do not commute. With this caveat in mind, we proceed as is usually done in the literature by choosing Gaussian normal coordinates (see below).

It is also convenient to define a template metric mimicking the main properties of a FLRW universe on large scales \[23, 24\] but encoding the small scale lumpy structures. In this way the averaged quantities will assume the usual meaning as in the traditional cosmological framework. The scale of the domain used in the averaging procedure is chosen as the cosmological volume over which it would be reasonable to recover homogeneity, i.e., somehow larger than $100 \, h^{-1}$ Mpc.

Let us briefly recall the essential points of Buchert’s averaging approach, referring the reader to \[25\] for details. For the sake of simplicity we turn our attention to Buchert’s original model (see \[5\] for a comprehensive review). This consists of a universe filled with an irrotational dust as the material source, with energy density $\rho$ and four-velocity $u^\mu$ satisfying $u_\mu u^\mu = -1$. The corresponding Einstein equations and stress-energy covariant conservation equation read

$$\begin{align}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 8\pi G \rho u_\mu u_\nu - \Lambda g_{\mu\nu}, \\
\nabla_\mu (\rho u^\mu u^\nu) &= 0,
\end{align}$$

where $\rho \equiv T_{\mu\nu} u^\mu u^\nu$. By adopting Gaussian normal coordinates it is possible to apply the standard ADM procedure for the 3+1 splitting of spacetime \[21\]. In these coordinates the spacetime manifold can be foliated with spacelike Cauchy hypersurfaces parametrized by the proper time $t$. In this framework the surfaces are comoving with the fluid in such a way that, casting the metric in the form

$$ds^2 = -dt^2 + g_{ij}(t, X^k) dX^i \otimes dX^j$$

we have $u^\mu = (1, 0, 0, 0)$ and $u^\nu \nabla_\nu u^\mu = 0$. The second fundamental form (extrinsic curvature) $K_{\mu\nu}$ of the geodesic normal slicing of spacetime is introduced as follows: Let $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ be the induced metric on the 3-surfaces. Then $K_{\mu\nu}$ is defined as the Lie derivative of this Riemannian metric in the time direction,

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_u h_{\mu\nu} = -\nabla_\mu u_\nu = -\frac{1}{2} \partial_t h_{\mu\nu}.$$  

Given the form of the metric \[8\], $K_{00}$ and $K_{0i}$ vanish while $K_{ij}$ can be expressed in terms of the expansion tensor $\theta_{ij}$, the expansion scalar $\theta \equiv \theta^i_i$, and the traceless shear tensor $\sigma_{ij}$ as

$$K_{ij} = -\theta_{ij} = -\left(\sigma_{ij} + \frac{\theta}{3} g_{ij}\right), \quad K \equiv K^i_i = -\theta \quad (i, j = 1, 2, 3).$$
Denoting with $D_{\mu}$ the derivative operator associated with the metric $h_{\mu\nu}$, it is possible to derive the Gauss–Codazzi relations between the curvature of the 3-surface, the extrinsic curvature and the spacetime curvature \cite{21}:

\begin{eqnarray}
\label{3}
R_{\mu\nu\rho\sigma} &=& R_{\alpha\beta\gamma\delta} h^{\alpha}_{\mu} h^{\beta}_{\nu} h^{\gamma}_{\rho} h^{\delta}_{\sigma} - K_{\mu\rho} K_{\nu\sigma} + K_{\mu\sigma} K_{\nu\rho}, \\
D_{\rho} K^{\rho}_{\nu} - D_{\nu} K &=& h^{\mu}_{\mu} R_{\mu\rho} u^{\rho}.
\end{eqnarray}

Saturating indices with the induced metric $h_{\mu\nu}$, it is possible to rearrange eq. \ref{11} as

\begin{eqnarray}
G_{\mu\nu} u^{\mu} u^{\nu} &=& \frac{1}{2} \left( (3) R + K^2 - K_{ij} K^{ij} \right),
\end{eqnarray}

where $(3) R$ is the scalar 3-curvature, i.e., the projection of the Ricci scalar onto the spatial hypersurface. On the other hand, using the definition of the Riemann tensor it follows that

\begin{eqnarray}
R_{\mu\nu} u^{\mu} u^{\nu} &=& K^2 - K_{\mu\nu} K^{\mu\nu} - \nabla_{\mu} (u^{\mu} \nabla_{\nu} u^{\nu}) + \nabla_{\nu} (u^{\mu} \nabla_{\mu} u^{\nu}),
\end{eqnarray}

with the last term vanishing because of the geodesic equation obeyed by the four-velocity of the dust. By combining \ref{14} with \ref{13} and taking into account the definition \ref{9} of extrinsic curvature, we are able to express the scalar curvature of spacetime as

\begin{eqnarray}
(4) R &=& (3) R + K^2 + K_{ij} K^{ij} - 2 \mathcal{L}_{\nu} K.
\end{eqnarray}

The Hamiltonian or energy constraint and the evolution equation for the expansion scalar (Raychaudhuri equation) can be derived from appropriate contractions of the Einstein equations: the Hamiltonian constraint is obtained by doubly contracting eq. \ref{6} with $u^{\mu}$ and using eq. \ref{13},

\begin{equation}
\frac{1}{2} \left( (3) R + K^2 - K_{ij} K^{ij} \right) = 8\pi G \rho + \Lambda,
\end{equation}

while the equation for the scalar expansion is found by tracing the Einstein equation. Taking into account eq. \ref{15} and the fact that $\mathcal{L}_{\nu} K = \partial_{\nu} K$, it follows that

\begin{equation}
(3) R + K^2 + K_{ij} K^{ij} - 2 \partial_{\nu} K = 8\pi G \rho + 4\Lambda.
\end{equation}

The scheme proposed by Buchert involves scalar quantities averaged over a compact domain $D$ with volume $V_D \equiv \int_D d^3X \sqrt{(3)g}$,

\begin{equation}
\langle \psi(t, X_i) \rangle_D \equiv \frac{1}{V_D} \int_D d^3X \sqrt{(3)g} \psi(t, X_i).
\end{equation}

Hence, in order to apply the averaging procedure, it is useful to re-arrange eqs. \ref{16} and \ref{17} taking into account the relations \ref{10}. In this way, we find the scalar equations

\begin{equation}
\frac{1}{2} \left( (3) R + \frac{2}{3} \theta^2 - 2\sigma^2 \right) = 8\pi G \rho + \Lambda,
\end{equation}

\begin{equation}
(3) R + \frac{4}{3} \theta^2 + 2\sigma^2 + 2\dot{\theta} = 8\pi G \rho + 4\Lambda,
\end{equation}

\begin{footnote}
Hereafter an overdot denotes differentiation with respect to the comoving time $t$ and the Latin indices $i$ and $j$ assume the values 1, 2, and 3.
\end{footnote}
where we have defined the shear scalar as $\sigma^2 \equiv \frac{1}{2}\sigma_{ij}\sigma^{ij}$. It is also useful to recall the energy conservation equation \([7]\), which takes the form
\[
\dot{\rho} = K\rho = -\theta \rho . \tag{21}
\]

In a spatially homogeneous and isotropic universe with curvature index $\kappa$ described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric \([8]\)
\[
d s^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-\kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \tag{22}
\]
and dominated by dust, one has
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2} , \tag{23}
\]
\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho + \frac{\Lambda}{3} , \tag{24}
\]
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} \rho = 0 . \tag{25}
\]

Using the averaging procedure, eqs. \([19]\)-\([21]\) can always be written in the form of a Friedmann-like system of averaged equations, following the operational definition \([18]\) and exploiting the non-trivial commutation relation that holds for any scalar quantity $\psi(t, X_i)$ \([25]\)
\[
\langle \psi(t, X_i) \rangle_D - \langle \dot{\psi}(t, X_i) \rangle_D = \langle \psi(t, X_i) \theta \rangle_D - \langle \psi(t, X_i) \rangle_D \langle \theta \rangle_D . \tag{26}
\]

Let us introduce also a dimensionless scale factor normalized by the volume $V_D$, of the region $D$ at some initial time $t_i$ as $a_D(t) \equiv (V_D/V_D)_{t_i}^{1/3}$, with the property that the averaged expansion rate is written as
\[
\langle \theta \rangle_D = \frac{V_D}{V_D} = \frac{3}{3} \frac{\dot{a}_D}{a_D} \equiv 3H_D . \tag{27}
\]

We define a “kinematical backreaction” term, vanishing on a FLRW background, as
\[
Q_D \equiv \frac{2}{3} \left( \langle \theta^2 \rangle_D - \langle \theta \rangle_D^2 \right) - 2 \langle \sigma^2 \rangle_D = \frac{2}{3} \langle \theta^2 \rangle_D - 2 \langle \sigma^2 \rangle_D - 6H_D^2 . \tag{28}
\]

The Einstein scalar equations and the covariant conservation equation now yield
\[
3 \left( \frac{\dot{a}_D}{a_D} \right)^2 - 8\pi G \langle \rho \rangle_D - \Lambda = -\frac{\langle (3) \mathcal{R} \rangle_D + Q_D}{2} , \tag{29}
\]
\[
3 \frac{\ddot{a}_D}{a_D} + 4\pi G \langle \rho \rangle_D - \Lambda = Q_D , \tag{30}
\]
\[
\langle \dot{\rho} \rangle_D + \langle \theta \rho \rangle_D = \langle \rho \rangle_D + 3 \frac{\dot{a}_D}{a_D} \langle \rho \rangle_D = 0 , \tag{31}
\]

\[\text{§} \] The Buchert scheme applies to vorticity-free spacetimes and it is not clear how to fit a small amount of rotation into a Buchert-like scheme. This issue deserves some attention in the future.
respectively. The energy constraint \( (29) \) and the Friedmann acceleration law \( (30) \) lead to a differential integrability condition involving \( Q_D \) and \( \langle (3) R \rangle_D \) that accounts for the coupling between 3-curvature and fluctuations:

\[
\frac{1}{a_D^6} \partial_t (Q_D a_D^6) + \frac{1}{a_D^2} \partial_t \left( \langle (3) R \rangle_D a_D^2 \right) = 0. \tag{32}
\]

The system of averaged equations is not closed because there are only three independent equations for the four unknown functions \( a_D, \langle \rho \rangle_D, Q_D, \langle (3) R \rangle_D \). This means that, in principle, different spacetimes could evolve in different ways even when they have the same average initial conditions. Extra assumptions are needed to close the system, for example assuming a certain effective cosmic equation of state, or demanding a particular functional relationship between \( Q_D \) and \( \langle (3) R \rangle_D \) (as it is done in \[25, 26\] in order to obtain scaling solutions).

3. Averaging procedure for scalar-tensor cosmology

It is convenient to write the field equations of scalar-tensor gravity in the form of effective Einstein equations, which allows for the direct application of Buchert’s formalism to this class of theories. It must be pointed out that choosing this form of the equations implies that the scalar field \( \phi \) plays the role of the inverse of a Newton “constant” now varying in space and time (the effective gravitational coupling in the action (1) is \( G_{\text{eff}} = \phi^{-1} \), although the coupling in a Cavendish experiment is instead \( G_{\text{eff}} = 1 / \phi \frac{2(\omega+2)}{2\omega+3} \) \[27\]). It is rather simple to notice that the presence of this extra field introduces a new ambiguity with respect to GR due to the non-linearity of the averaging procedure. In fact, the variation of the action (1) with respect to \( g^{\mu\nu} \) yields the field equations

\[
\phi G_{\mu\nu} = 8\pi \left( T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\phi)} \right), \tag{33}
\]

where \( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor and

\[
T_{\mu\nu}^{(\phi)} = \frac{\omega(\phi)}{\phi} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi \right) + \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi - \frac{V(\phi)}{2} g_{\mu\nu}. \tag{34}
\]

While it is common to divide by \( \phi \) to put this equation in the form of the effective Einstein equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}^{(m)} + \frac{\omega(\phi)}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi \right) + \frac{1}{\phi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right) - \frac{V(\phi)}{2\phi} g_{\mu\nu}, \tag{35}
\]

this operation does not commute with the spatial average if \( \partial \phi / \partial x^i \neq 0 \). As a result, once the scalar averaging has been performed, \( \langle \phi (4) R \rangle_D \neq \langle \phi \rangle_D \langle (4) R \rangle_D \). This problem does not appear in GR where the coupling is a true constant and is peculiar to scalar-tensor gravity. The outcomes of taking the average of eq. (33) or of eq. (35) are different. For ease of comparison with GR we choose to proceed by averaging eq. (35) but with a second caveat to keep in mind. Further, if one decides to adopt the Einstein conformal
frame instead of the Jordan frame, the relevant integro-differential equations can, in principle, have different solutions in the two frames. But this ambiguity remains even if we stay in the Jordan frame, depending on the choice one makes to use the scalar field directly linked to the gravitational sector or, as in our case, to recast the field equations as effective Einstein-like equations.

The variation of the action (1) with respect to the scalar field yields the equation of motion for $\phi$

$$\Box \phi = \frac{1}{2\omega(\phi) + 3} \left[ -8\pi\rho - \frac{d\omega}{d\phi} \nabla^\sigma \phi \nabla_\sigma \phi + \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right]. \quad (36)$$

The Hamiltonian constraint is obtained by double contraction of the previous equation with $u^\mu$ (time-time component of the field equations)

$$\frac{1}{2} (^{(3)}R + K^2 - K_{ij}K^{ij}) = \frac{8\pi\rho}{\phi} + \frac{\omega(\phi)}{2\phi^2} \dot{\phi}^2 + \frac{\omega(\phi)}{2\phi^2} g^{ij} \partial_i\phi \partial_j\phi$$

$$+ \frac{1}{\phi} \left( \ddot{\phi} + \Box \phi \right) + \frac{V(\phi)}{2\phi}, \quad (37)$$

while the evolution equation for the expansion scalar now reads

$$^{(4)}R = ^{(3)}R + K^2 + K_{ij}K^{ij} - 2\partial_iK$$

$$= -g^{\mu\nu} \left[ \frac{8\pi T_{\mu\nu}^{(m)}}{\phi} + \frac{\omega(\phi)}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\sigma \phi \nabla_\sigma \phi \right) \right.$$

$$\left. + \frac{1}{\phi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right) - \frac{V(\phi)}{2\phi} g_{\mu\nu} \right]$$

$$= \frac{8\pi\rho}{\phi} + \frac{\omega(\phi)}{\phi^2} \nabla^\mu \phi \nabla_\mu \phi + \frac{3\Box \phi}{\phi} + \frac{2V(\phi)}{\phi}. \quad (38)$$

By averaging the last two equations and using both the definition (25) of backreaction and the fact that $K^2 - K_{ij}K^{ij} = \frac{2}{3} \theta^2 - 2\sigma^2$, one obtains

$$\frac{1}{2} \left\langle ^{(3)}R \right\rangle_D + \frac{1}{2} Q_D + 3H_D^2 = 8\pi \left\langle \frac{\rho}{\phi} \right\rangle_D + \left\langle \frac{\omega(\phi)}{2} \frac{\dot{\phi}^2}{\phi^2} + g^{ij} \partial_i\phi \partial_j\phi \right\rangle_D$$

$$+ \left\langle \frac{\Box \phi + V(\phi)}{2\phi} \right\rangle_D, \quad (39)$$

$$\left\langle ^{(3)}R \right\rangle_D - Q_D + 6H_D^2 + 6 \frac{\ddot{a}_D}{a_D} = 8\pi \left\langle \frac{\rho}{\phi} \right\rangle_D + \left\langle \omega(\phi) \left( \frac{-\dot{\phi}^2 + g^{ij} \partial_i\phi \partial_j\phi}{\phi^2} \right) \right\rangle_D$$

$$+ \left\langle \frac{3\Box \phi + 2V(\phi)}{\phi} \right\rangle_D. \quad (40)$$

By combining the last two equations and using eq. (36) the cosmic acceleration is expressed as

$$\frac{\ddot{a}_D}{a_D} = -\frac{8\pi}{3} \left\langle \frac{\rho}{\phi} \left( \frac{\omega(\phi)}{2\omega(\phi) + 3} \right) \right\rangle_D + \frac{Q_D}{3} - \frac{1}{3} \left\langle \omega(\phi) \left( \frac{\dot{\phi}}{\phi} \right)^2 \right\rangle_D - \frac{1}{3} \left\langle \frac{\dot{\phi}}{\phi} \right\rangle_D. \quad (41)$$
\[-\frac{1}{6} \left\langle \frac{1}{2\omega(\phi) + 3} \frac{d\omega}{d\phi} \nabla^\sigma \phi \nabla_\phi \phi \right\rangle_D + \frac{1}{6} \left\langle \frac{1}{2\omega(\phi) + 3} \left( \frac{dV}{d\phi} + (2\omega(\phi) + 1) \frac{V}{\phi} \right) \right\rangle_D.\]

Since $\phi > 0$ and $\omega(\phi) > 0$ in order to keep the gravitational coupling positive, the positive energy density of dust in the first term on the right hand side causes deceleration.

The constraints on the magnitude of the factor $2(\omega + 2)/(2\omega + 3)$ depend on the range of the $\phi$. If the latter is comparable with the size of the solar system then the Cassini bound $\omega > 40000$ \cite{28} applies. However, this bound does not apply if the field is short-ranged or if endowed with a range depending on the environment (chameleon mechanism).

In an optimistic view, the backreaction term $Q_D$ is positive and contributes to acceleration, as generally argued in GR. However, this is not necessarily the case: in fact, prior to the 1998 discovery of the cosmic acceleration, the same backreaction term, with negative sign, was proposed as a solution to the dark matter problem (see \cite{29} and Sec. 5.5.2 of \cite{30}). This shows that the sign of $Q_D$ is highly uncertain. The third term on the right hand side of eq. (41) is definitely negative and contributes to decelerate the universe, while the signs of the fourth and fifth terms are undetermined.

There is little doubt that the terms involving the first and second derivatives of $\phi$ are small and, at best (i.e., when $\langle \ddot{\phi} \rangle_D < 0$) their effects conflict. However, the constraints on the temporal and spatial variation of $\phi$ after nucleosynthesis are rather poor. While the time variation of the gravitational coupling is constrained as $\left| \frac{\dot{G}}{G} \right| \simeq \left| \frac{\dot{\phi}}{\phi} \right| < H_0^{-1}$ (where $H_0$ is the present value of the Hubble parameter) \cite{15}, there is basically no constraint on the second time derivative of $\phi$.

The last term including the potential and its derivative is novel with respect to GR and could significantly affect the acceleration. While this could be interpreted as an obvious consequence of the fact that a potential can mimic a cosmological constant, we show later (see the case of $f(R)$ gravity discussed below) that it can be important and positive even in cases for which late time acceleration cannot be a priori expected from the form of the Lagrangian.

In summary, while no definitive conclusion can be reached on whether the inclusion of backreaction induces late time acceleration (as in the GR case), nonetheless there are encouraging new terms in scalar-tensor cosmology. Unfortunately no definitive answer on the relative magnitude and sign of the specific terms can be provided in such a general framework. Hence, in the following we shall consider specific implementation of the theory in which eq. (41) simplifies.

### 3.1. Brans–Dicke cosmology

As an example of the procedure developed, let us specialize the whole formalism to a true Brans–Dicke theory (i.e., $V \equiv 0$ and $\omega(\phi) \equiv \omega_0 = \text{constant}$) and let us also assume the scalar field to be spatially smooth on the scales of interest, $\phi = \phi(t)$. This is clearly an oversimplification but serves the purpose of illustration. This assumption
implies that all the averages involving the scalar field $\phi$ are domain-independent. In this context, the ambiguity in the choice of the representation described in the previous section is no longer present. Then, eqs. (39) and (40) become

$$\frac{1}{2} \langle (3) R \rangle_D + \frac{1}{2} Q_D + 3H_D^2 = 8\pi \frac{\langle \rho \rangle_D}{\phi} + \frac{\omega_0}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 - 3H_D \frac{\dot{\phi}}{\phi},$$

and

$$6\frac{\ddot{a}_D}{a_D} = -\langle (3) R \rangle_D + Q_D - 6H_D^2 + 8\pi \frac{\langle \rho \rangle_D}{\phi} - \omega_0 \frac{\dot{\phi}^2}{\phi^2} - 3 \frac{\dot{\phi} + 3H_D \phi}{\phi}. \quad (42)$$

The consistency relation between the Hamiltonian constraint and the Raychaudhuri equation can now be derived by differentiating the latter with respect to time and then substituting the result, the Hamiltonian constraint, and the equation of motion for the scalar field in the former. The result is

$$\frac{1}{a_D^2} \partial_t \left( Q_D a_D^6 \right) + \frac{1}{a_D^2} \partial_t \left( \langle (3) R \rangle_D a_D^2 \right) = \frac{2}{a_D^{6\omega_0 + 12}} \partial_t \left[ 8\pi \frac{\langle \rho \rangle_D}{\phi} a_D^{6\omega_0 + 12} \right] + \frac{1}{a_D^6} \partial_t \left[ \frac{\omega_0 \dot{\phi}^2}{\phi^2} a_D^6 \right] - \frac{6}{a_D^4} \partial_t \left[ \frac{\dot{\phi}}{\phi} H_D a_D^3 \right]. \quad (44)$$

As a check, it is noted that this equation reduces to the corresponding eq. (32) in the limit $\omega_0 \to \infty$, $\phi \approx \text{const.} + O \left( \frac{1}{\omega_0} \right)$ in which Brans–Dicke theory reduces to GR (this can be seen by using the form of the solution of eq. (31), $\langle \rho \rangle_D \propto a_D^{-3}$, in the first term on the right hand side of eq. (45)).

Let us consider a class of solutions in which the scalar field has the form

$$\phi(t) = \phi_0 + \phi_1 e^{-\beta t}, \quad (45)$$

where the requirement of a positive, non-vanishing scalar field implies $\phi_0, \beta > 0$ and $\phi_1 > -\phi_0$. Using the general solution of eq. (31) we can express the averaged energy density as $\langle \rho \rangle_D(t) = \langle \rho \rangle_D^0 a_D^{-3}(t)$, where the scale factor has been normalized at the starting time of the growth of structures (in our notation, $a_D(t = 0) = 1$ where $t = 0$ corresponds to the last scattering surface). Inserting this relationship into the equation of motion for $\phi$, it is possible to solve with respect to $a(t)$. The effective gravitational coupling is finite for both small and large times $t$, and the corresponding averaged scale factor is

$$a_D(t) = e^{\frac{\Delta t}{\beta}} (1 - \gamma t)^{1/3} \quad (46)$$

with

$$\gamma = \frac{8\pi \langle \rho \rangle_D^0}{\beta \phi_1 (2\omega + 3)}. \quad (47)$$

§ In the case of a massive dust, the limit of Brans–Dicke theory to GR is free of the ambiguities arising when $T^{(m)} = 0$ and the expansion $\phi = \text{const.} + O \left( \frac{1}{\omega_0} \right)$ is indeed correct (see [31] and references therein).
It is an easy task to show that late time accelerated solutions can be found for suitable values of the parameters. However, the physically motivated requirement that the backreaction is negligible at early stages further restricts the allowed range.

The following expressions for $\langle R \rangle_D$ and $Q_D$ are immediately obtained:

$$\langle R \rangle_D = \frac{\beta \phi_1 \gamma - 24\pi \langle \rho \rangle^0_D - 2\beta e^{\beta t} \phi_0 [\gamma (2 + \beta t) - \beta]}{2 (\phi_1 + e^{\beta t} \phi_0) (\gamma t - 1)},$$  

$$Q_D = \frac{\beta^2 \phi_1^2 \omega}{\phi_1 + e^{\beta t} \phi_0} + \frac{-8\pi \langle \rho \rangle^0_D + \beta \phi_1 [\gamma (2\beta t - 1) - 2\beta]}{2 (\phi_1 + e^{\beta t} \phi_0) (\gamma t - 1)} + \frac{1}{3} \left[ \frac{\beta^2}{\gamma t - 1} - \frac{2\beta^2}{(\gamma t - 1)^2} \right].$$  

The initial value of the backreaction term $Q_D$ could be different from zero (albeit small), as long as we assume a perturbed FLRW universe at the last scattering epoch. Furthermore, $Q_D$ approaches the asymptotic value $\beta^2/3$, giving a positive contribution to the acceleration.

3.2. Metric $f(R)$ gravity

We now consider the case of metric $f(R)$ gravity, described by the action

$$S' = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R) + S^{\text{(matter)}},$$

where $f(R)$ is a non linear function of its argument [20]. It is well known that this theory is equivalent to an $\omega = 0$ Brans–Dicke theory with Brans–Dicke scalar $\phi \equiv f'(R)$ and potential $V(\phi) = Rf'(R) - f(R)$ [19]. For the sake of illustration, let us take into account the Lagrangian density in the form $f(R) = R + \alpha R^n$ with $n > 1$ and $\alpha > 0$ as required for local stability [32]. Then, the potential can be expressed as

$$V(\phi) = \frac{n - 1}{n^{n-1} \alpha^{n-1}} (\phi - 1)^\frac{n}{n-1},$$

and eq. (11) reduces to

$$\frac{\dot{a}_D}{a_D} = -\frac{16}{9} \left\langle \frac{\rho}{\phi} \right\rangle_D + \frac{Q_D}{3} - \frac{1}{3} \left\langle \frac{\dot{\phi}}{\phi} \right\rangle_D + \frac{2n - 1}{18 n^{n-1}} \frac{1}{\alpha^{n-1}} \left\langle (\phi - 1)^\frac{1}{n-1} \right\rangle_D.$$  

$\alpha$ arises from quantum corrections and is presumably small, so it would seem that the last term on the right hand side of the previous equation is large. However, this is not the case because $(\phi - 1)^\frac{1}{n-1}$ is also small and contains the same power of $\alpha$: in fact, by expressing $(\phi - 1)$ as a function of $R$, the last term of eq. (52) is rewritten as $\frac{2n - 1}{18n} \langle R \rangle_D$. Nevertheless, it is relevant that this term is not suppressed by positive powers of $\alpha$, as

An example of such a solution can be found for the set of values $(\beta, \phi_0, \phi_1, \omega, \langle \rho \rangle^0_D) = (0.002, 750, -1, 40000, 1)$.  

\[parallel\]
one might expect, and hence it may contribute significantly to the cosmic acceleration. The third term on the right hand side, for small values of $\alpha$, is instead

$$-\frac{1}{3} \left\langle \frac{\dddot{\phi}}{\phi} \right\rangle_D \simeq -\frac{1}{3} \alpha n(n-1) \left\langle (n-2)R^{n-3}\dot{R}^2 + R^{n-2}\ddot{R} \right\rangle_D .$$

(53)

For the physically well-motivated case $n = 2$ associated to Starobinsky inflation in the early universe [18], this term reduces to $-\frac{2\alpha}{3} \left\langle \dot{R} \right\rangle_D$ and hence it is subdominant with respect to the last term of eq. (52). Finally for the first two terms on the right hand side of eq. (52) the same considerations presented after eq. (41) apply.

4. Conclusions

The increasing improvement in quality and quantity of the cosmological data motivates a proper evaluation of the backreaction of matter inhomogeneities. Hence, any test of alternative theories of gravitation will have to take into account possible corrections due to the backreaction mechanism, whether the latter are large or not. For this reason, we analyzed here the possibility of improving the averaging scheme in the prototypical alternative theories of gravity, the scalar-tensor ones.

Keeping this goal in mind and following the path outlined by Buchert and collaborators, we have derived two scalar equations (the Hamiltonian constraint and the equation for the scale factor) from contractions of the field equations written in the form of effective Einstein equations. The more general working frame exposed an intrinsic ambiguity of the averaging proposal related to the scalar degree of freedom in scalar-tensor theories. The ambiguity is twofold as it leads to different averaged equations for different conformal frames and, within a chosen frame, to different results depending on the way the field equations are cast at the beginning of the calculation. We made here the choice of working in the Jordan conformal frame and later on in the calculation the ansatz of a domain-independent scalar field allowed us to circumvent the ambiguity linked to the non-commutativity of the operations involved.

As in GR, the system of equations obtained is not closed, hence one extra assumption is needed in order to solve it. The backreaction term $Q_D$, and other terms as well, have signs that are undetermined and no clear effect. This is not too surprising, considering that a loss of information is unavoidable whenever an average is performed. Averaging makes it impossible to disentangle the individual contributions of inhomogeneities and anisotropies, but here even the collective effects are uncertain. While no definitive conclusion can be reached (as in the GR case), nonetheless there are encouraging new terms in scalar-tensor cosmology. In particular, we noticed that the term including the scalar field potential and its derivative could significantly affect the acceleration.

In order to gain a better understanding of the potentialities of the backreaction terms in eq. (41) to contribute significantly to late time acceleration we finally specialized to two specific sub-cases, namely Brans-Dicke and metric $f(R)$ gravity. In the first case
we have provided, as a proof of principle, a toy model solution which is accelerated at late times due to the presence of the Brans–Dicke scalar field $\phi$. In the second case, we have studied a polynomial Lagrangian using the connection between metric $f(R)$ and scalar-tensor theories. While it is natural to expect that higher order corrections to the Einstein–Hilbert Lagrangian would be suppressed by their small dimensional coefficients, we found that a generic $\alpha R^n$ term contributes via the potential term in eq. (41) without showing any suppression in $\alpha$. Moreover, the fact that this term is now proportional to the averaged Ricci scalar implies that it is not necessarily small at late times.

The analysis outlined here would certainly benefit from exact solutions — even simplified toy models such as Lemaitre–Tolman–Bondi solutions [33] — in order to better understand the role of matter inhomogeneities in scalar-tensor theories. The study of these exact models will be pursued elsewhere.

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References


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