Modified dispersion relations from the renormalization group of gravity

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We show that the running of gravitational couplings, together with a suitable identification of the renormalization group scale can give rise to modified dispersion relations for massive particles. This result seems to be compatible with both the frameworks of effective field theory with Lorentz invariance violation and deformed special relativity. The phenomenological consequences depend on which of the frameworks is assumed. We discuss the nature and strength of the available constraints for both cases and show that in the case of Lorentz invariance violation, the theory would be strongly constrained.

I. INTRODUCTION

The search for a quantum theory of gravity has been, for more than half a century, a driving force in theoretical physics. Unfortunately, this research has for a long time been frustrated by the intrinsic difficulty of testing theoretical predictions relevant at energy scales near the Planck scale, $M_P = G^{-1/2} = 1.22 \times 10^{19}$ GeV (we use units with $c = 1, \hbar = 1$). However this picture has recently been changed by the realization that several (though not all) quantum gravity models seem to predict a departure from exact Lorentz invariance due to ultraviolet physics at the Planck scale. This could lead either to effective field theories (EFTs) characterized by Planck suppressed Lorentz violations for elementary particles (see e.g., [2]) or to some new physics where the Lorentz symmetry is deformed in order to include an extra invariant scale (the Planck scale) apart from the speed of light. The latter framework is generally called deformed special relativity (DSR).

Common to all these models is that they seem to predict a modified form of the free particle dispersion relation, exhibiting extra momentum dependent terms, apart from the usual quadratic one occurring in the Lorentz invariant dispersion relation. In particular one most often considers violations or deformations of the boost subgroup, leaving rotational invariance unaffected and leading to an expansion of the dispersion relation in momentum dependent terms

$$E^2 = m^2 + p^2 + F(p, \mu, M_P)$$
$$= m^2 + p^2 + \sum_{n=1}^{\infty} \alpha_n(\mu, M_P) p^n, \quad (1)$$

where $p = \sqrt{||p||^2}$ and $\mu$ is some particle physics mass scale. In the following, we assume it to be equal to $m$, the mass of the particle.

In the case of an EFT with Lorentz invariance violation (LIV), strong constraints on the coefficients $\alpha_n$ for the cases $n = 1, 2, 3$ have been obtained [1], and there is some hope that LIV with $n = 4$ will be constrained in the near future by forthcoming experiments and improved observations [2].

In the case of deformed special relativity (DSR) constraints are more uncertain, as we are still lacking a satisfactory understanding of the theory in coordinate space and hence of the corresponding EFT. There are however conjectures about the phenomenological implications of such a framework and some constraints have been tentatively provided (e.g., for the GZK threshold and $n = 3$ [3]).

In this paper, we want to explore a mechanism that could lead to the emergence of such modified dispersion relations (MDRs), based on the idea that the spacetime structure could be an emergent concept. If so, it would be natural to expect that at sufficiently high energies the effective spacetime metric could become energy dependent. We shall see that sizeable effects could occur well below the Planck scale. It is interesting to note that such a framework closely resembles that of emergent effective geometries and Lorentz symmetries characterizing many condensed matter systems (see e.g., [5, 6]). There, linearized perturbations propagate on Lorentzian geometries in the low energy (phononic) limit but do show MDRs of the kind of Eq. (1) as the perturbation’s wavelength approaches the inter-molecular distance, or coherence length, below which the background cannot be considered a continuum. This phenomenon can be seen as an energy dependent background metric interpolating between a purely Lorentzian (and energy independent) form at very low energies, and a pre-geometric structure present at ultra-short length scales (sub-Planckian in the quantum gravity scenario).

Albeit intriguing, the above described phenomenology of condensed matter systems is still an analogy after all. So one might wonder if there is a framework within known quantum gravity models that could naturally produce such an energy dependence of the metric. Some EFT models with this property had been discussed in [5]. There it was argued that in certain gauge theories of gravity with torsion, the renormalization group (RG) flow of the couplings would produce a scale depen-
fluctuations of the fields with momenta above \( k \). Clearly, this effective action depends on energy scale \( \Lambda \). Wilson's insight is that at some quantum gravity phenomenology. We shall then analyze some constraints that can be cast on at MDRs for the propagation of massive particles. We shall then analyze some constraints that can be cast on such quantum gravity phenomenology.

II. THE RG OF GRAVITY

There are many different viewpoints on what to expect from a quantum theory of gravity, but a common feature to all is that they should reduce to general relativity at low energies. In quantum field theory, a model that holds only in a certain energy range is called an EFT. The best-known example is the nonlinear sigma model description of mesons, which is thought to be a low-energy approximation of QCD. It is now well understood that Einstein's theory can be treated in the same way [12]. Therefore, it is not necessary to regard Einstein's theory as a purely classical one: graviton loop effects can be computed, as long as the corresponding momentum integrations are cut off at some scale.

In any quantum field theory, whether fundamental or effective, the values of the coupling constants are not fixed but depend on the choice of a mass scale \( k \). Following Wilson's insight, an EFT describing physics at some energy scale \( k \) is the result of functionally integrating all fluctuations of the fields with momenta above \( k \) (thus, \( k \) can be regarded as an infrared cutoff). This results in an effective action \( \Gamma_k \) for the components of the fields with momenta lower than \( k \). A priori in an EFT, \( \Gamma_k \) will contain all the terms that are allowed by the symmetries of the theory. Thus, in the case of gravity, it can be written as

\[
\Gamma_k(g_{\mu\nu}) = \sum_i g_i(k)O_i, \tag{2}
\]

where \( O_i \) are integrals of scalar functions of the metric and its derivatives, and \( g_i \) are coupling constants. Clearly, this effective action depends on \( k \); the RG describes the way in which the couplings \( g_i \) flow in dependence of \( k \). It is important to stress that at this general level the physical meaning of \( k \) is not determined: for each physical process one will have to identify the relevant RG scale. This is a crucial step requiring physical insight.

In this paper, we restrict ourselves to the sub-Planckian (\( k^2 \ll G^{-1} \)) and small curvature (\( k^2 \gg R \)) regime, where it is usually believed that General Relativity is a good approximation (for different views see e.g. [13]). We therefore assume the validity of the Einstein–Hilbert action, possibly with a cosmological constant:

\[
\Gamma_k[g] = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} \left( 2\Lambda_k - R \right). \tag{3}
\]

The running of the gravitational couplings can be computed inserting this action in an exact RG equation [11, 14]. This approach can provide also nonperturbative information and has been applied in the search of a UV fixed point [15, 17, 18]. Let us stress, however, that we do not need to commit ourselves to any specific model of Planck scale physics, in particular we do not assume the existence of an UV fixed point.

In the regime we are interested in, one obtains from the RG equation the following \( \beta \)-functions for Newton’s constant and the cosmological constant [11]:

\[
k \frac{\partial}{\partial k} \left( \frac{1}{G_k} \right) = a k^2 \tag{4}
\]

\[
k \frac{\partial}{\partial k} \Lambda_k = b G_k k^4 \tag{5}
\]

where \( a \) and \( b \) are some order one, positive coefficients. The \( \beta \)-functions for gravity have also been discussed in different approaches (see e.g. [19, 20, 21, 22, 23, 24]).

The solutions to these ordinary differential equations are

\[
1 \frac{1}{G_k} = 1 \frac{1}{G_{k_0}} + a \frac{k^2}{2} - k_0^2 \tag{6}
\]

\[
\Lambda_k = \Lambda_{k_0} + b G_{k_0} \frac{k^4}{4} - b G_{k_0} \frac{k_0^4}{4} \tag{7}
\]

Here \( k_0 \) is the scale at which the initial conditions are set. In fact, \( G \) and \( \Lambda \) are measured on quite different scales. The value \( G^{-1} \approx M_P^2 \approx 1.49 \times 10^{38} \text{GeV}^2 \) is measured to be the same from laboratory up to planetary distance scales, whereas for the cosmological constant we have a value \( \Lambda_{k_0} \approx 1.75 \times 10^{-123} M_P^4 \) measured at the Hubble scale \( H_0 \approx 10^{-42} \text{GeV} \).

From equation \( \beta \) we see that the running of \( G \) is highly suppressed below \( M_P \) and hence will be neglected throughout. We see instead from Eq. \( \beta \) that the running of \( \Lambda \) becomes significant at energies of order \( k_T \approx \)

\[1\] Note that if we allow for the presence of minimally coupled matter fields, the form of equations \( \beta \) and \( \beta \) will not change but the values of \( a \) and \( b \) are affected [13]. We shall discuss the possible relevance of this fact in the conclusions.
$10^{-31}M_P \approx 10^{-3}$ eV or higher. (This corresponds to the “turning point”, in the language of [26].) This significant running of the cosmological constant at relatively low scales will play a crucial role in our analysis.

We have not yet provided a prescription to determine $k_0$ as a function of some physical scale. Given however that there is no strong evidence for a present running of the cosmological constant at cosmological scales, we will assume, for the moment, that $k_0$ is placed below $k_T$ far enough in the infrared to be always negligible. We shall check a posteriori that such an assumption is justified in the cases of our interest.

The equations of motion (EOM) at scale $k$ are obtained varying the effective action with respect to the metric,

$$\frac{\delta \Gamma_k}{\delta g_{\mu \nu}} = 0.$$

(8)

The solutions of the EOM at scale $k$ give the metric relevant for the physical process under consideration, with the couplings evaluated at $k$. In the theory with action [3] the EOM are

$$R^\mu_\nu [g_k] = \Lambda_k \delta^\mu_\nu.$$

(9)

Since $R^\mu_\nu [c g] = c^{-1} R^\mu_\nu (g)$ for any constant factor $c > 0$, equation (9) can be rewritten as

$$R^\mu_\nu [g_{k_0}] = \Lambda_{k_0} \delta^\mu_\nu = R^\mu_\nu \left[ \frac{\Lambda_k}{\Lambda_{k_0}} g_k \right].$$

(10)

where we have used the coordinate independence of $\Lambda_k$. Therefore, for any solution of equation (9) the inverse metric scales with the cosmological constant as [28]

$$g^\mu_\nu_k = \frac{\Lambda_k}{\Lambda_{k_0}} g^\mu_\nu_{k_0}.$$ 

(11)

We want now to analyze the consequences of such a scaling behaviour of the spacetime metric on the propagation of a free particle.

### III. MODIFIED DISPERSION RELATIONS FROM A “RUNNING” METRIC

Starting from Eq. (11) we can derive a MDR by contracting it with the particle’s four momentum and identifying $k$ with a function of the momentum. In the presence of an effective cosmological constant the solution of the EOM cannot be flat space. However, we want to work in a regime where the typical wavelength of the particle is much smaller than the characteristic curvature radius of spacetime, in our case $1/p \ll 1/\sqrt{R} \approx 1/\sqrt{\Lambda_k}$. We can then approximate $g^{\mu\nu}$ by a flat metric and equation (11) just results in an overall scaling of the latter. Of course, in order to check that the above condition holds, we need to know the relation between $p$ and $k$. We shall check a posteriori that this is indeed the case for our choice of $k$.

A global rescaling of the metric can be eliminated by a choice of coordinates, but this can only be done at a particular scale. We choose $g^{\mu\nu}_0 = \eta^{\mu\nu}$ (the usual Minkowski metric $(1, -1, -1, -1)$) for any $k = k_0 < k_T$. At scale $k > k_T$, we have the metric $g^{\mu\nu}_k$ defined by

$$g^{\mu\nu}_k = \frac{\Lambda_k}{\Lambda_{k_0}} \eta^{\mu\nu};$$

(12)

and contracting both sides of Eq. (12) with the particle four momentum we then obtain

$$m^2 = \frac{\Lambda_k}{\Lambda_{k_0}} \eta^{\mu\nu}_0 p_\mu p_\nu = \frac{\Lambda_k}{\Lambda_{k_0}} (E^2 - p^2),$$

(13)

where we have defined the mass to be $m^2 = g^{\mu\nu}_k p_\mu p_\nu$ and $p_\mu = (E, -\vec{p})$. So, using Eq. (12), one finally gets

$$E^2 - p^2 = \frac{\Lambda_{k_0}}{\Lambda_k} m^2 = \left( 1 + \frac{b}{4} \frac{\Lambda_k}{\Lambda_{k_0}} \right)^{-1} m^2$$

(14)

where $X = M^2_p/\Lambda_0 \approx 6 \cdot 10^{22}$.

In order to proceed further in our analysis we now need to clarify the relation between the RG scale $k$ and the particle momentum. Assuming that rotational invariance is preserved, one can predict that for a free particle the RG scale $k$ will be generically determined by the modulus of the particle’s three-momentum $p := \sqrt{\vec{p}^2}$ (or alternatively its energy given that they are practically the same, at first order, for high energy particles), its mass $m$, and possibly by the Planck scale. As we expect that any deviation from standard physics should be Planck suppressed, we can then write the following ansatz

$$k = \frac{p^\alpha m^\beta}{M^\alpha P_{\beta - 1}},$$

(15)

where $\alpha$ and $\beta$ are chosen to be positive integers. The above ansatz is of course inspired by the standard framework adopted in most of the quantum gravity phenomenology literature (see e.g. [1]) and for any $\alpha \neq 0$ it will lead to dispersion relations characterized by higher order terms in the momentum of the particle suppressed by appropriate powers of the Planck mass.

For sufficiently low momenta the dispersion relations so obtained will take the form

$$E^2 = p^2 + m^2 \left( 1 - \frac{b}{4} X \left( \frac{m}{M_P} \right)^{4\beta} \left( \frac{p}{M_P} \right)^{4\alpha} \right).$$

(16)

Note that due to the factor $m^2$ there is no modification of the dispersion relations for massless particles and that for a particle at rest $E = m$ as it is expected. Let us also emphasize that the above dispersion relation was derived assuming a point-like particle, as it is not clear at this stage which quantities might enter in the relation between $k$ and the physical momentum for composite particles. For this reason, we shall in what follows focus on electrons/positrons.
In applications of the RG to high-energy physics, where one considers mainly scattering processes, $k$ is usually identified with one of the Mandelstam variables of the process, a Lorentz invariant combination of the incoming particle momenta. From this point of view the most obvious and conservative choice would be to assume that $k$ is the unique Lorentz invariant function of the particle momentum, namely its mass. This would imply $(\alpha, \beta) = (0,1)$. Of course from the perspective of this work this is an uninteresting choice, as it implies that, for a given particle type, $k$ is fixed once for all: an ultra–high–energy particle would “feel” the same spacetime as one at rest. We shall then consider the case of $\alpha \neq 0$.

Conversely one might wonder if there could be some strong motivation to rule out a priori the mass dependence of relation (15). One possible argument can be based on the requirement that the natural condition $\Lambda_k = M_P \approx M_P^2$ holds. This implies that the corresponding physical momenta will be $p = M_P (M_P/m)^{(\beta/\alpha)}$. The case $\beta = 0$ is then the only one for which $k$ and $p$ would coincide at the Planck scale. Albeit appealing this feature of the $\beta = 0$ class of models does not seem sufficient for excluding a priori the other kinds of dispersion relations. We shall hence, for the moment, consider all the possible values of $\alpha$ and $\beta$ selecting them only on the basis of their phenomenological viability. However it is interesting to note that, in the end, such analysis will indeed select for us a dispersion relation belonging to the $\beta = 0$ class.

Before discussing the phenomenological viability of the above class of dispersion relations, it is perhaps important to stress that while the Planck scale dependence of (15) does imply a departure from standard GR at this scale, as generally expected, it does not necessarily conflict with the possible existence of an UV fixed point (16), (17), (18). In fact the fixed–point action will almost certainly not be the Einstein–Hilbert action (as this is the scale of the most energetic QED particles observed so far). Looking at Table I we see that the only case of phenomenological interest seems to be $(\alpha, \beta) = (2,0)$. Higher values of $(\alpha, \beta)$ are not a priori incompatible with observations, but at the moment lie beyond observational reach.

Before starting to consider the case $(\alpha, \beta) = (2,0)$ let us note however that the cases with $\alpha = 1$ are particularly interesting from a theoretical point of view as they would lead to an MDR of the form

$$E^2 = p^2 + m^2 + \eta_{\alpha=1} \frac{p^4}{M_P^2}$$

(17)

with $\eta_{\alpha=1} = -b/4 X (m/M_P)^{2+43}$. What is noticeable in our case is that the dimensionless coefficients $\eta$ do indeed contain, as conjectured (see e.g. (2)), powers of the small ratio $m/M_P$. These are however not necessarily leading, in the present framework, to an overwhelming suppression of the LIV term. In contrast, the presence of the huge numerical factor, $X$, that we inherited from the initial conditions (the observed value of the cosmological constant on cosmological scales) basically allows us to rule out the most obvious case $(\alpha, \beta) = (1,0)$ as this would lead to sizeable deviations from standard physics for any particle above $10^{-3}$ eV. If the observed $\Lambda_k$ contained also the contribution of some quintessence-like fields, the “true” cosmological constant at $k_0$ would be smaller hence leading to an even larger value of $X$.

The choice of parameters $(\alpha, \beta) = (2,0)$ gives

$$k = \frac{p^2}{M_P}$$

(18)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
<th>$\alpha = 3$</th>
<th>$\alpha = 4$</th>
</tr>
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<tbody>
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<td>$\beta = 0$</td>
<td>$3 \cdot 10^{-15}$</td>
<td>6.1</td>
<td>$7 \cdot 10^{-9}$</td>
<td>$3 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>$1 \cdot 10^{-7}$</td>
<td>$\cdot 10^{11}$</td>
<td>$2 \cdot 10^{13}$</td>
<td>$1 \cdot 10^{14}$</td>
</tr>
<tr>
<td>$\beta = 2$</td>
<td>$2 \cdot 10^{10}$</td>
<td>$\cdot 10^{15}$</td>
<td>$6 \cdot 10^{20}$</td>
<td>$4 \cdot 10^{20}$</td>
</tr>
</tbody>
</table>

TABLE I: The critical energies for order–one deviations from standard physics given in TeV for electrons/positrons at different combinations of $\alpha$ and $\beta$. The same order as the mass term, so that the approximation taken in order to derive Eq. (15) breaks down. This would indicate where the Planck–suppressed term starts introducing a running mass term for the particle and hence producing a detectable phenomenology for example via threshold reactions. Results for such critical values of the particle momentum for different choices of $\alpha$ and $\beta$ are given for the case of the electron (assuming $b = O(1)$) in Table I. The dispersion relations (16) will be of course phenomenologically acceptable if the modifications to standard physics arise only for very high energy particles. However we do also have to ask that the critical value of the momentum is not too high so that the corresponding MDR might lead to observable effects and consequently be subject to observational constraints. For example, in the case of QED, the most interesting MDRs will be those for which observable effects are expected at TeV energies, as this is the scale of the most energetic QED particles observed so far. Looking at Table I we see that the only case of phenomenological interest seems to be $(\alpha, \beta) = (2,0)$. Higher values of $(\alpha, \beta)$ are not a priori incompatible with observations, but at the moment lie beyond observational reach.

A. Phenomenological viability

A good indicator of the phenomenological viability of the above class of dispersion relations is easily obtained by considering when, for some choice of the parameters $\alpha$ and $\beta$, the Lorentz violating term becomes of

$$\eta_{\alpha=1} = -b/4 X (m/M_P)^{2+43}$$

(17)

What is noticeable in our case is that the dimensionless coefficients $\eta$ do indeed contain, as conjectured (see e.g. (2)), powers of the small ratio $m/M_P$. These are however not necessarily leading, in the present framework, to an overwhelming suppression of the LIV term. In contrast, the presence of the huge numerical factor, $X$, that we inherited from the initial conditions (the observed value of the cosmological constant on cosmological scales) basically allows us to rule out the most obvious case $(\alpha, \beta) = (1,0)$ as this would lead to sizeable deviations from standard physics for any particle above $10^{-3}$ eV. If the observed $\Lambda_k$ contained also the contribution of some quintessence-like fields, the “true” cosmological constant at $k_0$ would be smaller hence leading to an even larger value of $X$.

The choice of parameters $(\alpha, \beta) = (2,0)$ gives

$$k = \frac{p^2}{M_P}$$

(18)
and

\[ E^2 = m^2 + p^2 - \frac{b}{4} X \frac{m^2 p^8}{M_P^8}, \]  

(19)

which can be cast in the more suggestive form

\[ E^2 = m^2 + p^2 + \eta \frac{p^8}{M_P^8}, \]  

(20)

with \( \eta = -(b/4) X (m/M_P)^2 \).

Let us start noting that Eq. (18), together with Eq. (7), implies that at sub-Planckian energies \( (p \ll M_P) \) the de Broglie wavelength of the particle is always much smaller than the curvature radius of spacetime as we initially assumed. Furthermore it is also much smaller than the inverse of the RG parameter \( k \). This can be interpreted as saying that the particle has a somewhat lower resolution in probing spacetime than naively expected. We shall discuss this at length in the next section. Passing to the dispersion relation Eq. (19) we already saw (see Table I) that it leads to order-one deviations around 10 TeV. The reason for this lies again in the presence of the huge numerical factor \( X \) which is able to contrast the large Planck suppression. This feature makes the above dispersion relation compatible with current (low energy) observations while at the same time amenable to experimental constraints via high-energy astrophysics observations of QED phenomena. From the experimental point of view, \( (\alpha, \beta) = (2, 0) \) is therefore the most interesting value. We shall argue now that theoretically it is the best motivated.

\section*{B. Physical motivation for the case \( (\alpha, \beta) = (2, 0) \) }

In order to motivate physically the choice of the set of parameters \( (\alpha, \beta) = (2, 0) \) we shall start by addressing the question of the influence of the effective cutoff on the fluctuations of the gravitational field, which is relevant for the propagation of a free particle. As mentioned above, \( k \) marks the distinction between those modes that are integrated over and those that have to be treated classically in the resulting EFT.

As before, we assume that, in the absence of the particle, spacetime would be effectively flat (or anyway would have a curvature much smaller than \( m^2 \)). The particle produces a disturbance in the gravitational field in the form of local curvature and the fluctuations of the gravitational field will be affected by this curvature.

We have to consider the specific dynamics describing the effect of the particle on the gravitational field, which we have assumed to be given by Einstein’s equations. Stripped of all indices, these equations tell us that the second derivatives of the metric, or the square of its first derivatives, are of the order of \( G \rho \), where \( \rho \) is a typical component of \( T_{\mu \nu} \). The solution depends on the distance from the particle, and for a classical particle becomes singular near the origin. These issues do not arise when we take into account the quantum nature of the particle. The position of the quantum particle cannot be determined with a precision greater than the de Broglie wavelength \( \lambda = 1/p \). For an order of magnitude estimate, we can therefore spread the total energy and momentum \( p \) in a box of size \( 1/p \), so the order of magnitude of the diagonal components of the energy momentum tensor must be \( \rho \approx p^3 \). (We observe that this is also the vacuum energy density in a box of size \( 1/p \)). Einstein’s equations then give an estimate for the curvature

\[ (\partial g)^2 \approx R \approx G \rho \approx \frac{p^4}{M_P^2}. \]  

(21)

Of course, the gravitational field will fall off away from the particle; this equation gives just a characteristic value for the curvature very near the position of the particle. It says that a particle of momentum \( p \) excites Fourier modes of the gravitational field with momenta up to \( p^2/M_P \). Gravitational modes with higher momenta are essentially unaffected by the particle. It is therefore natural to assume that in the EFT these are the modes which have to be integrated over, meaning that the relevant cutoff is \( k \approx \rho^2/M_P \ll p \).

It is interesting to observe that the same argument, applied to a charged particle in an electromagnetic field, would yield a completely different result. In fact, the order of magnitude of the charge and current density is \( p^3 \), and from Maxwell’s equations one gets an estimate \( F \approx p^2 \). Since \( F \) has dimension of mass squared, we conclude that the characteristic momentum scale of the electromagnetic field generated by a particle of momentum \( p \) is \( k \approx \sqrt{F} \approx p \). This corresponds to the naive estimate \( \alpha = 1 \). Clearly, the different behavior is determined by the fact that the coupling constant of electromagnetism is dimensionless, while that of gravity has the dimension of area.

In closing this section, we observe that in the case of a Friedmann–Lemaître–Robertson–Walker metric, our ansatz \( k = \sqrt{G \rho} \) corresponds to the Hubble scale \(^3\). Interestingly this choice has also been advocated for applications to cosmology on the basis that the Bianchi–identity has to remain valid even in the presence of running coupling constants \([21, 23, 24]\). However, we stress that the framework presented here significantly differs from the one above. For example, in their argument the RG scale is assumed to be a function of the cosmological time \( k = k(t) \). This would be incompatible with Eq. (11) which explicitly relies on the coordinate independence of the cosmological constant. The same is valid for the

\(^2\) Note that the same cutoff has been derived in other frameworks in \([22, 33]\).

\(^3\) Incidentally this confirms that our assumption of neglecting \( k_0 \) in Eq. (7) was justified. In fact in this case \( k_0 \approx H_0 \approx 10^{-33} \) eV which is definitely negligible for any particle with energies well above few meV.
models discussed in [31,32], where spherical symmetry of spacetime leads to a radial dependence of the couplings giving rise to a modified action from which also Brans–Dicke theory can be obtained. In our argument, the cutoff \( k \) is determined by the properties of the object/apparatus that probes the spacetime metric, it does not depend \textit{a priori} on the characteristic scale of the universe.

IV. PHYSICAL INTERPRETATION AND PHENOMENOLOGY

The derived MDR has the form of a global scale transformation, so that a massless particle ends up probing the same spacetime no matter what energy it has. However, there will be phenomenological consequences for massive particles. To make some precise experimental predictions, we need to choose the framework in which we interpret this MDR. There are two main (inequivalent) candidates: EFT with LIV, or DSR. Both frameworks seem to be a priori compatible with our MDR. The main difference between the two theories is in the way in which momenta add up and possibly in the spacetime (non)commutativity.

A. EFT with LIV

The EFT framework has several advantages when discussing phenomenological consequences of LIV. Apart from being a well known and versatile framework it is also able to make sharp predictions as it allows to use the standard energy-momentum conservation and requires for its applicability just locality and local spacetime translational invariance above some length scale. In this context, one can apply the usual QFT tools, bearing in mind that the effective action will contain explicit Lorentz breaking terms, and hence cast constraints using experimental or observational tests. In particular for Lorentz violations at order \( O(p^3/M) \) and higher (i.e. induced by nonrenormalizable operators of mass dimension 5 or greater in the action) the most appropriate tests are coming from high-energy astrophysical observations (see e.g. [33]), as these are among the highest energy phenomena we can access nowadays.

Not all of the above–cited astrophysical tests can be applied to our framework. In particular cumulative effects based on the propagation of photons over cosmological distances [2,33] are unavailable as no modification is induced in dispersion relations of massless particles. Similarly some anomalous reactions, like the vacuum Čerenkov \( e^\pm \rightarrow e^\pm \gamma \) emission [34], are not allowed for “subluminal” dispersion relations of the leptons (and unmodified photons) like the one we found in Eq. (19).

Finally one might consider possible constraints coming from the shifting of normally allowed threshold reactions as the photon pair production, \( \gamma \gamma \rightarrow e^+e^- \), or the GZK reaction, \( p\gamma \rightarrow p\pi^0 \) (with \( \gamma \) a CMB photon, see e.g. [34,35]). Apart from the previously mentioned caveats related to the application of our MDR to composite particles, such a route is again unfeasible within our framework. In fact the analysis of such scattering reactions would require a new derivation of the relation between the RG parameter \( k \) and the physical momenta which (missing a better understanding of the theory) would require more arbitrary choices and assumptions on our side. Hence, given the above theoretical and observational uncertainties, characterizing these scattering reactions is beyond the scope of this paper. We can however provide very strong constraints on the MDR provided in Eq. (19) by considering the so called “photon decay”, \( \gamma \rightarrow e^+e^- \), usually forbidden by momentum conservation, and the synchrotron emission.

1. Photon decay

Within an EFT framework the photon decay process becomes possible above a certain threshold energy once one has dispersion relations violating Lorentz invariance. This threshold energy is given by the minimal momentum of the incoming photon such that the decay could happen preserving energy–momentum conservation. Following the steps from [34] and referring to eqs. (14) and (15), one obtains for the threshold momentum

\[
p_{\text{th}} = \left( \frac{10^{2}}{bX} \right)^{1/8} \left( \frac{M_{P}}{m_{e}} \right)^{\beta/2} M_{P}.
\]

In the case \( (\alpha, \beta) = (2, 0) \), \( b = 1 \) and using \( m_{e} \approx 0.511 \) MeV, the threshold energy for this process is \( p_{\text{th}} \approx 9.75 \) TeV. Moreover, following the steps in [2], one can calculate the decay rate of the photon, which turns out to be extremely fast, so much that a photon would not be able to propagate on any distance of astrophysical relevance. We then see that the observations of photons with energies above \( \approx 10 \) TeV propagating on astrophysical distances allow us to put upper bounds on the coefficient \( b \).

A strong constraint comes from the observation of high-energy \( \gamma \)-rays emitted by the Crab nebula [2]. The \( \gamma \)-ray spectrum of this source is very well understood. It results from a high–energy wind of electrons (and possibly positrons) which leads to a combination of synchrotron emission and inverse Compton scattering of (mainly the synchrotron) photons. The inverse Compton \( \gamma \)-ray spectrum so produced extends up to energies of at least 50 TeV. This implies that for these photons the threshold energy for their decay must be above 50 TeV. We must have

\[
b \leq \left( \frac{9.75 \text{ TeV}}{p_{\text{obs}}} \right)^{8},
\]

which for \( p_{\text{obs}} = 50 \) TeV gives a bound on \( b \) of order \( 10^{-6} \). This is clearly a very strong constraint since \( b \) is naturally of order one.
2. Synchrotron radiation

An even stronger constraint can be provided by the observation of high-energy synchrotron emission from the Crab nebula. Cycling electrons in a magnetic field \(B\) emit synchrotron radiation with a spectrum that sharply cuts off at a frequency \(\omega_c\) given by the formula

\[
\omega_c = \frac{3}{2} \frac{eB}{m_e} \gamma^3(E), 
\]

where \(\gamma(E) = (1 - v^2(E))^{-1/2}\) and \(v(E)\) is the electron’s group velocity. The formula (24) is based on the electron trajectory for a given energy in a given magnetic field, the radiation produced by a given current, and the relativistic relation between energy and velocity (see [2, 36] for a discussion about the validity of this formula in EFT with LIV and a detailed derivation of the constraint).

The maximum synchrotron frequency \(\omega_c^{\text{max}}\) is obtained by maximizing \(\omega_c\) with respect to the electron energy, which amounts to maximizing \(\gamma^3(E)/E\). Using the MDR (19) one can easily calculate the modified group energy, which amounts to maximizing by maximizing \(\eta\).

\[
\omega_c^{\text{max}} = 0.47 \frac{eB}{m_e} \left[ -\eta \left( \frac{m_e}{M_p} \right)^6 \right]^{-2/8}, \quad (25)
\]

where \(\eta = -b/4X (m_e/M_p)^2\). This maximum frequency is attained at the energy \(E_{\text{max}} = (-m_e^2 M_p^8/35\eta)^{1/8} \approx 4.2 b^{-1/8} \, \text{TeV}\).

The rapid decay of synchrotron emission at frequencies larger than \(\omega_c\) implies that most of the flux at a given frequency in a synchrotron spectrum is due to electrons for which \(\omega_c\) is above that frequency. Thus \(\omega_c^{\text{max}}\) must be greater than the maximum observed synchrotron emission frequency \(\omega_{\text{obs}}\). This yields the constraint

\[
b < \frac{4}{X} \left( \frac{M_p}{m_e} \right)^8 \left( \frac{0.47eB}{m_e \omega_{\text{obs}}} \right)^{8/2}. \quad (26)
\]

Using as in [36] the observation of synchrotron emission from the Crab nebula up to energies of about 100 MeV, and a conservative estimate of the magnetic field of 0.6 mG (this is the largest proposed value which yields the weakest constraint) we then infer\(^4\) that \(b \lesssim 2 \times 10^{-22}\). This constraint is so strong that one has to conclude that dispersion relations like (19) for leptons are ruled out by current astrophysical observations within an EFT framework\(^5\).

---

\(^4\) Note that possible complications related to different MDRs between electrons and positrons in EFT with LIV are not present here as the breakdown of Lorentz invariance is a geometric effect in our framework and as such will not distinguish between leptons of same mass.

\(^5\) One might wonder, given the strength of the synchrotron constraint for \((\alpha, \beta) = (2, 0)\), if this might also help constraining cases which require higher, but not totally unreasonable, critical energies like the cases \((\alpha, \beta) = (1, 1)\) and \((\alpha, \beta) = (3, 0)\). Unfortunately it is easy to check that this is not the case. For example, for \((\alpha, \beta) = (1, 1)\), the synchrotron constraint is just \(b \lesssim 10^{17}\) and for \((\alpha, \beta) = (3, 0)\) it is \(b \lesssim 10^{29}\).
worse, in this case we cannot resort to anomalous (normally forbidden) threshold reactions as the latter are not allowed in DSR either. The reason for this is simply that a kinematically forbidden reaction in the “platonic” variables \( \pi_p \) cannot be made viable just via a nonlinear redefinition of momenta. There have been attempts to consider constraints provided by shifts of normally allowed threshold reactions [4]. We note here that for such reactions the possible constraints are strongly dependent not just on kinematical considerations, but also on reaction rates which require some working framework for their derivation. Hence, missing a field theory description of DSR we cannot safely pose such constraints. Similar considerations hold for constraints based on the synchrotron emission [36].

V. CONCLUSION

We have shown how the RG of gravity could lead to MDRs for massive minimally coupled particles due to the effects of quantum fluctuations. We have argued that for a free particle the most plausible identification of the cutoff is \( k = p^2 / M_P \), where \( p \) is the particle three momentum, leading to sizeable effects in the region of \( p \approx 10 \) TeV for QED processes. To do this, we had to make several assumptions: we assumed the validity of Einstein’s theory of gravity from cosmological to particle physics scales \( (i.e. \text{ still much below the Planck energy}) \), we assumed that the cutoff is a function of three momentum squared rather than four momentum squared, and we took for granted the value \( \Lambda_0 \approx 10^{-85} \text{ GeV}^2 \) for the cosmological constant at cosmological scales. At no point was it necessary to assume the existence of a gravitational fixed point, as this concerns physics at or beyond the Planck scale.

Another implicit assumption was the identification of the components of the four momentum \( p^\mu = (E, \vec{p}) \), rather than \( p^\mu = (E, \vec{p}) \). The two identifications are not compatible if the metric is scale dependent. Had we chosen \( p^\mu = g^\mu_\nu p_\nu = (E, \vec{p}) \), we would have obtained a MDR of the type

\[
E^2 = p^2 + m^2 \left( 1 + \frac{b Xk^4}{4 M_P^4} \right). \tag{28}
\]

The main difference with respect to Eq. (19) is the positive sign in front of the correction. From an LIV EFT perspective, the immediate consequence is that the previously discussed synchrotron bound does not apply, so this MDR is not as constrained as the previous one (Eq. (19)). There is no photon decay either, however a vacuum Čerenkov effect may occur [1]. It can be shown that the latter can cast constraints on \( b \) of the same order as the photon decay case (cf Eq. (23)). Thus this case would also be quite constrained in the LIV interpretation. Note that a similar change in the sign of the LIV term in the MDR could also be due to a change in the sign of the coefficient \( b \) which can be induced by considering in the RG analysis also minimally coupled fermion fields [13]. Different initial assumptions may lead to effects that are either too strong to be compatible with current data, or too small to be detectable in the foreseeable future, or, hopefully, they might produce some interesting phenomenology.

The RG has been applied in an LIV context in [40] whose authors proved that Lorentz invariance can arise as a low–energy symmetry in an otherwise non–Lorentz invariant theory. Since our MDRs reduce to the standard ones at sufficiently low energies, our results are in agreement with theirs on this point, even though the formalism is quite different.

From the theoretical point of view, we cannot say at this point if the RG approach to gravity prefers EFT with LIV or DSR. Let us stress that our model is not a priori equivalent to any of the above frameworks. In fact in the case of EFT with LIV one assumes the existence of some aether field which allows one to construct LIV operators in the matter Lagrangians. However in our case the departure from standard special relativistic dynamics of matter is induced uniquely via the \( k \) dependence of the background metric and the fact that \( k \) is chosen not to be a Lorentz invariant. This can be seen as a special case of EFT with an aether field but it is not equivalent to it. (Note that if the aether field is taken to be dynamical then it will affect the RG flow, but this will be equivalent to the presence of an extra matter field.) Similarly we cannot a priori completely identify our framework with a DSR one since we cannot say at this stage if our effective geometry will also be accompanied by some alternative rule for the addition of momenta.

Which of the above possibilities would be actually realized within our framework could be probably assessed only after gaining a better understanding of EFT on running geometries, something that is still lacking at this time. Following the intuition that comes from the analogue models (where the underlying microscopic physics indeed violates Lorentz invariance), one would probably need to have some deeper understanding of the physics above the Planck scale (quantum gravity regime) to be able to distinguish between the two. In this sense the phenomenological analysis we have performed has to be taken as a first try aimed at seeing what constraints could be cast once this discrimination is done.

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