Dirac’s Observables for the Higgs Model: II) the non-Abelian

SU(2) Case.

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Abstract

We search a canonical basis of Dirac’s observables for the classical non-Abelian Higgs model with fermions in the case of a trivial SU(2) principal bundle with a complex doublet of Higgs fields and with the fermions in a given representation of SU(2). Since each one of the three Gauss law first class constraints can be solved either in the corresponding longitudinal electric field or in the corresponding Higgs momentum, we get a priori eight disjoint phases of solutions of the model. The only two phases with SU(2) covariance are
the SU(2) phase with massless SU(2) fields and the Higgs phase with massive SU(2) fields. The Dirac observables and the reduced physical (local) Hamiltonian and (nonlocal) Lagrangian of the Higgs phase are evaluated: the main result is the nonanalyticity in the SU(2) coupling constant, or equivalently in the sum of the residual Higgs field and of the mass of the SU(2) fields. Some comments on the function spaces needed for the gauge fields are made.

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I. INTRODUCTION

In a previous paper [1] (quoted as I), we evaluated the Dirac observables for the Abelian Higgs model [2] with fermions, in the case of a trivial U(1) principal bundle over a fixed \( x^\sigma \) slice of Minkowski spacetime, for both its electromagnetic and Higgs phases. Here we will study the simplest non-Abelian Higgs model (see for instance Ref. [3]) with fermions, namely the one associated with a trivial SU(2) principal bundle over a fixed \( x^\sigma \) slice of Minkowski spacetime with a complex doublet of Higgs fields and a set of fermion fields belonging to a representation \( \rho \) of SU(2). However, the same construction applies to every trivial principal \( G \)-bundle with a compact, semisimple, connected, simply connected structure Lie group \( G \) with suitable Higgs fields. As shown in Ref. [4], in this case one can find the Dirac observables of classical Yang-Mills theory with fermions, if the Yang-Mills gauge potentials belong to a suitable weighted Sobolev space in which the Gribov ambiguity is absent. See Ref. [5] for a review of the general methodology for finding Dirac’s observables of physical gauge systems.

II. THE LAGRANGIAN FORMALISM

By using the notations of Ref. [4], the SU(2) Higgs model is described by the following Lagrangian density \([\lambda > 0, \phi_o > 0]\)

\[
\mathcal{L}(x) = -\frac{1}{4g^2} F_{a\mu\nu}(x) F_a^{\mu\nu}(x) + [D_{\mu}^{(A)} \phi(x)]^\dagger D^{(A)}_{\mu} \phi(x) - V(\phi) + \]

\[+ \frac{i}{2} \bar{\psi}(x) [\gamma^\mu D^{(A)}_{\mu} - D^{(A)(\mu)}_{\mu} \gamma^\mu] \psi(x) - m \bar{\psi}(x) \psi(x), \tag{1}\]

where the Higgs field \( \phi(x) \) is a doublet of complex scalar fields \([D^{(A)}_{\mu}]^\dagger\) means transpose conjugate, not adjoint

\[
\phi(x) = (\phi_i(x)) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad \phi^\dagger(x) = (\phi_i^\dagger(x) \phi_2^\dagger(x))
\]

\[
(D^{(A)}_{\mu}\phi(x))_i = (\partial_{\mu} + \bar{A}_\mu(x))_{ij} \phi_j(x)
\]

\[
[D^{(A)}_{\mu}\phi(x)]^\dagger_i = \phi_j^\dagger(x) (\partial_{\mu} - \bar{A}_\mu(x))_{ji}
\]
\[ \tilde{A}_\mu(x) = A_{a\mu}(x)\tilde{T}^a, \quad \tilde{T}^a = -i\tau^a, \quad a = 1, 2, 3, \]

\[ V(\phi) = \lambda[\phi^\dagger(x)\phi(x) - \phi_0^2]^2 = \mu^2\phi^\dagger(x)\phi(x) + \lambda[\phi^\dagger(x)\phi(x)]^2 + \lambda\phi_0^4 = \]

\[ = -\frac{1}{2}m_H^2\phi^\dagger(x)\phi(x) + \lambda[\phi^\dagger(x)\phi(x)]^2 + \lambda\phi_0^2, \]

\[ \mu^2 = -2\lambda\phi_0^2 < 0, \quad m_H^2 = -2\mu^2 = 4\lambda\phi_0^4, \quad \phi_0 = \frac{m_H}{2\sqrt{\lambda}} = \sqrt{-\frac{\mu^2}{2\lambda}} \] (2)

The SU(2) generators are \( \tilde{T}^a = -i\tau^a = -(\tilde{T}^a)^\dagger \) [\( \tau^a \) are the Pauli matrices]. We take \( \mu^2 < 0 \), so that the potential \( V(\phi) \) has a set of absolute minima for \( \phi^\dagger\phi = \phi_0^2 \), parametrized by three phases [see later on Eq.(16)], and \( \phi_0 > 0 \), an arbitrary real number [\( < \phi > = \phi_0 \neq 0 \) at the quantum level: this is the gauge not-invariant formulation of the statement of symmetry breaking].

The gauge potentials \( A_\mu(x) = A_{a\mu}(x)\tilde{T}^a = g\tilde{A}_\mu(x) \) [\( g \) is the SU(2) coupling constant] belong to the adjoint representation of SU(2), whose Lie algebra \( \text{su}(2) \) has the structure constants \( c_{abc} = \epsilon_{abc} \); we have \( (\tilde{T}^a)^{bc} = \epsilon_{abc} \) and \( \tilde{T}^a = -(\tilde{T}^a)^\dagger \). The associated field strengths and covariant derivatives are

\[ F_{a\mu\nu}(x) = \partial_\mu A_{a\nu}(x) - \partial_\nu A_{a\mu}(x) + c_{abc}A_{b\mu}(x)A_{c\nu}(x), \quad F_{\mu\nu}(x) = F_{a\mu\nu}(x)\tilde{T}^a \]

\[ \hat{D}^{(A)}_{\muab} = \delta_{ab}\partial_\mu + c_{abc}A_{b\mu}(x), \] (3)

while the Bianchi identities are \( \hat{D}^{(A)}_{\muab}F^{\mu\nu} \equiv 0 \) [\( *F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \)].

The fermion fields \( \psi(x) = (\psi_{A\alpha}(x)) \) [\( \alpha \) are spinor indices] are Grassmann-valued and belong to a representation \( \rho \) of SU(2) with generators \( T^a = -(T^a)^\dagger \); the associated covariant derivative is

\[ D^{(A_{\rho})}_\mu(x)\psi(x) = (\partial_\mu + A^{(\rho)}_\mu(x))\psi(x) \]

\[ \bar{\psi}(x)D^{(A^{(\rho)})\dagger}_\mu(x) = \bar{\psi}(x)(\partial_\mu - A^{(\rho)}_\mu(x)) \]

\[ A^{(\rho)}_\mu(x) = A_{a\mu}(x)T^a. \] (4)
The SU(2) gauge transformations, under which the Lagrangian density is invariant, are defined in the following way:

\[ A_\mu(x) \mapsto A_\mu^U(x) = U^{-1}(x)A_\mu(x)U(x) + U^{-1}(x)\partial_\mu U(x) \]

\[ F_{\mu\nu}(x) \mapsto F_{\mu\nu}^U(x) = U^{-1}(x)F_{\mu\nu}(x)U(x) \]

\[ \phi(x) \mapsto \phi^U(x) = \tilde{U}^{-1}(x)\phi(x) \]

\[ \psi(x) \mapsto \psi^U(x) = U^{(\rho)-1}(x)\psi(x). \] (5)

The Euler-Lagrange equations are ["\overset{\cdots}{=}"] means evaluated on the extremals of the action

\[ S = \int d^4x L(x) \]

\[ L^a_\mu = g^2 \left( \frac{\partial L}{\partial A_{a\mu}} \right) \overset{\cdots}{=}, \quad J^a_\mu = i\bar{\psi}\gamma^\mu T^a \psi - \overset{\cdots}{=} \bar{\phi}[\tilde{T}^a D^{(A)\mu} - D^{(A)\mu}\tilde{T}^a]\phi \]

\[ \overset{\cdots}{L}_\psi = \frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \psi} = -\bar{\psi}[i(\partial_\mu - \tilde{A}_{a\mu}T^a)\gamma^\mu + m] \overset{\cdots}{=} 0 \]

\[ \overset{\cdots}{L}_\psi = \frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \psi} = [\gamma^\mu i(\partial_\mu + A_{a\mu}T^a) - m] \overset{\cdots}{=} 0 \]

\[ \overset{\cdots}{L}_{\phi_i} = \frac{\partial L}{\partial \phi_i} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi_i} = -[D^{(A)\mu} D^{(A)\mu}_\phi]_i \overset{\cdots}{=}, \quad \overset{\cdots}{L}_{\phi_i} = -\frac{\partial V(\phi)}{\partial \phi^*_i} \overset{\cdots}{=}, \quad \overset{\cdots}{L}_{\phi_i} \overset{\cdots}{=}, \quad \overset{\cdots}{L}_{\phi_i} \overset{\cdots}{=}. \] (6)

In absence of fermions, the solutions for the gauge potentials and the Higgs fields, corresponding to the vanishing of the \( \Theta^{\alpha\alpha}(x) \) component of the energy-momentum tensor [see later on Eq. (13)] and therefore of the total energy, are

\[ F^{a\mu\nu}(x) \overset{\cdots}{=} 0 \]

\[ D^{(A)\mu}_\nu \phi_j(x) \overset{\cdots}{=} 0 \]

\[ V(\phi) \overset{\cdots}{=} 0 \Rightarrow \phi^\dagger(x)\phi(x) = \phi^2_0. \] (7)

One such configuration is: \( A_{a\mu}(x) = 0 \) and \( \phi(x) = \tilde{\phi}_o \) [with \( \tilde{\phi}_o \) a given doublet, for instance \( \tilde{\phi}_o = \begin{pmatrix} 0 \\ \phi_o \end{pmatrix} \)]. If in a certain region of spacetime one has \( F^{a\mu\nu}(x) \neq (or =) 0 \) but the other
two equations (7) satisfied, one says that the fields are in the “Higgs vacuum”; see Ref. [6] for the use of this concept in the theory of non-Abelian monopoles.

Always in absence of fermions, the solution for the Higgs fields which is an absolute minimum of the potential \( V(\phi) \) is (see the next Section for the geometrical interpretation)

\[
\begin{align*}
D_{\mu ij} \phi_j(x) & = 0, \\
\frac{\partial V(\phi)}{\partial \phi} & = 0.
\end{align*}
\] (8)

While the second equation has the two solutions \( \phi = 0 \) and \( \phi = \phi_o \), from the first equation we get \( 0 = [D^{(A)}_\mu, D^{(A)}_\nu] \phi(x) = F_{\mu\nu}(x) \overline{T}^a \phi(x) \). Therefore, we get, besides the solutions either \( \phi = 0 \) and \( F \neq 0 \) or \( \phi = \phi_o \) and \( F = 0 \) of the Abelian case, the solution in which the components of \( F_{\mu\nu} \) corresponding to the generators of the unbroken subgroup \( H \subset G \) are non-zero; but in our case \( H = 0 \). Conditions like Eqs.(8) are imposed on the background fields [7], when the system is studied in an external field \( (A_\mu \mapsto A^{(ext)}_\mu) \), together with the requirement \( \tilde{D}^{(A^{(ext)})}_\mu F^{(ext)}_{\nu\rho}(x) = 0 \) (this implies that the external gauge field is Abelian up to a gauge transformation).

### III. DIFFERENTIAL GEOMETRIC SETTING

From a geometric point of view, see Refs. [8,9], in spontaneously broken gauge theories with symmetry-breaking Higgs fields one has:

i) A principal G-bundle \( P(M,G) \) over Minkowski spacetime \( M \) [or over one its fixed \( x^o \) slice \( R^3 \)], whose standard fiber is the structure group \( G \) [a compact, semisimple, connected Lie group]. We shall consider only trivial principal bundles \( P = M \times G \) and simply connected groups \( G \). A connection on \( P \) is described by a connection one-form \( \omega \) over \( P \) and with each global cross section \( \sigma : M \rightarrow P = M \times G \) is associated a global gauge potential \( A_\mu(x)dx^\mu = A^{(A)}_{\mu\nu}(x)\overline{T}^a dx^\mu = \sigma^* \omega \), which is a one-form over \( M \). The group \( G \) of gauge transformations connects all the gauge potentials on the same gauge orbit, associated with the given connection one-form \( \omega \) on \( P \), by considering all the possible global cross sections \( \sigma : M \rightarrow P \) (therefore \( P = M \times G \) can also be considered as the group manifold of \( G \).
ii) A bundle $E_\psi$, associated to $P = M \times G$, over $M$ with the same structure group $G$ and whose standard fiber is the vector space (with Grassmann-valued vectors as elements) of a representation $\rho$ of $G$ with generators $T^a$; its global cross sections describe the fermion fields $\psi(x)$ of the model.

iii) A bundle $E$, associated to $P = M \times G$, over $M$ with the same structure group $G$, whose standard fiber $F$ is the vector space of a representation $\rho'$ of $G$ with generators $\tilde{T}^a$; its global cross sections are the complex scalar (generalized) Higgs fields on $P$ of the model [10], which can also be described by special maps $\tilde{\phi} : P \rightarrow F$ [called tensorial 0-forms in Ref. [11]]; the usual Higgs fields on $M$ are $\phi : M \rightarrow F$, $\phi = \sigma^* \tilde{\phi}$, $\sigma : M \rightarrow P$. When $F$ is only a manifold (usually it is a vector space) with a $G$-action, the (generalized) Higgs field $\tilde{\phi}$ is called a “symmetry-breaking Higgs field” if $\tilde{\phi} : P \rightarrow F$ maps all of $P$ onto a single $G$-orbit $\xi$ with $\dim \xi \geq 1$ (more in general onto a union of $G$-orbits) of the $G$-action of $G$ on $F$ and each point of the $G$-orbit $\xi$ has an isotropy (or little or stability) group $H \subset G$ under the $G$-action; the spontaneously broken symmetries are the elements $a \in G$, $a \notin H$, because $a : \phi_o \in \xi \mapsto \phi \neq \phi_o$, where $\phi_o$ is a reference point in $\xi$ (called a “vacuum state” at the quantum level). This means that $\tilde{\phi} : P \rightarrow F$ identifies in $P = M \times G$ a residual symmetry group $H \subset G$ [$b \in H \Rightarrow b : \phi_o \in \xi \mapsto \phi_o$] and a corresponding subbundle $Q = M \times H$ [$Q$ must be such to satisfy the fundamental hypothesis that $g = h \oplus \mathcal{M}$ with $g$ the Lie algebra of $G$, $h$ the Lie algebra of $H$ and $\mathcal{M}$ a complementary space (with respect to an adjoint-invariant scalar product on $g$) invariant under the adjoint action of $H$).

On $M$ one adds to the Yang-Mills plus fermion Lagrangian density a term $[D_{\mu}^{(A)} \phi]^\dagger D^{(A)\mu} \phi + V(\phi)$ with $V(\phi)$ a suitable potential and looks for the subset of solutions of the Higgs Euler-Lagrange equations satisfying $D_{\mu}^{(A)} \phi = 0$ and $\frac{\partial V(\phi)}{\partial \phi} = 0$; the potential $V(\phi)$ must be such that it admits an absolute minimum (assumed equal to zero) for all those values of $\phi$ corresponding to symmetry-breaking Higgs fields: namely $V^{-1}(0)$ is spanned by $\phi = \sigma^* \tilde{\phi}$ with $\tilde{\phi} : P \rightarrow \xi \subset F$. The condition $D_{\mu}^{(A)} \phi = 0$ [11] of being covariantly constant with respect to the connection $\omega$ associated with the gauge potential $A_{\mu}$ is the necessary and sufficient condition [11] so that the connection $\omega$ on $P = M \times G$ be reducible to a connection.
on the subbundle $Q = M \times H$. If $i : Q \to P$ is the inclusion map, one has $i^* \omega = \omega_o + \gamma$ with $\omega_o$ a connection one-form on $Q$ and $\gamma$ being called a “tensorial one-form of type $\mu$”, where $\mu$ is the representation of $H$ on $\mathcal{M}$ induced by the representation $\rho'$ of $G$. See Ref. [12], for the description of spontaneous symmetry breaking with the Higgs mechanism in the context of general relativity.

In physical terms [3,13] $\gamma$ describes the massive gauge fields (whose mass depends on the representation $\rho'$ of $G$), while $\omega_o$ is the connection associated with those gauge potentials which remain massless; if $A_{a\mu}(x)$ are the original gauge potentials, with $a = 1, \ldots, \text{dim} \ g$, and if $b = 1, \ldots, \text{dim} \ h$ labels the generators of $h$, $\{A_{1\mu}, \ldots, A_{\text{dim} \ h \mu}\}$ are the massless gauge potentials and $\{A_{\text{dim} \ h + 1 \mu}, \ldots, A_{\text{dim} \ g \mu}\}$ are the massive ones.

Note that in this geometrical construction one never speaks of massless Goldstone bosons (with the quantum numbers of the broken generators at the quantum level) associated with the spontaneous symmetry breaking from $G$ to $H$ (here $G$ is regarded as the global rigid symmetry group contained in the group of gauge transformations $G[4]$); as shown in Ref. [14] the Goldstone bosons (and the associated infrared singularities) are hidden in the unphysical gauge degrees of freedom of the model (they are a subset of the Higgs fields) present in gauge theories due to the Gauss’ laws. As we shall see in the last Section, the discussion of the Gauss law first class constraints is not trivial as in absence of spontaneous symmetry breaking with the Higgs mechanism: the reduction to Dirac’s observables (equivalent to the unitary gauge but without the introduction of any gauge-fixing) elucidates the real meaning of the statement “the would-be Goldstone bosons are eaten by those gauge bosons which become massive”. After the reduction one is left with only a subset of physical Higgs fields which depend on the original representation $\rho'$ of $G$.

In this paper, we shall consider the simplest case of $G=\text{SU}(2)$ and $H=0$, so that $P = M \times G$ is reduced to $Q = M \times \{0\}$ and no massless gauge fields are left.
IV. THE HAMILTONIAN FORMALISM

The canonical momenta implied by the Lagrangian density (1) are

\[
\pi^a_0(x) = \frac{\partial L(x)}{\partial \partial_0 A_{ao}(x)} = 0
\]

\[
\pi^k_a(x) = \frac{\partial L(x)}{\partial \partial_0 A_{ak}(x)} = -g^{-2} F^o_{ak}(x) = g^{-2} E^k_a(x)
\]

\[
\pi_{A\alpha}(x) = \frac{\partial L(x)}{\partial \partial_0 \psi_{A\alpha}(x)} = -\frac{i}{2} (\bar{\psi}(x)\gamma_o)_{A\alpha}
\]

\[
\bar{\pi}_{A\alpha}(x) = \frac{\partial L(x)}{\partial \partial_0 \bar{\psi}_{A\alpha}(x)} = -\frac{i}{2} (\gamma_o \psi(x))_{A\alpha}
\]

\[
\pi_{\phi_i}(x) = \frac{\partial L(x)}{\partial \partial_0 \phi_i(x)} = \left[D^{(A)}_\alpha \phi(x)\right]_i
\]

\[
\bar{\pi}_{\phi^*_i}(x) = \frac{\partial L(x)}{\partial \partial_0 \phi^*_i(x)} = \left[D^{(A)}_\alpha \phi(x)\right]_i.
\]

They satisfy the standard Poisson brackets

\[
\{ A_{\alpha\mu}(\vec{x}, x^\alpha), \pi^\nu_p(\vec{y}, x^\alpha) \} = \delta_{\alpha\nu} \delta^3(\vec{x} - \vec{y})
\]

\[
\{ \psi_{A\alpha}(\vec{x}, x^\alpha), \pi_{B\beta}(\vec{y}, x^\alpha) \} = \{ \bar{\psi}_{A\alpha}(\vec{x}, x^\alpha), \bar{\pi}_{B\beta}(\vec{y}, x^\alpha) \} = -\delta_{AB} \delta_{\alpha\beta} \delta^3(\vec{x} - \vec{y})
\]

\[
\{ \phi_i(\vec{x}, x^\alpha), \pi_{\phi^*_j}(\vec{y}, x^\alpha) \} = \{ \phi^*_i(\vec{x}, x^\alpha), \pi_{\phi_j}(\vec{y}, x^\alpha) \} = \delta_{ij} \delta^3(\vec{x} - \vec{y}).
\]

Following Ref. [4], the second-class constraints

\[
\pi_{A\alpha}(x) + \frac{i}{2} (\bar{\psi}(x)\gamma_o)_{A\alpha} \approx 0, \quad \bar{\pi}_{A\alpha}(x) + \frac{i}{2} (\gamma_o \psi(x))_{A\alpha} \approx 0,
\]

are eliminated by going to Dirac brackets; then the surviving variables \(\psi_{A\alpha}(x), \bar{\psi}_{A\alpha}(x)\) satisfy (for the sake of simplicity we still use the notation \(\{,\}\) for the Dirac brackets)

\[
\{ \psi_{A\alpha}(\vec{x}, x^\alpha), \bar{\psi}_{B\beta}(\vec{y}, x^\alpha) \} = -i (\gamma^\alpha)_{\alpha\beta} \delta_{AB} \delta^3(\vec{x} - \vec{y})
\]

\[
\{ \psi_{A\alpha}(\vec{x}, x^\alpha), \psi_{B\beta}(\vec{y}, x^\alpha) \} = \{ \bar{\psi}_{A\alpha}(\vec{x}, x^\alpha), \bar{\psi}_{B\beta}(\vec{y}, x^\alpha) \} = 0.
\]

The resulting Dirac Hamiltonian density is (after an allowed integration by parts; \(\lambda_{ao}(x)\) is a Dirac multiplier)

\[
\mathcal{H}_D(x) = \frac{1}{2} \sum_a \left[ g^2 \pi^2_a(x) + g^{-2} \bar{B}^2_a(x) \right] +
\]
first class with the only nonvanishing Poisson brackets of the model, are ambiguous. They can be thought either as elliptic equations in the non-pi momenta, all the three equations are solved in phases. Only two of these phases preserve SU(2) covariance: i) the SU(2) phase, in which a nontrivial zeroth homotopy group) of the Gauss laws and, therefore, eight inequivalent solutions (the space of solutions has generators are \[ \sigma \]. Since the conserved energy-momentum and angular momentum tensor densities and Poincaré fields become massive (spontaneous symmetry breaking through the Higgs mechanism). See after Eq.(24) for more details.

The time constancy of the primary constraints \( \pi_a^o(x) \approx 0 \) yields the Gauss law secondary constraints

\[
\Gamma_a(x) = - \partial \cdot \pi_a(x) - c_{abc} \bar{A}_b(x) \cdot \pi_c(x) + i \psi^\dagger(x) T^a \psi(x) +
\]

\[
\quad + \frac{i}{2} [\pi_{ai}(x) (\tau^a)_{ij} \phi_j(x) - \pi_{a*} j (\tau^a)_{ij} \phi_j^*(x)] \approx 0. \tag{13}
\]

The \( \Gamma_a(x) \)'s are constants of the motion and the six constraints \( \pi_a^o(x) \approx 0, \Gamma_a(x) \approx 0 \) are first class with the only nonvanishing Poisson brackets

\[
\{ \Gamma_a(\vec{x},x^0), \Gamma_b(\vec{y},x^0) \} = c_{abc} \Gamma_c(\vec{x},x^0) \delta^3(\vec{x} - \vec{y}) \tag{14}
\]

The equations \( \Gamma_a(x) = 0 \), namely the acceleration independent Euler-Lagrange equations of the model, are ambiguous. They can be thought either as elliptic equations in the non-Abelian electric fields \( \pi_a(\vec{x}) = g^{-2} \bar{E}_a(x) \) or as algebraic equations for three of the Higgs momenta \( \pi_{a*}^i(x), \pi_{a*}^i(x) \). Since each equation has this ambiguity, we find that the model admits eight (modulo identifications) disjoint sets of solutions (the space of solutions has a nontrivial zeroth homotopy group) of the Gauss laws and, therefore, eight inequivalent phases. Only two of these phases preserve SU(2) covariance: i) the SU(2) phase, in which all the three equations are solved in \( \pi_a(x) \) and in which the SU(2) fields remain massless; ii) The Higgs phase, in which all the equations are solved in the Higgs momenta and the SU(2) fields become massive (spontaneous symmetry breaking through the Higgs mechanism). See after Eq.(24) for more details.

Let us make a digression on the choice of the boundary conditions on the various fields. Since the conserved energy-momentum and angular momentum tensor densities and Poincaré generators are \( [\sigma^{\mu \nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu], \sigma^i = \frac{1}{2} \epsilon^{ijk} \sigma^{jk}, \tilde{\alpha} = \gamma^0 \tilde{\gamma}, \beta = \gamma^0 \]
\[
\Theta^{\mu\nu}(x) = g^{-2}(F_{\alpha}^{\mu}(x)F_{\alpha}^{\nu}(x) + \frac{1}{4}\eta^{\mu\nu}F_{\alpha}^{\alpha\beta}(x)F_{\alpha\beta}(x)) + \frac{i}{2}\bar{\psi}(x)[\gamma^\mu D(\alpha)_{\nu} - D(\alpha)_{\nu}[\gamma^\alpha, \gamma^\beta]]\psi(x) + (D^{(A)\mu}\phi(x)) \dagger D^{(A)\nu}\phi(x) + (D^{(A)\nu}\phi(x)) \dagger D^{(A)\mu}\phi(x) - \eta^{\mu\nu}[(D^{(A)\alpha}\phi(x)) \dagger D^{(A)\alpha}\phi(x) - V(\phi)],
\]

\[
\mathcal{M}^{\mu\alpha\beta}(x) = x^\alpha T^{\mu\beta}(x) - x^\beta T^{\mu\alpha}(x) + \frac{1}{4}\bar{\psi}(x)(\gamma^\mu \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \gamma^\mu)\psi(x),
\]

\[
\partial_\nu T^{\mu\nu}(x) = 0, \quad \partial_\mu \mathcal{M}^{\mu\alpha\beta}(x) = 0,
\]

\[
P^\mu = \int d^3x \Theta^{\mu0}(\vec{x}, x^0),
\]

\[
J^{\mu0} = \int d^3x \mathcal{M}^{0\mu}(\vec{x}, x^0),
\]

\[
P^\alpha = \int d^3x \left\{ \frac{1}{2} \sum_a [g^2 \pi^2_a(\vec{x}, x^0) + g^2 \vec{B}_a^2(\vec{x}, x^0)] + \pi_\phi(\vec{x}, x^0)\pi_\phi(\vec{x}, x^0) + (\vec{D}^{(A)}\phi(\vec{x}, x^0)) \dagger \vec{D}^{(A)}\phi(\vec{x}, x^0) + V(\phi) + \frac{i}{2}\bar{\psi}(\vec{x}, x^0)[D^{(A)\alpha}/D^{(A)\alpha}]\psi(\vec{x}, x^0) \right\}
\]

\[
P^i = \int d^3x \left\{ (\vec{\pi}_a(\vec{x}, x^0) \times \vec{B}_a(\vec{x}, x^0)) + \pi_\phi(\vec{x}, x^0)\pi_\phi(\vec{x}, x^0) + (\vec{D}^{(A)\alpha}/\vec{D}^{(A)\alpha})\pi_\phi(\vec{x}, x^0) + \frac{i}{2}\bar{\psi}(\vec{x}, x^0)[D^{(A)\alpha}/D^{(A)\alpha}]\psi(\vec{x}, x^0) \right\}
\]

\[
J^i = \frac{1}{2}e^{ijk}J^{jk} = \int d^3x \left\{ [\vec{x} \times (\vec{\pi}_a(\vec{x}, x^0) \times \vec{B}_a(\vec{x}, x^0))] + [\vec{x} \times (\vec{\pi}_\phi(\vec{x}, x^0)\vec{D}^{(A)}\phi(\vec{x}, x^0) + (\vec{D}^{(A)}\phi(\vec{x}, x^0)) \dagger \vec{\pi}_\phi(\vec{x}, x^0)] + \frac{i}{2}\bar{\psi}(\vec{x}, x^0)[\vec{x} \times (\vec{D}^{(A)\alpha} + \vec{D}^{(A)\alpha})] \dagger \psi(\vec{x}, x^0) + \frac{1}{2}\bar{\psi}(\vec{x}, x^0)\sigma^i \psi(\vec{x}, x^0) \right\}
\]

\[
K^i = J^{0i} + x^0 P^i = \int d^3x x^i \Theta^{i0}(\vec{x}, x^0),
\]

following Ref. [3], we will assume boundary conditions [r = |\vec{x}|] A_{a0}(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} a_0/r^{1+\epsilon}, \bar{A}_a(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} a_0/r^{2+\epsilon}, \pi_a(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} p_a^0/r^{1+\epsilon} + O(r^{-2}), \bar{\pi}_a(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} e_a^0/r^2 + O(r^{-3}), \lambda_{a0}(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} e_{a0}/r^{1+\epsilon} + O(r^{-2}), \psi(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} \chi/r^{3/2+\epsilon} + O(r^{-2}), \phi(\vec{x}, x^0) \rightarrow_{r \rightarrow \infty} \text{const.} +
\[ \varphi/r^{2+\epsilon} + O(r^{-3}) \] [the constant allows the existence of minima \( \phi_o \) of the potential \( V(\phi) \)], \( \pi_\phi(\vec{x}, x^o) \to r \to \infty \zeta/r^{2+\epsilon} + O(r^{-3}) \), so that \( \Gamma_a(\vec{x}, x^o) \to r \to \infty \gamma_a/r^{3+\epsilon} + O(r^{-4}) \) and the Poincaré generators are finite. If we assume that the gauge transformations behave as \( U(\vec{x}, x^o) \to r \to \infty \text{const.} + O(r^{-1}), U^{(\rho)}(\vec{x}, x^o) \to r \to \infty \text{const.} + O(r^{-1}), \bar{U}(\vec{x}, x^o) \to r \to \infty \text{const.} + O(r^{-1}) \), the previous boundary conditions on the fields are preserved by the gauge transformations and the non-Abelian SU(2) charges (see the last Section) transform covariantly under them. Let us remark that the previous boundary conditions are adapted to the fixed \( x^o \), not Lorentz covariant, Hamiltonian formalism; however, they become natural in its covariantization based on the reformulation of the theory on spacelike hypersurfaces.

Let us make a technical remark about the choice of the function space the Yang-Mills gauge potentials belong to in spontaneously broken gauge theories with the Higgs mechanism. As in Ref. [4], we consider only trivial principal bundles and we exclude monopole solutions, but now it is not clear what to do with the Gribov ambiguity, because the condition \( D_\mu^{(A)} \phi(x) = 0 \) of Eq. (8) for the reducibility of the connection \( \omega \) on \( P = M \times G \) to a connection \( \omega_o \) on the subbundle \( Q = M \times H \) (here \( Q = M \times \{0\} \)) implies the existence of gauge symmetries (nontrivial stability group of a gauge potential) so that the needed space of connections on \( P = M \times G \) cannot contain only fully irreducible connections [as shown in Ref. [4], only in this case the Gribov ambiguity is absent (the stability groups of the gauge potentials (gauge symmetries) and of the field strengths (gauge copies) are trivial) and this is obtainable by the choice of special weighted Sobolev spaces]. If we formulate the theory in ordinary Sobolev spaces, as it is usually done, we do not have problems in the Higgs phase, because there the reduction associated with the Gauss’ laws first class constraints is purely algebraic. The problem with the Gribov ambiguity arises in the reduction of the SU(2) phase and of the mixed non-SU(2)-covariant ones. However these phases are not physical, so that we do not worry about them [however there can be a cosmological use [17], for explaining the observed cosmological baryon density, of the phase transition restoring the ordered SU(2) phase from the disordered Higgs one (what about the mixed phases?)]; see
also Ref. [7] for the phase transition restoring the SU(2) symmetry in presence of a constant external electromagnetic field. The only problem is that the formal proofs of renormalizability need all the phases and, therefore, they have to face the problems connected with the Gribov ambiguity.

V. THE HIGGS PHASE

In this paper we shall study only the Higgs phase, because the SU(2) phase can be reduced by combining the methods of Ref. [1,4].

The parametrization of the Higgs fields suitable to the Higgs phase and realizing the spontaneous symmetry breaking by a choice of a reference point in the degenerate set of minima of the classical potential $V(\phi) [\phi^i\phi = \phi_o^2]$ is

$$\phi(x) = e^{i\tilde{T}a\theta_a(x)} \begin{pmatrix} 0 \\ \phi_o + \frac{1}{\sqrt{2}} H(x) \end{pmatrix} = e^{i\tilde{T}a\theta_a(x)} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} = \tilde{U}_\theta(x) \tilde{\phi}(x)$$

$$\tilde{T}a\theta_a(x) = -i\tilde{\tau}^a \cdot \tilde{\theta}(x), \quad v = \sqrt{2}\phi_o, \quad \tilde{\phi}(x) = [\phi_o + \frac{1}{\sqrt{2}} H(x)] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (16)$$

with $\theta_a(x), H(x)$ real fields $[H(\vec{x}, x^0) \to_{r \to \infty} h/r^{2+\epsilon} + O(r^{-3})], \theta_a(\vec{x}, x^0) \to_{r \to \infty} \zeta_a/r^{2+\epsilon} + O(r^{-3})$.

The value $\phi = 0$ is not covered by these radial coordinates; for the sake of simplicity we take a positive value $\phi_o > 0$ for the arbitrary symmetry breaking reference point in the set of minima of the potential: this set is spanned by varying the angular variables $\theta_a$, so that the $\theta_a$’s are the would-be Goldstone bosons; the symmetry group SU(2) is completely broken and there is no residual stability group of the points of minimum.

The parametrization of Eq. (16) requires a restriction to Higgs fields which have no zeroes, namely $\phi^i(x)\phi(x) \neq 0 [H(x) \neq -v = -\sqrt{2}\phi_o]$, and with nonsingular phases $\theta_a(x)$’s because we assumed a trivial SU(2) principal bundle. The analogue of the quantum statement of symmetry breaking, i.e. that the theory is invariant under a group G but not the ground state, is replaced by the choice of the parametrization (16) with a given $\phi_o$, i.e. by the choice
of a family of solutions of the Euler-Lagrange equations associated with Eq. (1) not invariant under SU(2).

Since the parametrization has the form of a gauge transformation, we have

\[
D^{(A)}_{\mu}(x) = U_\theta(x)D^{(\hat{A}^\theta)}_{\mu}(x) = U_\theta(x)\left[ \frac{1}{\sqrt{2}} \partial_{\mu}H(x) + (\phi_o + \frac{1}{\sqrt{2}} H(x)) \hat{A}^\theta_{\mu}(x) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

\[
[D^{(A)}_{\mu}(x)]^{\dagger} = \begin{pmatrix} 0 & 1 \end{pmatrix} \left[ \frac{1}{\sqrt{2}} \partial_{\mu}H(x) - (\phi_o + \frac{1}{\sqrt{2}} H(x)) \hat{A}^\theta_{\mu}(x) \right] U_\theta^{-1}(x),
\]

\[
\phi^{\dagger} \phi = \bar{\phi}^{\dagger} \bar{\phi} = (\phi_o + \frac{1}{\sqrt{2}} H)^2.
\] (17)

By using \( \tau_a \tau^b = \delta_{ab} + i \epsilon_{abc} \tau^c \), \( \begin{pmatrix} 0 & 1 \end{pmatrix} \hat{T}^{a} \hat{T}^{d} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{4}(1 + \tau^2)_{cd} \), \( e^{\theta_b \hat{T}^b} = e^{-\frac{1}{2}\theta_{bh} \tau^b} = \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \hat{n}_b \tau^b \) [with \( \sum_b \hat{n}_b^2 = 1 \), \( \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2} \), \( \hat{n}_a = \theta_a/\theta \)], we get

\[
\hat{A}^\theta_{\mu}(x) = \hat{U}_\theta^{-1}(x)(\hat{A}_{a\mu}(x) \hat{T}^a + \partial_\mu \hat{U}_\theta(x)) =
\]

\[
= \{[\cos \theta(x) \delta_{ab} + 2 \sin^2 \frac{\theta(x)}{2} \hat{n}_a(x) \hat{n}_b(x) + \sin \theta(x) \epsilon_{abc} \hat{n}_c(x)] A_{b\mu}(x) +
\]

\[
+ [\hat{n}_a(x) \partial \theta(x) \frac{\partial \theta(x)}{\partial \theta_b} + \sin \theta(x) \frac{\partial \hat{n}_a(x)}{\partial \theta_b} - 2 \sin^2 \frac{\theta(x)}{2} \epsilon_{acd} \hat{n}_c(x) \frac{\partial \hat{n}_d(x)}{\partial \theta_b}] \partial_\mu \theta_b(x) \}\hat{T}^a,
\] (18)

\[
\begin{pmatrix} 0 & 1 \end{pmatrix} \left[ \hat{A}^\theta_{\mu}(x) \hat{A}^{\theta \mu}(x) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} =
\]

\[
= -\frac{1}{4} [A_{a\mu}(x) +
\]

\[
+ (\hat{n}_a(x) \frac{\partial \theta(x)}{\partial \theta_b} + \sin \theta(x) \frac{\partial \hat{n}_a(x)}{\partial \theta_b} + 2 \sin^2 \frac{\theta(x)}{2} \epsilon_{acd} \hat{n}_c(x) \frac{\partial \hat{n}_d(x)}{\partial \theta_b}] \partial_\mu \theta_b(x)]^2,
\] (19)

so that the Lagrangian density (I) becomes

\[
\mathcal{L}(x) = -\frac{1}{4g^2} F_{a\mu}(x) F^{a\mu}(x) +
\]

\[
+ \frac{1}{4} (\partial_\mu + \frac{1}{\sqrt{2}} H(x))^2 \times
\]

\[
[A_{a\mu}(x) + (\hat{n}_a(x) \frac{\partial \theta(x)}{\partial \theta_b} + \sin \theta(x) \frac{\partial \hat{n}_a(x)}{\partial \theta_b} + 2 \sin^2 \frac{\theta(x)}{2} \epsilon_{acd} \hat{n}_c(x) \frac{\partial \hat{n}_d(x)}{\partial \theta_b}) \partial_\mu \theta_b(x)]^2 +
\]

\[
+ \frac{1}{2} \partial_\mu H(x) \partial^\mu H(x) - \frac{\lambda}{2} H^2(x)(\frac{1}{\sqrt{2}} H(x) + 2 \phi_o)^2 +
\]

\[
+ \frac{i}{2} \bar{\psi}(x) \gamma^\mu (\partial_\mu + A_{a\mu}(x) T^a) - (\partial_\mu - A_{a\mu}(x) T^a \gamma^\mu) \psi(x) - m \bar{\psi}(x) \psi(x),
\] (20)
The new Higgs canonical momenta are

\[
\pi_H(x) = \frac{\partial L(x)}{\partial \partial_0 H(x)} = \partial^0 H(x)
\]

\[
\pi_{\theta_a}(x) = \frac{\partial L(x)}{\partial \partial_0 \theta_a(x)} = \frac{1}{2} [\phi_a + \frac{1}{\sqrt{2}} H(x)]^2 \times V_{ab}(\theta(x))[\tilde{A}_{ab}(x) + V_{bc}(\theta(x))\partial_c \theta(x)],
\]

\[
\pi_{\phi_a}(x) - \frac{1}{\sqrt{2}} \pi_H(x) - (\phi_o + \frac{1}{\sqrt{2}} H(x)) \times \left[ \cos \theta(x) \delta_{ab} + 2 \sin^2 \theta(x) \frac{\partial \tilde{n}_a(x)}{\partial \theta_b} + \sin \theta(x) \epsilon_{abc} \tilde{n}_c(x) \right] \tilde{A}_{ab}(x) + \
\left[ \tilde{n}_a(x) \frac{\partial \theta}{\partial \theta_b} + \sin \theta(x) \frac{\partial \tilde{n}_a(x)}{\partial \theta_b} - 2 \sin^2 \theta(x) \frac{\partial \epsilon_{abc} \tilde{n}_c(x)}{\partial \theta_b} \right] V_{br}^{-1}(\theta(x)) \times \
\left[ V_{rs}^{-1}(\theta(x)) \frac{2 \pi_{\theta}(x)}{(\phi_o + \frac{1}{\sqrt{2}} H(x))^2} - \tilde{A}_{ro}(x) \right] e^{-\theta_d(x)\tilde{T}^d}
\]

\[
= e^{\theta_d(x)\tilde{T}^d} \left[ \frac{1}{\sqrt{2}} \pi_H(x) + (\phi_o + \frac{1}{\sqrt{2}} H(x)) \times \left[ \cos \theta(x) \delta_{ab} + 2 \sin^2 \theta(x) \frac{\partial \tilde{n}_a(x)}{\partial \theta_b} + \sin \theta(x) \epsilon_{abc} \tilde{n}_c(x) \right] \tilde{A}_{ab}(x) + \
\left[ \tilde{n}_a(x) \frac{\partial \theta}{\partial \theta_b} + \sin \theta(x) \frac{\partial \tilde{n}_a(x)}{\partial \theta_b} - 2 \sin^2 \theta(x) \frac{\partial \epsilon_{abc} \tilde{n}_c(x)}{\partial \theta_b} \right] V_{br}^{-1}(\theta(x)) \times \
\left[ V_{rs}^{-1}(\theta(x)) \frac{2 \pi_{\theta}(x)}{(\phi_o + \frac{1}{\sqrt{2}} H(x))^2} - \tilde{A}_{ro}(x) \right] \tilde{T}^d \right] \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \tag{22}
\]

The relation between the old and the new Higgs momenta is

\[
\Rightarrow \partial_\theta \theta_a = V_{ab}^{-1}(\theta(x))[V_{bc}^{-1}(\theta(x))\frac{2 \pi_{\theta}(x)}{(\phi_o + \frac{1}{\sqrt{2}} H(x))^2} - \tilde{A}_{ab}(x)]. \tag{21}
\]

The new Dirac Hamiltonian density is

\[
H_D(x) = \frac{1}{2} \sum_a [g^2 \tilde{\pi}_a^2(x) + g^{-2} \tilde{B}_a^2(x)] + 
\]
\[\begin{align*}
+ [2V_{ac}^{-1}(\theta(x)) - V_{ca}^{-1}(\theta(x))]V_{cb}^{-1}(\theta(x)) \frac{\pi_{\theta_a}(x)\pi_{\theta_b}(x)}{(\phi_o + \frac{1}{\sqrt{2}}H(x))^2} + \\
+ \frac{1}{4} [\phi_o + \frac{1}{\sqrt{2}}H(x)]^2 \sum_a [\tilde{A}_a(x) + V_{ab}(\theta(x))\tilde{\theta}_b(x)]^2 + \\
+ \frac{i}{2} \tilde{\psi}(x)[\gamma \cdot (\tilde{\partial} + \tilde{A}_a(x)T^a) - (\tilde{\partial} - \tilde{A}_a(x)T^a) \cdot \tilde{\gamma}] \psi(x) + m\tilde{\psi}(x)\psi(x) - \\
- A_{ao}(x) [-\tilde{D}_{ab} \cdot \pi_b(x) + i\tilde{\psi}(x)\gamma^aT^a\psi(x) + \pi_{\theta_b}(x)V_{ba}^{-1}(\theta(x))] + \lambda_{ao}(x)\pi_o^a(x). \tag{23}
\end{align*}\]

The Gauss law secondary constraints take the form

\[\Gamma_a(x) = -\tilde{\partial} \cdot \pi_a(x) - c_{abc}\tilde{A}_b(x) \cdot \pi_c(x) + i\psi\gamma^aT^a\psi(x) + \pi_{\theta_b}(x)V_{ba}^{-1}(\theta(x)) \approx 0 \tag{24}\]

which can be trivially solved in the Higgs momenta \(\pi_{\theta_a}(x)\).

We can now see explicitly the existing phases:

i) the SU(2) symmetric phase, with no broken generator and all the fields \(A_{\mu a}\) massless, in which Eqs.(24) is solved for \(\pi_a\).

ii) 3 phases with SU(2) broken to two not-commuting U(1)’s [one broken and two unbroken generators], in which Eqs.(24) are solved in two of the \(\pi_a\)’s and one of the \(\pi_{\theta_a}\)’s. The three phases are: a) \(A_{1\mu}, A_{2\mu}\) massless and \(A_{3\mu}\) massive; b) \(A_{3\mu}, A_{1\mu}\) massless and \(A_{2\mu}\) massive; c) \(A_{2\mu}, A_{3\mu}\) massless and \(A_{1\mu}\) massive. Naturally there are many more possibilities, because one could choose any combination of the \(A_{\mu a}\)’s as the massive field.

iii) 3 phases with SU(2) broken to U(1) [two broken and one unbroken generator], in which Eqs.(24) are solved in one of the \(\pi_a\)’s and two of the \(\pi_{\theta_a}\)’s. The three phases are: a) \(A_{1\mu}\) massless and \(A_{2\mu}, A_{3\mu}\) massive; b) \(A_{3\mu}, A_{1\mu}\) massless and \(A_{2\mu}\) massive; c) \(A_{2\mu}\) massless and \(A_{3\mu}, A_{1\mu}\) massive. Again it is arbitrary which combination of the \(A_{\mu a}\)’s is chosen to remain massless.

iv) the Higgs phase with SU(2) totally broken and all the \(A_{\mu a}\) massive. Eqs.(24) are solved in the \(\pi_{\theta_a}\).
is invariant under gauge transformations, let us do the field-dependent gauge transformation \( \tilde{U}_o^{-1}(x) \) [like for going to the unitary gauge] on Eq.(18). The final result is

\[
L'(x) = -\frac{1}{4g^2} F'_{a\mu
u} F'_a{}^{\mu\nu}(x) + \phi_o^2(1 + \frac{1}{\sqrt{2}\phi_o} H(x))^2 \sum_a A'_a(x) + \\
+ \frac{1}{2} \partial_\mu H(x) \partial^\mu H(x) - 2(\phi_o \sqrt{\lambda})^2 H^2(x)(1 + \frac{1}{2\sqrt{2}\phi_o} H(x))^2 + \\
+ \frac{i}{2} \bar{\psi}'(x) [\gamma^\mu (\partial_\mu - \tilde{A}'_{a\mu}(x)T^a) - (\partial_\mu - \tilde{A}'_{a\mu}(x)T^a) \gamma^\mu] \psi'(x) - m \bar{\psi}'(x) \psi'(x),
\]

with the new Lagrangian density depending only on the configuration variables

\[
A'_{a\mu}(x) = (A'_\mu(x)), \\
\psi'(x) = \tilde{U}_o^{-1}(x) \psi(x) \\
H(x).
\]

The new canonical momenta are

\[
\pi'_a(x) = 0 \\
\vec{\pi}'_a(x) = g^{-2} \vec{E}'_a(x) \\
\pi_H(x) = \partial^\rho H(x),
\]

plus the fermion momenta. The resulting Dirac Hamiltonian density is

\[
H'_D(x) = \frac{1}{2} \sum_a [g^2 \vec{\pi}''_a(x) + g^{-2} \vec{B}''_a(x)] - \\
- \phi_o^2(1 + \frac{1}{\sqrt{2}\phi_o} H(x))^2 \sum_a [A''_{a\rho}(x) - \tilde{A}''_a(x)] + \\
+ \frac{1}{2} [\pi_H^2(x) + [\tilde{\partial}H(x)]^2] + 2(\phi_o \sqrt{\lambda})^2 H^2(x)(1 + \frac{1}{2\sqrt{2}\phi_o} H(x))^2 + \\
+ \frac{i}{2} \bar{\psi}'(x) [\gamma \cdot (\tilde{\partial} + \tilde{A}'_a(x)T^a) - (\tilde{\partial} - \tilde{A}'_a(x)T^a) \cdot \gamma] \psi'(x) + m \bar{\psi}'(x) \psi'(x) - \\
- A''_{a\rho}(x) [-\tilde{D}_{ab} \cdot \vec{\pi}'_b(x) + i \psi^\dagger(x) T^a \psi'(x)] + \lambda_{ao}(x) \pi_a^\rho(x).
\]

The time constancy of the primary constraints \( \pi_a^\rho(x) \approx 0 \) generates the secondary constraints
\[ \zeta_a(x) = 2[\phi_o + \frac{1}{\sqrt{2}}H(x)]^2A'_{ao}(x) - \bar{\partial} \cdot \bar{\pi}_a(x) - c_{abc}\bar{A}_b(x) \cdot \bar{\pi}_c(x) + i\psi^\dagger(x)T^a\psi'(x) \approx 0. \quad (29) \]

The time constancy of the constraints \( \zeta_a(x) \approx 0 \) determines the three Dirac multipliers \( \lambda_{ao}(x) \). Therefore, we get three pairs of second class constraints \( \pi'_a(x) \approx 0, \zeta_a(x) \approx 0 \), eliminating the three pairs \( A'_{ao}(x), \pi'_a(x) \), of canonical variables by going to Dirac brackets. The final canonical basis of Dirac’s observables is

\[ \bar{A}_a(x), \quad \bar{\pi}_a(x) = g^{-2}\bar{E}_a(x), \quad H(x), \quad \pi_H(x), \quad \psi'_{Aa}(x), \quad \bar{\psi}'_{Aa}(x); \quad (30) \]

the physical fields \( \bar{A}_a, \psi' \), have been dressed with a cloud of Higgs would-be Goldstone fields \( \theta_a(x) \).

By going to Dirac brackets with respect to the second class constraints, we find the physical reduced Hamiltonian density of the Higgs phase [we rescale the fields: \( \bar{A}_a \mapsto g\bar{A}_a, \bar{E}_a = g^{-1}\bar{E}_a \mapsto \bar{\pi}_a = \bar{E}_a, \psi' \mapsto \psi \) so that \( F_{a\mu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g c_{abc}A_{b\mu}A_{c\nu} \) and \( A_{ao} \equiv \frac{1}{2}[\bar{\partial} \cdot \bar{\pi}_a + c_{abc}\bar{A}_b \cdot \bar{\pi}_c - ig\psi^\dagger T^a\psi]/g^2[\phi_o + \frac{1}{\sqrt{2}}H]^2] \]

\[ \mathcal{H}(\text{Higgs})(x) = \frac{1}{2} \sum_a[\bar{\pi}_a^2(x) + \bar{B}_a^2(x)] + \frac{1}{2}m_A^2(1 + \frac{|g|}{m_A}H(x))^2 \sum_a \bar{A}_a^2(x) + \]
\[ + \frac{1}{2}[\pi_H^2(x) + (\bar{\partial}H(x))^2] + \frac{1}{2}m_H^2H^2(2)(1 + \frac{|g|}{2m_A}H(x))^2 + \]
\[ + \sum_a \frac{[\bar{\partial} \cdot \bar{\pi}_a(x) + gc_{abc}\bar{A}_b(x) \cdot \bar{\pi}_c(x) - ig\psi^\dagger(x)T^a\psi(x)]^2}{2m_A^2(1 + \frac{|g|}{m_A}H(x))^2} + \]
\[ + \frac{i}{2} \bar{\psi}(x)[\bar{\gamma} \cdot (\bar{\partial} + g\bar{A}_a(x)T^a) - (\bar{\partial} - g\bar{A}_a(x)T^a) \cdot \bar{\gamma} \psi(x) + m\bar{\psi}(x)\psi(x), \quad (31) \]

where the original parameters \( \phi_o, \lambda \), have been replaced by the masses of the gauge and residual Higgs fields

\[ m_A = \sqrt{2}|g|\phi_o, \quad m_H = 2\phi_o\sqrt{\lambda}, \]
\[ \Rightarrow \phi_o = \frac{m_A}{\sqrt{2}|g|}, \quad \lambda = \frac{g^2m_H^2}{2m_A^2}. \quad (32) \]

We see that the Hamiltonian is local, because the Higgs mechanism produces a local, even if not polynomial, self-energy term. However, this self-energy term yields a non-local relation between \( \partial^a \bar{A} \) and \( \bar{\pi}_a \) (like in the Abelian case of I)
\[ \partial^o A^i_a(\vec{x}, x^o) = \{ A^i_a(\vec{x}, x^o), \int d^3y H^{(\text{Higgs})}_{\text{phys}}(\vec{y}, x^o) \} = \]
\[ = -\pi^i_a(\vec{x}, x^o) + \partial^i(\vec{\partial} \cdot \vec{\pi}_a(\vec{x}, x^o) + gc_{abc} \vec{A}_b(\vec{x}, x^o) \cdot \vec{\pi}_c(\vec{x}, x^o) - ig\psi^\dagger(\vec{x}, x^o) T^a \psi(\vec{x}, x^o) \]
\[ \frac{1}{(m_A + |gH(x)|^2)} \]
\[ \Rightarrow \pi^i_a(x) = -\partial^o A^i_a(x) - \]
\[ \partial^i \frac{1}{\Delta + (m_A + |gH(x)|^2)} \times \]
\[ [\vec{\partial} \cdot \partial^o \vec{A}_a(x) - gc_{abc} \vec{A}_b(\vec{x}, x^o) \cdot \vec{\pi}_c(\vec{x}, x^o) + ig\psi^\dagger(\vec{x}, x^o) T^a \psi(\vec{x}, x^o)] , \]
\[ \Rightarrow [ \quad Z^{(A)}_{ab}(\vec{x}, x^o) + (m_A + |gH(\vec{x}, x^o)|^2 ) \frac{2\delta_{ab}}{\Delta} \Delta + (m_A + |gH(\vec{x}, x^o)|^2 ) \]
\[ (\partial^o \vec{\partial} \cdot \vec{A}_a(\vec{x}, x^o) - gc_{abc} \vec{A}_b(\vec{x}, x^o) \cdot \vec{\pi}_c(\vec{x}, x^o) + ig\psi^\dagger(\vec{x}, x^o) T^a \psi(\vec{x}, x^o) ) = \]
\[ = \partial^o \vec{\partial} \cdot \vec{A}_a(\vec{x}, x^o) + gc_{abc} \vec{A}_b(\vec{x}, x^o) \cdot \partial^o \vec{\pi}_c(\vec{x}, x^o) + ig\psi^\dagger(\vec{x}, x^o) T^a \psi(\vec{x}, x^o), \quad (33) \]

Here \( Z^{(A)}_{ab} \) is the following operator [see Eq.(3-12) of the second paper in Ref. \[4\] for the determination of its Green function \( G^{(A,Z);}_{o,ab} \)]

\[ Z^{(A)}_{ab}(\vec{x}, x^o) = \delta_{ab} + gc_{abc} \vec{A}_c(\vec{x}, x^o) \cdot \frac{\vec{\partial}}{\Delta}, \]
\[ Z^{(A)}_{ab}(\vec{x}, x^o) G^{(A,Z);}_{o,abc}(\vec{x}, \vec{y}, x^o) = \delta_{ac} \delta^3(\vec{x} - \vec{y}), \]
\[ G^{(A,Z);}_{o,ab}(\vec{x}, \vec{y}, x^o) = -\vec{\partial}_x \cdot G^{(A,Z);}_{ab}(\vec{x}, \vec{y}, x^o) = \]
\[ = -\vec{\partial}_x \cdot [\vec{c}(\vec{x} - \vec{y}) (P e^{(3)}_{\vec{y}} d\vec{z} \vec{A}_a(\vec{z}, x^o) T^a)_{ab}] \quad (34) \]

with the path ordering along the flat geodesic and with \( \vec{c}(\vec{x}) = \frac{\vec{\partial}}{\Delta} \delta^3(\vec{x}) = \frac{\vec{x}}{4\pi|\vec{x}|^2} \).

To invert the operator \( Z^{(A)}_{ab} + \delta_{ab}(m_A + |g|H(x))^2 \frac{1}{\Delta} \), we need its Green function \( G^{(A,Z);}_{ab} \), which is given by (assuming that the perturbation of the Higgs field is small so that the series converges)


\[ T_{ab}(\vec{x}, x^o) = (m_A + |g|H(\vec{x}, x^o))^2 \frac{1}{\Delta_{\vec{x}}}. \quad (35) \]

Finally we get
\[ \pi_a^i(x) = -\partial^i A_a^i(x) - \partial^i \frac{1}{\Delta_x} \int d^3y G^{(A,Z)}_{ab}(\vec{x}, \vec{y}, x^o) \\\nabla_i (\partial^0 \vec{\partial}_y \cdot A_a^i(\vec{y}, x^o) + gc_{bab} \vec{A}_b(\vec{y}, x^o) \cdot \partial^i \vec{A}_a(\vec{y}, x^o) + ig \psi^\dagger(\vec{y}, x^o) T^a \psi(\vec{y}, x^o), \quad (36) \]

and a nonlocal Lagrangian density describing only the Higgs phase

\[ L_{phys}^{(Higgs)}(x) = \psi^\dagger(x)[i \partial^0 - \vec{\alpha} \cdot (i \vec{\partial} + e A_a(x) T^a) - \beta m] \psi(x) - \frac{1}{2} \frac{1}{\Delta_x} \int d^3y G^{(A,Z)}_{ab}(\vec{x}, \vec{y}, x^o) \partial^i \vec{\partial}_y \cdot \vec{A}_b(\vec{y}, x^o) + gc_{bac} \vec{A}_c(\vec{y}, x^o) \cdot \partial^i \vec{A}_a(\vec{y}, x^o) + ig \psi^\dagger(\vec{y}, x^o) T^b \psi(\vec{y}, x^o)] \times \\\n(\Delta + (m_A + |g| H(\vec{x}, x^o)) \int d^3z G^{(A,Z)}_{ac}(\vec{x}, \vec{z}, x^o) \partial^i \vec{\partial}_z \cdot \vec{A}_c(\vec{z}, x^o) + gc_{crs} \vec{A}_r(\vec{z}, x^o) \cdot \partial^i \vec{A}_s(\vec{z}, x^o) + ig \psi^\dagger(\vec{z}, x^o) T^c \psi(\vec{z}, x^o)) - \frac{1}{2} m_A^2 (1 + \frac{|g|}{m_A} H(x))^2 \sum_a \vec{A}_a^2(x) - \frac{1}{2} \sum_a \vec{B}_a^2(x) + \frac{1}{2} \partial_\mu H(x) \partial^\mu H(x) - \frac{1}{2} m_H^2 H^2(x)(1 + \frac{|g|}{2 m_A} H(x))^2. \quad (37) \]

Let us remark that in those points \( x^\mu \) where \( H(x) = -m_A/|g| = -\sqrt{2} \phi_o \) [which were excluded to exist not to have problems with the origin of the radial coordinates of Eq.(16)] we would recover massless SU(2) gauge theory, so that the numerator of the self-energy term in Eq.(32) must vanish, being the Gauss law of the massless theory. Therefore we should not have a singularity in these points, but new physical effects like non-Abelian vortices \[18\] in analogy to the Nielsen-Olesen vortices of the Abelian case (see I); however now one needs to consider nontrivial SU(2) bundles even in absence of non-Abelian monopoles \[6,19\]. See also Ref. \[20\], where there is mass generation without the Higgs mechanism from the requirement of integrability (absence of essential singularities) of the equations of motion.

Let us remark that the self-energy appearing in Eq.(31) is local and that, in presence of fermion fields, it contains a 4 fermion interaction, which has appeared from the nonperturbative solution of the Gauss laws and which is a further obstruction to the renormalizability of the reduced theory (equivalent to the unitary gauge, but without having added any gauge-fixing), which already fails in the unitary physical gauge due to the massive vector boson propagator not fulfilling the power counting rule; as said in Ref. \[13\], this is due to the fact
that the field-dependent gauge transformation relating $\vec{A}$ and $\vec{A}'$ in Eq. (26) is not unitarily implementable. It is interesting to note that all the interaction terms of the residual Higgs field $H(x)$ in Eq. (32) show that it couples to the ratio $|g|/m_A$.

As in the Abelian case of I, one can consistently eliminate [21] the residual Higgs field $H(x)$ by adding with a multiplier the constraint $H(x) \approx 0$ at the physical Hamiltonian: its time constancy would produce the constraint $\pi_H(x) \approx 0$, and $\partial^o \pi_H(x) \approx 0$ would determine the multiplier; the final Hamiltonian density would be

$$H^{(H=0)}(x) = \frac{1}{2} \sum_a [g^2 \pi_a^2(x) + g^{-2} \vec{B}_a^2(x)] + \frac{1}{2} m_A^2 \sum_a \vec{A}_a^2(x) +$$

$$+ \frac{[\vec{\partial} \cdot \vec{\pi}_a(x) + c_{abc} \vec{A}_b(x) \cdot \vec{\pi}_c(x) - ig \psi^\dagger(x) T^a \psi(x)]^2}{2m_A^2} +$$

$$+ \frac{i}{2} \vec{\psi}(x)[\gamma \cdot (\vec{\partial} + g \vec{A}_a(x) T^a) - (\vec{\partial} - g \vec{A}_a(x) T^a) \cdot \gamma] \psi(x) + m \vec{\psi}(x) \psi(x), \quad (38)$$

The elimination of $H(x)$ reproduces the reduction to Dirac’s observables of the massive non-Abelian vector theory (like in the Abelian case of I) or Proca field theory, whose Lagrangian density, in absence of fermions, is (see for instance Ref. [22])

$$\mathcal{L}^{(\text{mass})}(x) = -\frac{1}{4} F_{\mu\nu}(x) F_{a}^{\mu\nu}(x) + \frac{1}{2} M^2 A_{a\mu}(x) A_a^\mu(x). \quad (39)$$

The elimination of $H(x)$ can also be thought as a limiting classical result of the so-called “triviality problem” [triviality of the $\lambda \phi^4$ theory [24]], which however would imply a quantization (but how?) of the Higgs phase alone without the residual Higgs field, so that also its quantum fluctuations would be absent. Instead these fluctuations are the main left quantum effect in the limit $m_H \to \infty$, which is known to produce [24], in the non-Abelian case, a gauge theory coupled to a nonlinear $\sigma$-model, equivalent [22] to a massive Yang-Mills theory. Indeed, in absence of fermions, the Lagrangian density [1] can be rewritten in terms of a linear $SU(2)_L \times SU(2)_R \sigma$-model $[\lambda = m_H^2/4\phi_o^2]$

$$\tilde{\mathcal{L}}(x) = -\frac{1}{4} F_{\mu\nu}(x) F_a^{\mu\nu}(x) + [D^{(A)}_\mu(\phi(x))]^\dagger (D^{(A)}_\mu(\phi(x) - \frac{m_H^2}{4\phi_o^2}[\phi^\dagger(x) \phi(x) - \phi_0^2])^2 =$$

$$= -\frac{1}{4} F_{\mu\nu}(x) F_a^{\mu\nu}(x) + \frac{1}{2} Tr[D^{(A)}_\mu(x) D^{(A)}_\mu M(x)] - \frac{m_H^2}{4\phi_o^2} \frac{1}{2} Tr M^\dagger(x) M(x) - \phi_0^2]$$
\[ M(x) = \begin{pmatrix} \phi_1(x) & -\phi_2^*(x) \\ \phi_2(x) & \phi_1^*(x) \end{pmatrix} = \sigma(x) + i\vec{\tau} \cdot \vec{\zeta}(x). \]  

(40)

For \( m_H \to \infty \) one has \( M^\dagger M \to \phi_2^2 \) (i.e. \( H(x) \to 0 \); strongly interacting symmetry breaking sector), so that one gets \( M(x) = \phi_o U(x) \) with \( U^\dagger U = 1 \) and \( \tilde{\mathcal{L}} \to -\frac{1}{4} F^2 + \frac{\phi_2^2}{2} Tr [D^{(A)}_\mu U^\dagger D^{(A)\mu} U] \), which is the Lagrangian density of the nonlinear \( SU(2)_L \times SU(2)_R \) \( \sigma \)-model (broken to \( SU(2)_{L+R} \) by \( U^\dagger U = 1 \)) with either \( U(x) = e^{i\vec{\tau} \cdot \vec{\rho}(x)} \) or \( U(x) = \sqrt{1 - \vec{\zeta}(x)^2} + i\vec{\tau} \cdot \vec{\zeta}(x) \).

As in I, one could add to either Eq.(1) or Eq.(39) a term \( \frac{1}{2} \vec{\partial} A_{ao}(x) \cdot \vec{\partial} A_{ao}(x) \) [which could be made Lorentz-covariant by reformulating the theory on spacelike hypersurfaces \(^{[13,11]}\), so that the local self-energy terms in Eqs.(31) and (38) would be replaced by \( \frac{1}{2} \tilde{\mathcal{L}} \cdot \tilde{\pi}_a(x) + c_{abc} \tilde{A}_b(x) \cdot \tilde{\pi}_c(x) - ig\psi^\dagger(x) T^a \psi(x) \) \( \frac{1}{\Delta + m_H^2 (1 + \frac{\text{Im} H(x)}{m_H^2})^2} \frac{1}{\Delta + m_H^2} \tilde{\mathcal{L}} \cdot \tilde{\pi}_a(x) + c_{abc} \tilde{A}_b(x) \cdot \tilde{\pi}_c(x) - ig\psi^\dagger(x) T^a \psi(x) \) and \( \frac{1}{2} \tilde{\mathcal{L}} \cdot \tilde{\pi}_a(x) + c_{abc} \tilde{A}_b(x) \cdot \tilde{\pi}_c(x) - ig\psi^\dagger(x) T^a \psi(x) \) \( \frac{1}{\Delta + m_H^2 (1 + \frac{\text{Im} H(x)}{m_H^2})^2} \frac{1}{\Delta + m_H^2} \tilde{\mathcal{L}} \cdot \tilde{\pi}_a(x) + c_{abc} \tilde{A}_b(x) \cdot \tilde{\pi}_c(x) - ig\psi^\dagger(x) T^a \psi(x) \), respectively.

VI. COMMENTS

i) As in the Abelian case of I, the covariant R-gauge-fixings \(^{[25]}\)
\[ \partial^\mu A_{ao}(x) + \xi \theta_a(x) \approx 0, \]  

(41)

used in the proof of renormalizability and in the evaluation of radiative corrections, are ambiguous like the Gauss laws: they can be solved either in the Higgs fields (would-be Goldstone bosons) \( \theta_a(x) \) [Higgs phase] or in \( A_{ao}(x) \) [SU(2) phase] or in a mixed way [the other four mixed phases]. It turns out that in the proofs of renormalizability one is mixing all the existing disjoint phases (all of them are not physical except the Higgs one; at most the SU(2) phase could be relevant in cosmology, but not the mixed non-SU(2)-covariant ones), and only at the end, in the limit \( \xi \to \infty \), one is recovering the Higgs phase.

ii) Let us now consider the non-Abelian SU(2) charges, whose existence is implied by the Yang-Mills Noether identities \(^{[4,26]}\).
The gauge invariance of \( L(x) \) under the infinitesimal gauge transformations \( \delta A_{a\mu}(x) = \partial_\mu \alpha_a(x) + c_{abc} A_{b\mu}(x) \alpha_c(x) = \hat{D}^{(A)}_{\mu a b} \alpha_b(x) \), \( \delta \psi(x) = -\alpha_a(x) T^a \psi(x) \), \( \delta \bar{\psi}(x) = \bar{\psi}(x) \alpha_a(x) T^a \), \( \delta \phi(x) = -i \alpha_a(x) \tilde{T}^a \phi(x) \), \( \delta \phi^\dagger = i \phi^\dagger \alpha_a(x) \tilde{T}^a \), produces the Noether identities

\[
0 \equiv \delta L = \frac{\partial L}{\partial A_{a\mu}} \delta A_{a\mu} + \frac{\partial L}{\partial \partial_\nu A_{a\mu}} \delta \partial_\nu A_{a\mu} + \frac{\partial L}{\partial \bar{\psi}} \delta \bar{\psi} + \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\partial L}{\partial \phi^\dagger} \delta \phi^\dagger = g^{-2} L^\mu_a \delta A_{a\mu} + \delta \bar{\psi} \bar{\psi} - L \psi \delta \psi + \delta \phi^\dagger \phi \phi^\dagger_1 L_\phi^\dagger_1 + L_\phi^\dagger \delta \phi_1 + \partial_\mu G^\mu
\]

\[
G^\mu = \alpha_a G^\mu_{1a} + \partial_\mu \alpha_a G^\mu_{oa} = -g^{-2} F_a^{\mu \nu} \delta A_{a\nu} - \frac{i}{2} [\delta \bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu \delta \psi] - [D^{(A)}_{\mu \phi}]^\dagger \phi^\dagger + \partial_\mu D^{(A)}_{\mu \phi}
\]

\[
G^\mu_{oa} = -g^{-2} F_a^{\mu \nu}
\]

\[
G^\mu_{1a} = -g^{-2} c_{abc} F_b^{\mu \nu} A_{c\nu} - i \bar{\psi} \gamma^\mu T^a \psi + [D^{(A)}_{\mu \phi}]^\dagger \bar{\phi}^\dagger T^a \psi + \phi^\dagger \tilde{T}^a D^{(A)}_{\mu \phi} = -g^{-2} c_{abc} F_b^{\mu \nu} A_{c\nu} - J^\mu_a
\]

\[
\partial_\mu G^\mu = \partial_\mu \partial_\nu \alpha_a G^\mu_{oa} + \partial_\mu \alpha_a [\partial_\nu G^\mu_{oa} + G^\mu_{1a}] + \alpha_a \partial_\mu G^\mu_{1a} \equiv -g^{-2} L^\mu_a \delta A_{a\mu} + \partial_\mu \bar{\psi} \bar{\psi} - \delta \bar{\psi} \bar{\psi} - L_\phi^\dagger \delta \phi_1 - \delta \phi^\dagger \phi \phi^\dagger_1 \equiv 0. \tag{42}
\]

The last line implies the Noether identities [(\( \mu \nu \)] and [\( \mu \nu \)] mean symmetrization and antisymmetrization respectively]

\[
G^\mu_{oa} \equiv 0
\]

\[
\partial_\nu G^\mu_{oa} \equiv -G^\mu_{oa} + g^{-2} L^\mu_a = g^{-2} (L^\mu_a + c_{abc} F_a^{\mu \nu} A_{a\nu}) + i \bar{\psi} \gamma^\mu T^a \psi + \phi^\dagger \tilde{T}^a D^{(A)}_{\mu \phi} - [D^{(A)}_{\mu \phi}]^\dagger \tilde{T}^a \phi
\]

\[
\partial_\mu G^\mu_{1a} \equiv g^{-2} c_{abc} A_{b\mu} L^\mu_c + L_\psi T^a \psi + \bar{\psi} T^a L_\psi + L_\phi^\dagger \tilde{T}^a \phi_1 - \phi^\dagger_1 \tilde{T}^a L_\phi^\dagger_1 \equiv 0 \tag{43}
\]

and, from the last two lines of these equations, the contracted Bianchi identities

\[
\hat{D}^{(A)}_{\mu} L^\mu - g^2 \bar{T}^a (L_\psi T^a \psi + \bar{\psi} T^a L_\psi + L_\phi^\dagger \tilde{T}^a \phi_1 - \phi^\dagger_1 \tilde{T}^a L_\phi^\dagger_1) \equiv 0. \tag{44}
\]
The following subset of Noether identities reproduces the Hamiltonian constraints

\[
\pi_a^\alpha = G_a^{(\alpha \alpha)} \equiv 0
\]

\[
0 \equiv \partial^\alpha \pi_a^\alpha = -\Gamma_a - g^{-2} L_a^\alpha \equiv -\Gamma_a.
\] (45)

The strong improper conservation laws \[24\] \( \partial_\mu V_a^\mu \equiv 0 \), implied by Eqs.(43), identify the strong improper conserved currents (strong continuity equations)

\[
-V_a^\mu = \partial_\nu G_a^{\mu \nu} = g^{-2} \partial_\nu F_a^{\mu \nu} = \partial_\nu U[a^{\mu \nu}] \equiv g^{-2} c_{abc} F_b^{\mu \nu} A_{cv} + j_a^\mu =
\]

\[
= g^{-2} c_{abc} F_b^{\mu \nu} A_{cv} + i \bar{\psi} \gamma^\mu T^a \psi + [D^{(A)} \phi^\dagger \tilde{T}^a \phi - \phi^\dagger \tilde{T}^a D^{(A)} \mu \phi =
\]

\[
= -G_a^\mu = g^{-2} c_{abc} F_b^{\mu \nu} A_{cv} + j_{F_a}^\mu + j_{KG_a}^\mu,
\] (46)

with the superpotential \( U[a^{\mu \nu}] = -g^{-2} F^{a \mu \nu} \). In the last line, \( j_{F_a}^\mu = i \bar{\psi} \gamma^\mu T^a \psi \) and \( j_{KG_a}^\mu = [D^{(A)} \phi^\dagger \tilde{T}^a \phi - \phi^\dagger \tilde{T}^a D^{(A)} \mu \phi \) are the charge currents of the fermion field and of the complex Klein-Gordon Higgs fields respectively, while \( j_a^\mu \) is the total current of Eq.(43).

The associated weak improper conservation laws are \( \partial_\mu G_a^{\mu \alpha} = 0 \) [it is obtained by using the second line of Eqs.(43)]. If \( Q_a \) are the weak improper conserved non-Abelian Noether charges and \( Q_a^{(V)} \) the strong improper conserved ones, we get [its meaning is equivalent to \( \int d^3x \Gamma_a(\vec{x}, x^o) = 0 \)]

\[
Q_a = -g^{-2} \int d^3x G_a^{\alpha}(\vec{x}, x^o) = -c_{abc} \int d^3x F_b^{\alpha k}(\vec{x}, x^o) A_c^k(\vec{x}, x^o) + g^2 \int d^3x j_a^\alpha(\vec{x}, x^o) =
\]

\[
= \int d^3x \left[-c_{abc} \vec{A}_b(\vec{x}, x^o) \cdot \vec{E}_c(\vec{x}, x^o) + ig^2 \bar{\psi}^\dagger(\vec{x}, x^o) T^a \psi(\vec{x}, x^o) -
\]

\[
- g^2 (\pi_\phi(\vec{x}, x^o) \pi^\dagger(\vec{x}, x^o) - \phi^\dagger(\vec{x}, x^o) \pi_\phi(\vec{x}, x^o)) \right] =
\]

\[
= g^2 \int d^3x \left[-c_{abc} \vec{A}_b(\vec{x}, x^o) \cdot \vec{E}_c(\vec{x}, x^o) + ig^2 \bar{\psi}^\dagger(\vec{x}, x^o) T^a \psi(\vec{x}, x^o) +
\]

\[
+ \pi_\theta(\vec{x}, x^o) V_{ba}^{-1}(\theta(\vec{x}, x^o)) \right] = Q_{F_a} + Q_{\theta a}^\alpha =
\]

\[
\equiv Q_a^{(V)} = g^2 \int d^3x V_a^{\alpha}(\vec{x}, x^o) =
\]

\[
= \int d^3x \partial^k F_a^{\alpha k}(\vec{x}, x^o) = g^2 \int d^3x \bar{\pi}_a(\vec{x}, x^o) = \int d^3 \bar{S} \cdot \vec{F}_a(\vec{x}, x^o),
\] (47)

where \( Q_{F_a} \) and \( Q_{\theta a} \) are the non-Abelian charges (in units of \( g \)) of the fermion fields and of the complex Higgs field. They are gauge covariant due to the assumed bound-
ary conditions [4]. For $\alpha = \alpha_a T^a = \text{const.}$, one speaks of improper "global" [or "rigid" or "first kind"] gauge transformations [under them the gauge potentials transform gauge covariantly [4]: $A \mapsto U^{-1} \tilde{A} U$]; the generator of the infinitesimal improper gauge transformations is $G[\alpha] = - \int d^3 x \alpha_a(\vec{x}, x^o) \Gamma_a(\vec{x}, x^o) = - \int d^3 x \tilde{\pi}_a(\vec{x}, x^o) \tilde{D}_{ab} \alpha_b(\vec{x}, x^o) - \tilde{\partial} \cdot (\alpha_a(\vec{x}, x^o) \tilde{\pi}_a(\vec{x}, x^o)) = \alpha_a = \text{const.} - \int d^3 x c_{abc} \tilde{\pi}_a(\vec{x}, x^o) \cdot \tilde{A}_b \alpha_c + \sum_a \alpha_a Q_a^{(V)}$.

In the SU(2) phase, $Q_a^{(V)} = 0$ because the electric fields $\vec{E}_a$ decay at infinity due to the generated mass $m_A$ of the SU(2) fields (the interactions have become short-range). Therefore, in presence of spontaneous symmetry breaking through the Higgs mechanism, we loose the Gauss theorem; Eq.(47) only says that $Q_{F a} = - Q_{\theta a}$, like in the Abelian case of I, before doing the canonical reduction to the Dirac observables; but this is the statement that each fermion and vector field is going to be dressed by a Higgs cloud of would-be Goldstone bosons $\theta_a(x)$. In the Higgs sector, the original SU(2) local gauge symmetry is reduced to a global one, under which the Lagrangian density (37) is invariant. This implies that the non-Abelian charges $Q_{F a}$ become ordinary Noether constants in the Higgs phase. Their expression in terms of Dirac observables (in units of $g$) is

$$Q_{F a} = \int d^3 x [c_{abc} \tilde{A}_b(\vec{x}, x^o) \cdot \tilde{E}_c(\vec{x}, x^o) + i \bar{\psi}_a^{+}(\vec{x}, x^o) T^a \psi_a(\vec{x}, x^o)]$$

$$\{Q_{F a}, Q_{F b}\} = c_{abc} Q_{F c}. \quad (48)$$

iii) Let us remark that, since in the Higgs phase the Gauss law constraints are solved algebraically in the Higgs momenta, we did not need to make additional assumptions about the functional space the gauge potentials and gauge transformations belong to as in Ref. [4] to avoid Gribov ambiguity. However the assumptions of Ref. [4] are necessary to find
the Dirac observables of the other phases, because only in this way all forms of Gribov ambiguity are avoided: in this way neither any gauge potential nor any field strength has a nontrivial stability subgroup of gauge transformations [otherwise, for gauge potentials one has “gauge symmetries” and the associated stratification of the constraint manifold and of the reduced phase space, while for field strengths one has the problem of “gauge copies”]. In particular, without these assumptions, there exist special gauge potentials \( \tilde{A}_{\mu}(x) \) with nontrivial (i.e. different from the center of the group of gauge transformations) stability subgroup; the gauge transformations \( \tilde{U} \) belonging to these subgroups are covariantly constant \( [\hat{D}_\mu(\tilde{A})\tilde{U} = 0] \) and are called “gauge symmetries” of these gauge potentials, which correspond to reducible connections on the principal bundle of the theory. For \( G = SU(2) \) the only possible stability subgroups are \( U(1) \) and \( Z_2 \); with each inequivalent \( U(1) \subset SU(2) \) is associated a stratum of reducible gauge potentials (connected by gauge transformations) with reduced structure group \( H=U(1) \) and with stability subgroup (or stabilizer) of gauge transformations \( Z_{SU(2)}[U(1)] = U(1) \) \( [Z_G(H) = \{ a \in G \mid ab = ba \ for each \ b \in H \} \) is the centralizer of \( H \) in \( G \). For \( G \neq SU(2) \), if \( H \) is the reduced structure group of a stratum \( (H \ also \ is \ the \ holonomy \ group \ of \ all \ the \ connections \ in \ the \ stratum) \), then its gauge potentials have \( Z_G(H) \) as stability subgroup of gauge transformations with \( H \neq Z_G(H) \) in general; the selection of one of these strata, when the function space allows their existence, is a kind of symmetry breaking since \( G \) (the structure group of the main stratum without gauge symmetries) is reduced to \( H \); if also the field strength has a stability subgroup \( G_F \) of gauge transformations \( (\hat{U} \in G_F \ iff [\hat{U}, F_{\alpha\beta}\hat{T}_{\alpha}] = 0) \), one has \( G_F \supseteq Z_G(H) \) with \( H \subset G \) (and, if \( \pi_1(G) \neq 0 \), each stratum has disjoined components) and there are “gauge copies” \( \hat{A} = \hat{U}^{-1}A\hat{U} + \hat{U}^{-1}d\hat{U} \) of each gauge potential \( A \) in the stratum with the same field strength \( F_{\alpha\beta}[\hat{A}] = F_{\alpha\beta}[A] \). Only in function spaces allowing the existence only of irreducible gauge potentials [all of them have \( G \) as holonomy group], one has \( H=\pi \) and \( G_F = Z_G(G) = Z_G \) [\( Z_G \) is the center of \( G \) and there is no Gribov ambiguity.

As shown in Ref. \[27\], the existence of reducible gauge potentials has implications for the non-Abelian charges associated with them. Let \( \tilde{A}_\mu = \tilde{A}_{\mu\bar{a}}\hat{T}_{\bar{a}} \) be a gauge potential
with holonomy group $H \subset G$ [$\hat{T}^a$ are the generators of the Lie subalgebra $g_H$ of the Lie algebra $g$ of $G$] and stability subgroup $Z_G(H)$. Since $\hat{A}_{\mu} \in g_H$, one has that also the corresponding Euler-Lagrange equations belong to $g_H$: $L_{\mu}^{\alpha} \hat{T}^a = (\hat{D}^{(A)}_{\nu\bar{a}b} F_{\bar{b}}^\nu + g^2 J_1^{\mu} \hat{T}^a \in g_H$.

If $U = 1 + \alpha \hat{T}^a \in Z_G(H)$ is an infinitesimal gauge symmetry of $\hat{A}_{\mu}$, then $\hat{D}^{(A)}_{\muab} \alpha_b = 0$. In the Noether identities of Eq.\(\text{(12)}\) one has $G_{\mu}^{\alpha} \equiv 0$, $G_{1a}^{\mu} \hat{T}^a = J_1^{\mu} \hat{T}^a \in g_Z(G(H))$ [the Lie algebra of $Z_G(H)$], $\partial_{\mu} G^\mu = \partial_{\mu} \alpha_a G_{1a}^{\mu} + \alpha_a \partial_{\mu} G_{1a}^{\mu} \equiv L_{A} \delta \psi - \delta \psi L_{\phi} - L_{\phi} \delta \phi_i - \delta \phi_i L_{\phi^* i}$, so that Eqs.\(\text{(13)}\) are replaced by

$$G_{1a}^{\mu} \hat{T}^a \equiv J_1^{\mu} \hat{T}^a \in g_Z(G(H))$$

$$\partial_{\mu} G_{1a}^{\mu} \hat{T}^a \equiv (L_{A} \psi + \psi T^a \psi + L_{\phi} \phi_i - \phi_i L_{\phi^* i}) \hat{T}^a \equiv 0.$$

There is only the weak improper conserved Noether charge

$$Q_a \hat{T}^a = \int d^3 x J_a^0 (\bar{\epsilon}, x^0) \hat{T}^a \in g_Z(G(H)).$$

This charge has in general two components, $Q_a \hat{T}^a = Q_1 + Q_2 \in g_Z(G(H))$, with $Q_1 = Q_{1a} \hat{T}^a \in g_Z(G(H)) \cap g_H$ and $Q_2 = Q_{2a} \hat{T}^a \hat{T}^a \in g_Z(G(H))$ but not in $g_H$. While $Q_2$ is only a Noether charge determined by the matter fields [a flavourlike charge in the terminology of Ref. [27]], there is another form associated with $Q_1$, because one can use in it the $g_H$-valued Euler-Lagrange equations $L_{\mu}^{\alpha} \hat{T}^a \equiv 0$: $Q_2 = \int d^3 x J_a^0 \hat{T}^a \equiv -g^2 \int \hat{D}^{(A)}_{\nu\bar{a}b} F_{\bar{b}}^\nu \hat{T}^a$. Therefore, $Q_1$ may be reexpressed only in terms of the Yang-Mills field [itis a dynamical charge in the terminology of Ref. [27]] as in the standard case of the Gauss theorem, and it can be shown [27] that the charges $Q_{1a} \hat{T}^a$ are U(1)-charges corresponding to the Abelian part of $Z_G(H)$; instead the charges $Q_{2a}$ are non-Abelian.

In our case with $G$=SU(2), the only possibility is $H = Z_G(H) = U(1)$ and there is only an Abelian charge $Q_1$ associated with reducible gauge potentials.

iv) As said in I, to recover Lorentz covariance one has to reformulate the theory on spacelike hypersurfaces as shown in Refs. [15,16,17]. This reformulation produces an ultraviolet cutoff, because a classical unit of length $\rho = \sqrt{-W^2/P^2}$ emerges [the space domain over which the noncovariance of the canonical center of mass of the field configuration extends],
when we restrict to massive Poincaré representations, $P^2 > 0$, $W^2 \neq 0$. However, it is not yet clear how to use this physical cutoff in the quantization of nonlocal and nonpolynomial theories.

v) In a future paper we will unify the results of this paper and of Refs. [4,1] to determine the Dirac observables of the standard $SU(3) \times SU(2) \times U(1)$ model.
REFERENCES


