Abstract

By introducing generalized Bäcklund Transformations depending on arbitrary functions, wave and localized soliton solutions of the Davey-Stewartson equations are generated. Moreover explicit soliton solutions of the Hamiltonian DSI and DSIII equations are obtained.

1 Introduction

Many efforts have been made in the last years to extend the soliton theory to the non linear evolution equations (NLEEs) in 2 + 1 (two spatial and one temporal) dimensions. The Spectral Transform (ST), which is the principal mathematical tool of the soliton theory, has been extended to dispersive NLEEs in 2 + 1 dimensions in the first ’80 years, but it was generally admitted the lack of two dimensional localized solitons. Only recently, in 1988, it has been discovered by Boiti-Léon-Martina-Pempinelli that all the equations in the hierarchy related to the Zakharov-Shabat (ZS) hyperbolic spectral problem in the plane have exponentially localized soliton solutions. The most representative equation in the hierarchy is the Davey-Stewartson I (DSI) equation, which provides a two dimensional generalization of the non linear Schrödinger (NLS) equation. These 2-dimensional soliton solutions (also called dromions) display a richer phenomenology than in 1 + 1 dimensions. The scattering of the solitons can be inelastic and they can change shape and also exchange mass. They can also simulate inelastic scattering processes of quantum particles as creation and annihilation, fusion and fission, and interaction with virtual particles. Moreover, as a relevant application of the spectral method proposed by Sabatier and further developed by Boiti-Pempinelli-Sabatier, it has been stated that an additional NLEE, called Davey-Stewartson III (DSIII) equation admits also localized soliton solutions with properties similar to those of the DSI equation. Localized soliton solutions for the DSI equation were first found by using gauge Bäcklund Transformations (BT). Later these solutions were also obtained by means of the Inverse Spectral Transform (IST) and direct methods. Now we want to reobtain them by using a generalized
The advantage of this method is that in such a way it is possible to obtain soliton solutions for DSI and DSIII equations in the so-called Hamiltonian case. For this very interesting case we have not presently at our disposal the ST or the dressing method, so to get explicit solutions we must generalize the Bäcklund gauge in order to include the special form of the boundaries in the Hamiltonian case.

Finally let us note that a review on multidimensional localized solitons and more references are given in [14].

2 Davey-Stewartson I and III equations

Let us write the DSI equation in his more general two component form in characteristic coordinates

$$iQ_t + \sigma_3(Q_{uu} + Q_{vv}) + [A, Q] = 0$$ (2.1)

$$\begin{pmatrix} A^{(1)}_u & 0 \\ 0 & A^{(2)}_v \end{pmatrix} = -\frac{1}{2} \sigma_3 \begin{pmatrix} (Q^2)_v & 0 \\ 0 & (Q^2)_u \end{pmatrix}$$ (2.2)

where

$$Q = \begin{pmatrix} 0 & q(u, v, t) \\ r(u, v, t) & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A^{(1)}(u, v, t) & 0 \\ 0 & A^{(2)}(u, v, t) \end{pmatrix}$$ (2.3)

and $u$ and $v$ are the characteristic coordinates $u = x + y$, $v = x - y$.

The equation DSI is compatible with the reduction

$$q = \varepsilon r, \quad \varepsilon = \pm$$ (2.4)

(where $\overline{r}$ denotes the complex conjugate of $r$), which furnishes the so-called reduced DSI equation, describing physical situations as in hydrodynamics or in plasma physics.

The equation DSI can be obtained [6, 7] as the compatibility condition between two Lax operators $T_1$ and $T_2$ which commute in the “weak” sense [15]

$$[T_1, T_2] \phi = 0$$ (2.5)

$T_1$ is the ZS hyperbolic operator in the plane

$$T_1 \phi = \left\{ 2 \left( \frac{\partial_t}{\partial_u} \begin{array}{cc} 0 & \partial_u \\ 0 & \partial_v \end{array} \right) + Q \right\} \phi = 0$$ (2.6)

and $T_2$ has the form

$$2T_2 \phi \equiv \left\{ i\partial_t + \partial_u^2 - \partial_v^2 + A + \left( \begin{array}{cc} 0 & -q_u \\ r_v & 0 \end{array} \right) \right\} \phi = -k^2 \sigma_3 \phi$$ (2.7)

Let us remember that the boundary conditions of the auxiliary field $A$ can be arbitrarily chosen. The equation DSI in his standard version, i.e. with the boundary written as

$$A^{(1)}(u, v, t) = -\frac{1}{2} \int_{-\infty}^{v} du' (Q^2)_v + a^{(1)}_0(v, t)$$ (2.8)

$$A^{(2)}(u, v, t) = \frac{1}{2} \int_{-\infty}^{v} du' (Q^2)_u + a^{(2)}_0(u, t),$$ (2.9)
admits localized soliton solutions \[1\]. The one-soliton solution has the form
\[
q = -\frac{2}{D} \lambda I \eta e^{i\theta}, \quad r = -\frac{2}{D} \mu I \rho e^{-i\theta}
\] (2.10)
where
\[
D = 2\gamma (\cosh \xi_1 + \cosh \xi_2 + e^{\xi_2})
\] (2.11)
\[
\xi_1 = -\mu u - \lambda_I v + 2(\lambda_R \lambda_I + \mu_R \mu_I) t
\] (2.12)
\[
\xi_2 = \mu u - \lambda_I v + 2(\lambda_R \lambda_I - \mu_R \mu_I) t
\] (2.13)
\[
\theta = \mu_R u + \lambda_I v + (\lambda_I^2 - \lambda_R^2 + \mu_I^2 - \mu_R^2) t
\] (2.14)
\[
\gamma = \frac{1}{4} \eta \rho.
\] (2.15)
The complex parameters \(\lambda = \lambda_R + i\lambda_I, \mu = \mu_R + i\mu_I\) are the discrete eigenvalue of the associated Zakharov–Shabat spectral problem and \(\rho, \eta\) are arbitrary complex constants satisfying the conditions \(\gamma \in \mathbb{R}\) and \((1 + \gamma) > 0\).

Another relevant choice for the boundaries can be done
\[
A^{(1)} = -\frac{1}{4} \left( \int_{-\infty}^{u} + \int_{+\infty}^{u} \right) du' (Q^2)_v + A^{(1)}_0(v, t)
\] (2.16)
\[
A^{(2)} = \frac{1}{4} \left( \int_{-\infty}^{v} + \int_{+\infty}^{v} \right) dv' (Q^2)_u + A^{(2)}_0(u, t).
\]
In this case one can introduce the Hamiltonian
\[
H = \iint dudv \left[ r(\partial_u^2 + \partial_v^2)q - \frac{1}{4} qr(\partial_u \partial_u^{-1} + \partial_v \partial_v^{-1})qr + (A^{(1)}_0 - A^{(2)}_0)qr \right]
\] (2.17)
and the canonical Poisson bracket
\[
\{F, G\} = i \iint dudv \left[ \frac{\delta F}{\delta q} \frac{\delta G}{\delta r} - \frac{\delta F}{\delta r} \frac{\delta G}{\delta q} \right]
\] (2.18)
where \(q\) and \(r\) are the conjugate variables. Then the equations of motion
\[
q_t = \{q, H\}, \quad r_t = \{r, H\}
\] (2.19)
yield the DSI equation. It has been shown that the DSI equation for \(A^{(1)}_0 \equiv A^{(2)}_0 \equiv 0\) is completely integrable in the hamiltonian sense \[16, 17\], so this case is called Hamiltonian. Quantum extensions have also been done. However the problem of defining a ST is completely open.

There is another NLEE that can be associated to ZS hyperbolic spectral operator in the plane and admits localized soliton solutions, the so-called DSIII equation \[6, 7\]. Indeed the hamiltonian version of this equation appears already in \[18\] and a bihamiltonian version in \[19\]. The DSIII equation has the form
\[
iQ_t + \sigma_3(Q_{vv} - Q_{uu}) + [A, Q] = 0
\] (2.20)
\[
\begin{pmatrix}
A^{(1)}_u \\
A^{(2)}_v
\end{pmatrix} = -\frac{1}{2} \begin{pmatrix}
(Q^2)_v & 0 \\
0 & (Q^2)_u
\end{pmatrix}.
\] (2.21)
Also this equation is compatible with the reduction \( q = \varepsilon \). It can be obtained as the “weak” compatibility condition between two operator \( T_1 \) and \( T_2 \). \( T_1 \) is given, as before, by (2.6). \( T_2 \) in this case takes the form

\[
2T_2 \varphi \equiv \left\{ i \partial_t + \partial_u^2 + \partial_v^2 + A + \begin{pmatrix} 0 & q_u \\ r_v & 0 \end{pmatrix} \right\} \varphi = -k^2 \varphi. \tag{2.22}
\]

Also the DSIII equation in his standard version, i.e. with the boundaries chosen as

\[
A^{(1)}(u, v, t) = -\frac{1}{2} \int_{-\infty}^{u} du' (Q^2)_v + a_0^{(1)}(v, t) \tag{2.23}
\]

\[
A^{(2)}(u, v, t) = -\frac{1}{2} \int_{-\infty}^{v} dv' (Q^2)_u + a_0^{(2)}(u, t), \tag{2.24}
\]

admits localized soliton solutions \([6, 7]\), that have the same shape of those of the DSI equation, but different time evolution. The one-soliton solution has the form

\[
q = -\frac{2}{D} \lambda \eta e^{i\theta}, \quad r = -\frac{2}{D} \mu \rho e^{-i\theta} \tag{2.25}
\]

where

\[
D = 2\gamma (\cosh \xi_1 + \cosh \xi_2) + e^{\xi_2} \tag{2.26}
\]

\[
\xi_1 = -\mu \mu_t u - \lambda \lambda_t v + 2(\lambda R \lambda_t - \mu R \mu_t) \tag{2.27}
\]

\[
\xi_2 = \mu \mu_t u - \lambda \lambda_t v + 2(\lambda R \lambda_t + \mu R \mu_t) \tag{2.28}
\]

\[
\theta = \mu \mu_R u + \lambda \lambda_R v + (\lambda^2 - \lambda_R^2 - \mu^2 + \mu_R^2) \tag{2.29}
\]

\[
\gamma = \frac{1}{4} \eta \rho. \tag{2.30}
\]

As in the DSI case, the complex parameters \( \lambda = \lambda_R + i\lambda_I, \mu = \mu_R + i\mu_I \) are the discrete eigenvalue of the associated Zakharov–Shabat spectral problem and \( \rho, \eta \) are arbitrary complex constants satisfying the conditions \( \gamma \in \mathbb{R} \) and \( \gamma(1 + \gamma) > 0 \).

Moreover the choice for the boundaries

\[
A^{(1)} = -\frac{1}{4} \left( \int_{-\infty}^{u} + \int_{+\infty}^{u} \right) du' (Q^2)_v + A^{(1)}_0(v, t) \tag{2.31}
\]

\[
A^{(2)} = -\frac{1}{4} \left( \int_{-\infty}^{v} + \int_{+\infty}^{v} \right) dv' (Q^2)_u + A^{(2)}_0(u, t)
\]

furnishes the Hamiltonian case. If we introduce the Hamiltonian

\[
H = \iint dvdu \left[ r(-\partial_u^2 + \partial_v^2)q + \frac{1}{4} qr(\partial_u \partial_v - \partial_u \partial_v^{-1})qr + (A^{(1)}_0 - A^{(2)}_0)qr \right], \tag{2.32}
\]

the equations of motion \( q_t = \{q, H\} \) and \( r_t = \{r, H\} \) yield the DSIII equation.

Finally let us make some comments. The DSI and DSIII equations belong to the same principal spectral problem (2.6), so the direct and the inverse problems are the same for these two equations. On the contrary, they have different auxiliary spectral problems, so the evolution of the ST and the dispersion relation are different. Also, as we have seen, there is a simple modification for the boundary conditions. The conclusion is that all the results known for the DSI equation can be easily extended to the DSIII equation [6].
3 Solutions via Bäcklund Transformations

Given a solution $Q$ of the DSI (or DSIII) equation, we want to generate a new solution $Q'$ of the same equation by introducing a convenient gauge operator $B$ which transforms by means of

$$\psi' = B(Q', Q)\psi$$  \hspace{1cm} (3.1)

the matrix solution $\psi$ of the principal spectral problem

$$T_1(Q)\psi = 0$$  \hspace{1cm} (3.2)

for $Q$ to the matrix solution $\psi'$ of the same spectral problem for $Q'$

$$T_1(Q')\psi' = 0.$$  \hspace{1cm} (3.3)

It is easy to verify that if $B$ satisfies

$$T_1(Q')B(Q', Q) - B(Q', Q)T_1(Q) = 0$$  \hspace{1cm} (3.4)

$$T_2(Q')B(Q', Q) - B(Q', Q)T_2(Q) = 0,$$  \hspace{1cm} (3.5)

then $T_1(Q')$ and $T_2(Q')$ satisfy the same compatibility condition and therefore $Q'$ satisfies the same equation as $Q$. The above equations furnish, respectively, the so-called space and time component of the BT.

We are interested in the most general Bäcklund gauge as polynomial of the first order in $\partial_y$

$$B(Q', Q) = \alpha\partial_y + B_0(Q', Q)$$  \hspace{1cm} (3.6)

with $\alpha$ a constant diagonal matrix and $B_0$ a matrix. By inserting it in (3.4), where $T_1$ is the ZS hyperbolic spectral operator (2.6), we get

$$B(Q', Q) = \alpha\partial_y - \frac{1}{2}\sigma_3(Q'\alpha - \alpha Q) - \frac{1}{2}\sigma_3\alpha I(Q'^2 - Q^2) + \beta$$  \hspace{1cm} (3.7)

and the space component of the Bäcklund transformation

$$Q'\left[\beta - \frac{1}{2}\alpha\sigma_3 I(Q'^2 - Q^2)\right] - \left[\beta - \frac{1}{2}\alpha\sigma_3 I(Q'^2 - Q^2)\right]Q$$

$$- \frac{1}{2}\sigma_3(Q'\alpha - \alpha Q)_x - \frac{1}{2}(Q'\alpha + \alpha Q)_y = 0.$$  \hspace{1cm} (3.8)

The matrix operator $I$ is defined by

$$I = (\partial_x + \sigma_3\partial_y)^{-1}$$  \hspace{1cm} (3.9)

and the diagonal matrix $\beta$ is subjected to the constraint

$$(\partial_x + \sigma_3\partial_y)\beta = 0,$$  \hspace{1cm} (3.10)

i.e. it is of the form

$$\beta = \begin{pmatrix} \beta_1(v, t) & 0 \\ 0 & \beta_2(u, t) \end{pmatrix}$$  \hspace{1cm} (3.11)

where $\beta_1$ and $\beta_2$ are arbitrary functions. Note that for $\beta$ a space dependence is allowed in contrast with the $1 + 1$ dim case where $\beta$ is the constant of integration. This freedom will be used for getting soliton solutions for Hamiltonian DSI and DSIII equations. By inserting (3.7) in (3.5) we get the time component of the
BT, which is equivalent to the DSI (or DSIII) equation for $Q'$ plus two additional equations that can be used for determining the auxiliary field $A'$ and the admissible \( \beta \)'s. For details for the DSI equation see [13,14]. The final result is that \( \alpha \) and \( \beta \) can be written as
\[
\alpha = 1, \quad \beta = \begin{pmatrix} \lambda(v,t) & 0 \\ 0 & \mu(u,t) \end{pmatrix}
\] (3.12)
in the Bäcklund gauge \( B = B(Q,Q';\lambda,\mu) \). Note that \( \lambda \) and \( \mu \) generalize the parameters of the BT in 1 + 1 dimensional case.

For \( \lambda \) and \( \mu \) complex constants, the BT furnishes the localized soliton solutions for DSI (or DSIII) equation for the choice \( Q = 0, A = 0 \) [1, 8].

But to obtain soliton solutions for the Hamiltonian DSI (or DSIII) equation, we must use

1. the space dependence of \( \lambda \) and \( \mu \)
2. \( A \) can be \( \neq 0 \) when \( Q = 0 \).

In the case where \( \lambda \) and \( \mu \) have a space dependence, the calculations are very complicated and some constraint equations appear. We will give here only the principal results. Details for the DSI case can be found in [13]. We will restrict here at this equation. Results for the DSIII equation can be obtained with some modifications from those of the DSI equation.

The starting solution is \( Q = 0 \) and \( A = \text{diag}(A_{00}^{(1)}(v,t), A_{00}^{(2)}(u,t)) \) where \( A_{00}^{(i)} \) are arbitrary boundary values. If we choose \( A_{00}^{(1)} \) real and moving with constant speed
\[
A_{00}^{(1)}(v,t) = A_{00}^{(1)}(v + 2\phi t) \\
A_{00}^{(2)}(u,t) = A_{00}^{(2)}(u + 2\theta t),
\] (3.13)
a new solution \( Q, A \) can be explicitly written
\[
q = \frac{W(\eta, D)}{ED + \rho \eta/4} \exp[-i(\epsilon + \delta)], \quad r = -\frac{W(\rho, \mathcal{E})}{\mathcal{E}D + \rho \eta/4} \exp[i(\epsilon + \delta)]
\] (3.14)
with boundary values
\[
A_{00}^{(1)} = A_{00}^{(1)} + 2\partial_v^2 \log(ED + \rho \eta/4)|_{v=0}, \\
A_{00}^{(2)} = A_{00}^{(2)} - 2\partial_u^2 \log(ED + \rho \eta/4)|_{u=0}
\] (3.15)
and
\[
\delta = \phi v + (\phi^2 - \phi_0)t + \delta_0, \quad \epsilon = \theta u + (\theta^2 - \theta_0)t + \epsilon_0,
\] (3.17)
where \( \phi_0, \theta_0, \delta_0, \epsilon_0 \) are real constants. For the reduction \( r = \epsilon \eta \), the wronskian \( W \) becomes constant
\[
W(\eta, D) = 2a, \quad W(\rho, \mathcal{E}) = -2\epsilon a, \quad a \in \mathbb{R}.
\] (3.18)

The real functions \( D, \eta \) and \( \mathcal{E}, \rho \) satisfy the stationary Schrödinger equations
\[
D_{vv} + (A_{00}^{(1)} - \phi_0) D = 0, \\
\eta_{vv} + (A_{00}^{(1)} - \phi_0) \eta = 0
\] (3.19)
and
\[ E_{uu} - (A^{(2)}_{00} + \theta_0) E = 0, \]
\[ \rho_{uu} - (A^{(2)}_{00} + \theta_0) \rho = 0. \]  
(3.20)

The localized soliton solution is reobtained with a special choice of the functions \( D, \eta, E, \rho \) and with \( A^{(1)}_{00} = A^{(2)}_{00} = 0 \). We are interested in the Hamiltonian case
\[ A^{(1)}_0 \equiv A^{(2)}_0 \equiv 0 \]  
(3.21)
in the reduced case. Let us introduce two functions \( \eta_0(v + 2\varphi t) \) and \( \rho_0(u + 2\vartheta t) \) defined as follows
\[ A^{(1)}_{00} = 2\varphi^2 \log \eta_0, \quad A^{(2)}_{00} = -2\vartheta^2 \log \rho_0. \]  
(3.22)

We have to solve a complicated nonlinear system of coupled equations with constraints for \( D, \eta, \eta_0 \) and \( E, \rho, \rho_0 \). If we add the additional constraints
\[ \lim_{v \to \pm \infty} \frac{\rho}{\mathcal{E}} = -2(\rho_1 \pm \rho_2), \]  
(3.23)
\[ \lim_{u \to \pm \infty} \frac{\eta}{\mathcal{D}} = -2(\eta_1 \pm \eta_2) \]  
(3.24)
\((\rho_i \text{ and } \eta_i \text{ are real constants to be determined}), \) the system decouples and we obtain the solution
\[ q = \frac{2a \varphi_2 \eta_2 \rho_0 \exp[-i(\epsilon + \delta)]}{\sinh \alpha \sinh \beta + (\rho_1 \sinh \alpha + \varphi' \rho_2 \cosh \alpha)(\eta_1 \sinh \beta + \varphi'' \eta_2 \cosh \beta)} \]  
(3.25)

where \((\varphi')^2 = 1, (\varphi'')^2 = 1\) and \(\alpha, \beta\) are related to \(D\) and \(E\) respectively. \(\eta_0, \alpha, \rho_0, \beta\) are determined by
\[ \frac{\partial^2 \eta_0}{\partial \varphi^2} + a^2 \rho^2 \eta_0^5 - \phi_0 \eta_0 = 0, \quad \partial_\varphi \alpha = \varphi' \rho_2 \eta_0^2. \]
\[ \frac{\partial^2 \rho_0}{\partial \varphi^2} + a^2 \rho^2 \rho_0^5 - \theta_0 \rho_0 = 0, \quad \partial_\varphi \beta = -\theta_0 \varphi' \rho_2 \rho_0^2 \]  
(3.26)

with the consistency conditions
\[ \lim_{v \to \pm \infty} (\rho_1 + \varphi' \rho_2 \coth \alpha) = -(\eta_1 \pm \eta_2)^{-1}, \]  
(3.27)
\[ \lim_{u \to \pm \infty} (\eta_1 + \varphi'' \eta_2 \coth \beta) = -(\rho_1 \pm \rho_2)^{-1}. \]  
(3.28)

The above ordinary differential equations for \(\eta_0\) and \(\rho_0\) can be explicitly integrated in terms of elementary or classical transcendental functions and it is easy to verify the consistency conditions.

Let us consider two cases in particular
1. \[ \partial_\varphi \eta_0 \equiv 0, \quad \partial_\varphi \rho_0 \equiv 0, \quad \phi_0 = \lambda_0^2 > 0, \quad \theta_0 = \mu_0^2 > 0. \]  
(3.29)

We get
\[ q(u, v, t) = 2|\lambda_0 \mu_0|^{1/2} \exp[-i(\epsilon + \delta)] \cosh \xi \]  
(3.30)

where
\[ \xi = \mu_0(u + 2\vartheta t - \omega_0) - \lambda_0(v + 2\varphi t - v_0), \]
\[ \epsilon = \theta_0 u + (\varphi^2 - \mu_0^2)t + \epsilon_0, \]
\[ \delta = \phi_0 v + (\varphi^2 - \lambda_0^2)t + \delta_0, \]  
(3.31)
so \(q\) in this case is a wave soliton solution.
\[ \partial_v \eta_0 \neq 0, \quad \partial_u \phi_0 \neq 0, \quad \phi_0 < 0, \quad \theta_0 < 0. \quad (3.32) \]

In this case the solution can be explicitly written \[\wp, \zeta, \sigma\] in terms of the \(\wp\), \(\zeta\), \(\sigma\)-Weierstrass elliptic functions and describes an infinite wave with a periodically modulated amplitude.

references