Noncommutative Synchrotron

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Abstract

We study the departures from the classical synchrotron radiation due to noncommutativity of coordinates. We find that these departures are significant, but do not give tight bounds on the magnitude of the noncommutative parameter. On the other hand, these results could be used in future investigations in this direction. We also find an acausal behavior for the electromagnetic field due to the presence in the theory of two different speeds of light. This effect naturally arises even if only $\theta^{12}$ is different from zero.

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1 Introduction

The idea of noncommutative space-time coordinates in physics dates back to the 1940's [1]. Recently, due to the discovery of Seiberg and Witten [2] of a map (SW map) that relates noncommutative to commutative gauge theories, there has been an increasing interest in studying the impact of noncommutativity on fundamental as well as phenomenological issues. Two directions seem to us very central: on the one hand, the clear understanding of the spatiotemporal symmetry and unitarity properties of these theories [3], [4], [5], [6], [7], [8], [9]; on the other hand the hunting for experimental evidence (see e.g. [10]), possibly in simple theoretical set-ups [11].

The aim of the present paper is to study the effects of noncommuting space-time coordinates on synchrotron radiation in classical electrodynamics. The motivations are twofold: i) we want to see the fundamental effects of noncommutativity, such as acausality, and violation of Lorentz and scale invariance, practically at work in a simple case; ii) we want to explore the physics of synchrotron radiation in the hope that more stringent limits on the magnitude of the noncommutative parameter $\theta$ could be set in this case (for a nice account on the current bounds see for instance [4]).

The current views on the space-time properties of noncommutative field theories are essentially three: i) with the only exception of space-time translations, spatiotemporal symmetries are manifestly violated (see for instance [3]), and sometimes the artifact of the so-called “observer transformations” has to be introduced [3], [12]; ii) full Lorentz invariance (including parity and time-reversal) is imposed on (a dimensionless) $\theta^{\mu\nu}$, leading to a quantum space-time with the same classical global symmetries [6]; iii) being the noncommutative field theory an effective theory of the fundamental string theory, space-time symmetries are not a big issue. For a quite comprehensive review see for instance [13]. We believe that a lot more work is needed to fully understand this very fascinating matter.

What we intend to do here is to take a practical view and tackle the problem of finding corrections to the spectrum of synchrotron radiation induced by noncommutativity at first order in $\theta$. We shall find that, in our approximations, these corrections act as a powerful “amplifier” of the effects of noncommutativity: independently from the actual value of $\theta$ the synchrotron radiation amplifies the effects of a factor $O(10^{13})$. On the other hand, due to the current bounds on $\theta$ this amounts to a correction of $O(10^{-10})$ of the commutative counterpart, hence we are still far from possible testable effects. We also see in this analysis that some surprising acausal behaviours naturally arise even if only the space-space component $\theta^{12}$ is taken to be different from zero. The effect is due to the presence of two different speeds of light in this theory. This result is in contrast with the general belief that acausality effects should arise in this context only when $\theta^{0i} \neq 0$. We take this last result as a confirmation that the issues of space-time properties are far from being clarified.

The theory we shall be dealing with in this paper is affected with serious problems in the quantum phase (see e.g. [14], [15]). For instance, the truncation of the theory at first order in $\theta$ leads to infrared instabilities at the quantum level [10], and in the limit $\theta \rightarrow 0$ the commutative quantum electrodynamics is not recovered [10]. These obnoxious features seem to be related to an unusual correspondence among the ultraviolet and infrared perturbative regimes of the quantum theory. It is still unclear whether this correspondence is an artifact of the perturbative calculations or a more fundamental (hence more serious) problem. For instance in [17] it is shown that there are scalar field theories where the connection is actually absent. These facts evidently mean that the quantum theory is still a “work in progress”,
and we shall not further address these important matters here. We shall instead focus on the classical theory, in the hope that a meaningful quantum theory might be discovered in the future, and that such a theory could have the classical model we are about to use as a limit. After all classical general relativity is widely used and experimentally tested even if a sound quantum theory of gravity still does not exist (and may as well not exist at all).

In the next Section we shall recall the main ingredients of noncommutative electrodynamics [11], and set the notation. In Sections 3 we shall exhibit the electromagnetic potentials for the noncommutative synchrotron, while in Section 4 we shall give the approximate expressions for the electric and magnetic fields to estimate the leading corrections to synchrotron radiation. Finally, in Section 5 we shall draw our conclusions.

2 Noncommutative Electrodynamics

For us the noncommutativity of space-time coordinates will be expressed in the simplest possible fashion, the canonical form [18], given by

\[ x^\mu \ast x^\nu - x^\nu \ast x^\mu = i\theta^\mu\nu, \]  

(1)

where the Moyal-Weyl \(\ast\)-product of any two fields \(\phi(x)\) and \(\chi(x)\) is defined as

\[ (\phi \ast \chi)(x) \equiv \exp\left\{ \frac{i}{2} \theta^\mu\nu \partial_\mu \partial_\nu \right\} \phi(x)\chi(y)|_{y \rightarrow x} \]  

(2)

\(\theta^{\mu\nu}\) is \(c\)-number valued, the Greek indices run from 0 to \(n-1\), and \(n\) is the dimension of the space-time. This approach, of course, does not contemplate all the possible ways noncommutativity of the coordinates could take place. For instance, two equally valid, if not more general, approaches are the Lie-algebraic and the coordinate-dependent (\(q\)-deformed) formulations [18], and many other approaches exist. Nonetheless, the canonical form is surely the most simple and the basic features of noncommutativity are captured in this model.

The action for the noncommutative Maxwell theory for \(n = 4\) is

\[ \hat{I} = -\frac{1}{4} \int d^4x \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}, \]  

(3)

where \(\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu], \hat{A}_\mu\) can be expressed in terms of a U(1) gauge field \(A_\mu\) and of \(\theta^{\mu\nu}\) by means of the SW map [2], \(\hat{A}_\mu(A, \theta)\). Note that \(\hat{A}_\mu(A, \theta) \rightarrow A_\mu\) as \(\theta^{\mu\nu} \rightarrow 0\), hence, in that limit, \(\hat{F}_{\mu\nu} \rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\).

Let us now recall some useful results, valid at all orders in \(\theta\). The Noether currents for space-time transformations (full conformal group) for the noncommutative electrodynamics described by Eq. (3) were obtained in [5] were obtained in [5]

\[ J^\mu_f = \Pi^{\mu\nu} \delta_f A_\nu - \hat{\mathcal{L}} f^\mu, \]  

(4)

where \(\hat{\mathcal{L}} = \int d^4x \hat{\mathcal{L}}, \Pi^{\mu\nu} = \delta \hat{\mathcal{L}}/\delta \partial_\mu A_\nu,\) and for translations (the only symmetric case) \(f^\mu = a^\mu\), where \(a_\mu\) are the infinitesimal parameters. By making use of the gauge-covariant transformations [5] one finds the conserved energy-momentum tensor [5]

\[ T^{\mu\nu} = \Pi^{\mu\rho} \Gamma^\rho_{\rho\nu} - \eta^{\mu\nu} \hat{\mathcal{L}}, \]  

(5)
whose symmetry is, of course, not guaranteed as in the commutative case ($\Pi^{\mu\nu} = -F^{\mu\nu}$), and the conservation holds when the equations of motion
\[ \partial_\mu \Pi^{\mu\rho} = 0, \]
are satisfied. Thus, in full generality, the conserved Poynting vector is
\[ \vec{S} = \frac{c}{4\pi} \vec{D} \times \vec{B} = \frac{c}{4\pi} \vec{E} \times \vec{H}, \]
where
\[ D^i \equiv \Pi^{i0} \quad \text{and} \quad H^i \equiv \frac{1}{2} \epsilon^{ijk} \Pi_{jk}. \]
For our purpose it suffices to treat the simplest case of noncommutative electrodynamics at order $O(\theta)$, coupled to an external current, described by
\[ \hat{I} = -\frac{1}{4} \int d^4x \left[ F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta} F_{\alpha\beta} F^{\mu\nu} F_{\mu\nu} + 2 \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu} + J_\mu \hat{A}^\mu \right], \]
where we made use of the $O(\theta)$ SW map
\[ \hat{A}_\mu(A, \theta) = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}), \]
and of the $*$-product defined in Eq. (2). From now on our considerations will be based on such a $O(\theta)$ theory.

In Ref. [11] it was found that, in the presence of a background magnetic field $\vec{b}$, and in absence of external sources ($J_\mu = 0$), the $O(\theta)$ plane-wave solutions exist. The waves propagating transversely to $\vec{b}$ travel at the modified speed $c' = c(1 - \vec{b}_T \cdot \vec{b}_T)$ (where $\vec{b} \equiv (\theta^1, \theta^2, \theta^3)$, with $\theta^{ij} = \epsilon^{ijk} \theta^k$, and $\theta^{0i} = 0$) while the ones propagating along the direction of $\vec{b}$ still travel at the usual speed of light $c$.

The plane-waves, unfortunately, do not give a stringent bound on $\theta$. As a matter of fact, with the current bound of $10^{-2}$ (TeV)$^{-2}$ [21], one would need a background magnetic field of the order of 1 Tesla over a distance of 1 parsec to appreciate the shift of the interference fringes due to the modified speed of noncommutative light. It is then of strong interest to find more stringent phenomenological bounds on the noncommutative parameters. In the next Sections we shall study the synchrotron radiation in the hope to ameliorate those bounds.

In order to do that let us recall the linearized constitutive relations among the fields following from the modified Maxwell Lagrangian in Eq. (9) [11]
\[ D^i = \epsilon^{ij} E^j \quad \text{and} \quad H^i = (\mu^{-1})^{ij} B^j, \]
where
\[ \epsilon^{ij} \equiv a \delta^{ij} + \theta^i b^j + \theta^j b^i, \quad (\mu^{-1})^{ij} \equiv a \delta^{ij} - (\theta^i b^j + \theta^j b^i), \]
\[ a = (1 - \vec{b} \cdot \vec{b}). \] Since $F_{\mu\nu} = \partial_\mu A_\nu$ still holds,
\[ \vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi, \]
the Bianchi identities are not modified. On the other hand, the dynamical Maxwell equations when a source is added, become
\[ \partial_\mu \Pi^{\mu\nu} = J^\nu + \theta^\alpha \nu J^\sigma \partial_\alpha A_\sigma + \theta^{\alpha \sigma} \partial_\nu (A_\alpha J^\nu) , \] (14)
leading to (for \( \theta^0 = 0 \))
\[ \vec{\nabla} \cdot \vec{D} = 4\pi[\rho + \vec{\theta} \cdot (\vec{\nabla} \times (\rho \vec{A}))] , \] (15)
\[ \left( \vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial}{\partial t} \vec{D} \right)^i = \frac{4\pi}{c} \left[ J^i + \theta^i (\epsilon^{ijk} J^j \partial_k A_\sigma + \epsilon^{jik} \partial_k (A^j J^i)) \right] , \] (16)
where the Latin indices run from 1 to 3.

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By using the potentials in Eq. (13), and the Lorentz gauge \( \partial_\mu A^\mu = 0 \), the equations of motion (15) and (16) become
\[ a \Box \Phi + (\theta^i b^j + \theta^j b^i) \left[ \partial_i \partial_j \Phi + \frac{1}{c} \frac{\partial}{\partial t} (\partial_i A_j) \right] = -4\pi[\rho + \vec{\theta} \cdot (\vec{\nabla} \times (\rho \vec{A}))] , \] (17)
\[ a \Box A^i + (\theta^i b^j + \theta^j b^i) \left[ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} A_j + \partial_j (\vec{\nabla} \vec{A}) \right] + \epsilon^{ikm} (\theta^m b^j + \theta^j b^m) \epsilon^{jlp} \partial_k \partial_l A_p \]
\[ = -\frac{4\pi}{c} \left[ J^i + \theta^i (\epsilon^{ijk} J^j \partial_k A_\sigma + \epsilon^{jik} \partial_k (A^j J^i)) \right] , \] (18)
where \( \Box \equiv -c^{-2} \partial^2_t + \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3} \).

For our purpose we use the following settings:

- Charged particle moving (circularly) in the plane \((1,2)\) with speed \(c\beta_i(t) = \dot{r}_i(t)\), \(i = 1, 2\), i.e.
  \[ J_\mu = e c \beta_\mu \delta(x_3) \delta^{(2)}(\vec{x} - \vec{r}(t)) , \] (19)
  where \(\vec{r}(t)\) is the position of the particle, and \(\beta_\mu = (1, \vec{\beta})\) (hence \(J_3 = 0\));
- \( \vec{b} = (0, 0, b) \), background magnetic field speeding up the particle;
- \( \vec{\theta} = (0, 0, \theta) \), i.e. \(\theta^3\) is the only non-zero component on \(\theta^\mu\nu\), this is the simplest possible case to see the effects of noncommutativity.

With these settings \(\epsilon^{ij} = a \delta^{ij} + \delta^{i3} \delta^{j3} \lambda\), where \(\lambda \equiv 2\theta b\), \(a = (1 - \theta b) = (1 - \lambda/2)\).

Furthermore, we are interested in evaluating the larger noncommutative departures from the synchrotron spectrum, hence contributions which are of order higher that \(O(c/R)\), where \(c\) is the electric charge, and \(R\) is the distance from the source, will be neglected. Let us write the general solutions of Eqs. (17) and (18) as \(A_\mu = A_\mu^{(0)} + \theta (A_\mu^{(\theta)} + \bar{A}_\mu^{(\theta)})\), where \(A_\mu^{(0)}\) is the solution for \(\theta = 0\), \(A_\mu^{(\theta)}\) is the correction obtained neglecting the \(O(\theta)\) contributions to the coupling with the external currents, and \(\bar{A}_\mu^{(\theta)}\) is the correction
coming solely from the \(O(\theta)\) contributions to the coupling with the external currents. It is easy to see that \(\tilde{A}_\mu^{(\theta)}\) is \(O(e/R)^2\), while \(A_\mu^{(\theta)}\) is \(O(e/R)\).

Taking all of the above into account, the approximations made lead us to write Eqs. (17), and (18) as

\[
\Box A_1 + \lambda \partial_2(\partial_1 A_2 - \partial_2 A_1) = -\frac{4\pi}{c} \tilde{J}_1 ,
\]

\[
\Box A_2 + \lambda \partial_1(\partial_2 A_1 - \partial_1 A_2) = -\frac{4\pi}{c} \tilde{J}_2 ,
\]

\[
\Box A_3 - \frac{\lambda}{c^2} \partial_1^2 A_3 + \frac{1}{c} \lambda \partial_3 \partial_1 \Phi = 0 ,
\]

\[
\Box \Phi + \lambda (\partial_3^2 \Phi + \frac{1}{c} \partial_1 \partial_3 A_3) = -4\pi \tilde{\rho} ,
\]

where \(\tilde{J}_i \equiv J_i/a, \ i = 1, 2\), and \(\tilde{\rho} \equiv \rho/a\). We notice here that: i) the \(1 \leftrightarrow 2\) symmetry for Eqs. (20) and (21), due to the rotation symmetry still present on the plane for the noncommutative case; ii) Eqs. (22) and (23) couple the components \(A_1\) and \(A_2\), while Eqs. (24) and (25) couple the components \(A_3\) and \(\Phi\). When one solves the equations for \(A_3\) and \(\Phi\), one sees that \(A_3 \sim O(\lambda)\). This gives a negligible contribution \(O(\lambda^2)\) to \(\Phi\), but is an effect completely due to noncommutativity, absent in the standard theory. As a matter of fact we have \(A_3 \neq 0\) even if the current \(\tilde{J}\) is taken to lay in the plane \((1, 2)\).

Indeed, by writing Eqs. (20), (21), (22), and (23) in the space of momenta we obtain to order \(O(\lambda)\)

\[
A_1(\vec{k}, \omega) = -\frac{4\pi}{c} \tilde{J}_2(\vec{k}, \omega)\lambda k_1 k_2 + \tilde{J}_1(\vec{k}, \omega)[a(\omega^2/c^2 - \vec{k}^2) + \lambda \vec{k}_3^2] \quad (24)
\]

\[
A_2(\vec{k}, \omega) = A_1(\vec{k}, \omega)(1 \leftrightarrow 2) \quad (25)
\]

\[
A_3(\vec{k}, \omega) = -\lambda \frac{4\pi \tilde{\rho}(\vec{k}, \omega)k_3\omega}{[\omega^2/c^2 - \vec{k}^2 + \lambda \omega^2][a(\omega^2/c^2 - \vec{k}^2 - \lambda \vec{k}_3^2)]} \quad (26)
\]

\[
\Phi(\vec{k}, \omega) = -\frac{4\pi \tilde{\rho}(\vec{k}, \omega)}{(\omega^2/c^2 - \vec{k}^2 - \lambda \vec{k}_3^2)} \quad (27)
\]

or

\[
A_\mu(\vec{k}, \omega) \equiv G_{\mu\nu}(\vec{k}, \omega)J_\nu(\vec{k}, \omega) \quad (28)
\]

where \(\tilde{J}_0 \equiv c \tilde{\rho}\). We can now identify the Green’s functions as

\[
G_{\mu\nu}(\vec{R}; \tau) = \frac{1}{4\pi^3} \int d^3k d\omega e^{i(\omega - \vec{k} \vec{R})} G_{\mu\nu}(k, \omega) \quad (29)
\]

where \(\vec{R} \equiv \vec{x} - \vec{x}', \ \tau \equiv t - t', \) and we made use of the fact that translation invariance is still present for the noncommutative theory.

The non-zero Green’s functions are then

\[
G_{00}(\vec{R}; \tau) = \frac{1}{R} \delta(\tau - R/c) - \lambda \left(\frac{1 - c\tau/R}{R} \delta(\tau - R/c) + \frac{\tau}{R} \delta'(\tau - R/c)\right) \quad (30)
\]

\[
G_{11}(\vec{R}; \tau) = \frac{1}{R} \delta(\tau - R/c) + \lambda \frac{c\tau}{2R^2} \delta(\tau - R/c) = G_{22}(\vec{R}; \tau) \quad (31)
\]

\[
G_{30}(\vec{R}; \tau) = -\lambda \frac{c\tau}{2R^2} \delta(\tau - R/c) - \frac{\lambda}{2R} \delta'(\tau - R/c) \quad (32)
\]
where $R \equiv |\vec{R}|$, the prime on the delta function means derivative with respect to its argument. It is interesting to notice that the effect of the noncommutativity appears as a $\delta'$ coming from the shifted poles in the $\omega$ integrals. This means that the difference in the propagation speeds, $c$ and $c'$, will be converted into a pre-acceleration effect due to $\ddot{\beta}$ (see below) at one single speed of light $c$.

One can now compute the electric and magnetic fields from the general expression for the potentials given by

$$A_\mu (\vec{x}, t) = \frac{1}{c} \int d^3x' dt' G_{\mu\nu} (\vec{x} - \vec{x'}; t - t') J_\nu (\vec{x'}, t') .$$  

From the structure of the Green’s functions in (30)-(32) one can see that

$$A_\mu = A_\mu^{(0)} + \lambda A_\mu^{(\lambda)} , \quad \vec{E} = \vec{E}^{(0)} + \lambda \vec{E}^{(\lambda)} , \quad \vec{B} = \vec{B}^{(0)} + \lambda \vec{B}^{(\lambda)}$$

where $A_3^{(0)} = 0$, and $\vec{B}^{(0)} = \vec{n} \times \vec{E}^{(0)}$.

The electric and magnetic fields have quite involved expressions, and the part proportional to $\lambda$ contains a term of the form

$$\left[ \frac{1}{c (1 - \vec{n} \cdot \vec{\beta})} \frac{d}{dt'} \left( \frac{1}{c (1 - \vec{n} \cdot \vec{\beta})} \frac{d}{dt'} \frac{\vec{n} \cdot c (t - t')}{1 - \vec{n} \cdot \vec{\beta}} R \right) \right]_{ret} ,$$

where $\vec{n} = \vec{R}/R$, and $\left[ \right]_{ret}$ are the usual retarded quantities. We see here the announced contributions proportional to the derivative of the acceleration. As discussed earlier, the $\ddot{\beta}$ contribution arises as an effect of the conversion of the two speeds of lights $c$ and $c'$ in the poles of the Green’s function into a single speed $c$ with a derivative of the delta function $\delta' (\tau - R/c)$. We have taken this view, rather than retaining both speeds, in order to better compare our results with the experiments.

Let us say here that these terms, which are introduced solely by noncommutativity, recall the familiar acausal scenario of the Abraham-Lorentz pre-acceleration effects for the classical self-energy of a point charge [19]. Even if not directly connected to the Abraham-Lorentz case, this feature is quite surprising in this context. As a matter of fact, we naturally obtain this effect by retaining only one space component of $\theta^{\mu\nu}$, while it seems that one should expect such behaviors only for non-zero time components of $\theta^{\mu\nu}$.

### 4 Corrections to Synchrotron Spectrum

In order to compare our results with the standard ones, we want now to compute the effects of noncommutativity on the synchrotron radiation in the experimentally relevant case defined by the following approximations:

- Ultra-relativistic motion $\beta = v/c \to 1$;
- Radiation observed in the plane $(1, 2)$, and far from the source $|\vec{x}| \sim R >> |\vec{r}(t)|$.

The power radiated in the direction $\vec{n}$ is

$$\frac{dP(t)}{d\Omega} = R^2 [\vec{S} \cdot \vec{n}] ,$$

$$dP(t) = R^2 [\vec{S} \cdot \vec{n}] ,$$

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where all the quantities \((\vec{n}, \vec{\beta}, \hat{\beta}, R)\) are in the plane \((1, 2)\), and we used the modified Poynting vector given in Eq. (17)

\[
\vec{S} = \frac{c}{4\pi} \vec{B} \times \vec{B}.
\]

One can easily verify that in these approximations \(A_3^{(\lambda)}\) does not contribute to the radiated power in Eq. (30), and for the evaluation of the order of magnitude of the leading noncommutative correction to the power one can use the following approximate expressions for the electric and magnetic fields

\[
\vec{E}(\vec{x}, t) \sim e \left[ \frac{1}{c \zeta} \frac{d}{dt} \vec{n} - \frac{\vec{\beta}}{cR} \right] \left[ 1 + \frac{\lambda}{c \zeta} \frac{d}{dt} \left( \vec{n} \frac{c(t - t')}{\zeta R} \right) \right]_{\text{ret}},
\]

\[
\vec{B}(\vec{x}, t) \sim e \left[ \frac{1}{c \zeta} \frac{d}{dt} \vec{\beta} \times \vec{n} \right] \left[ 1 + \frac{\lambda}{c \zeta} \frac{d}{dt} \left( \vec{\beta} \times \vec{n} \frac{c(t - t')}{\zeta R} \right) \right]_{\text{ret}},
\]

where \(\zeta = 1 - \vec{n} \cdot \vec{\beta}\). The expressions (37) and (38) for the electric and magnetic fields, in the limit \(\lambda = 0\), reproduce the standard results for the terms \(O(1/R)\) in the ultra-relativistic limit [19], which are the only relevant ones for the evaluation of the synchrotron radiation.

By using the expressions (37) and (38) for the electric and magnetic fields, and retaining only the leading contributions for large \(R\), and \(\beta \to 1\), it turns out that [20]

\[
\frac{d}{d\Omega} P(t) \equiv |\vec{E}(t)|^2 \sim \left( \frac{e^2}{4\pi c} \right)^{1/2} \left[ (1 + 3\lambda/2\zeta) \vec{n} \times (\vec{\beta} \times \vec{n}) \right]_{\text{ret}}^2,
\]

where \(\vec{E}(t) \equiv \sqrt{c/4\pi} [R(\vec{E}^{(0)} + \lambda/2\vec{E}^{(\lambda)})]_{\text{ret}}\).

The energy radiated in the plane is then [19]

\[
\frac{d}{d\Omega} I(\omega) = 2|\vec{E}(\omega)|^2,
\]

where \(\vec{E}(\omega)\) is the Fourier transform of \(\vec{E}(t)\) given by

\[
\vec{E}(\omega) = \left( \frac{e^2}{8\pi^2 c} \right)^{1/2} \int dt' e^{-i\omega(t' - \vec{n} \cdot \vec{r}(t'))} \vec{n} \times (\vec{n} \times \vec{\beta}) [-i\omega(1 + 3\lambda/2\zeta) + \frac{3\lambda}{2\zeta^3} \vec{n} \cdot \vec{\beta}].
\]

In the ultra-relativistic approximations two are the characteristic frequencies for the synchrotron: the cyclotron frequency \(\omega_0 \sim c/|\vec{n}|\), and the critical frequency \(\omega_e \sim 3\omega_0 \gamma^3\). In order to consider only the radiation in the plane, we shall work in the range of frequencies \(\omega \gg \omega_0\), for which the latitude \(\vartheta \sim \pi/2\) [19].

In this setting the leading terms for the energy radiated in the plane are [19], [20]

\[
\frac{d}{d\Omega} I(\omega) \sim \frac{e^2}{3\pi^2 c} \left( \frac{\omega}{\omega_0} \right)^2 \gamma^{-4} \left[ K_{2/3}(\xi) [1 + \lambda(1 + 6\gamma^2)] + \lambda \frac{24\gamma^5 \omega_0}{\omega} K_{1/3}(\xi) K_{2/3}(\xi) \right],
\]

where

\[
\xi = \frac{\omega}{3\omega_0 \gamma^{-3}},
\]

\(K_{2/3}(\xi), K_{1/3}(\xi)\) are the modified Bessel functions. The formula in Eq. (42) reproduces the standard results in the case \(\lambda = 0\). When \(\omega << \omega_c, \xi \to 0\), and \(K_\nu(\xi) \sim \xi^{-\nu}, \nu = 2/3, 1/3\).
When $\omega_0 << \omega << \omega_c$, i.e. $1 << \omega/\omega_0 << \gamma^3$,\n\[ X \equiv \frac{dI(\omega)/d\Omega}{dI(\omega)/d\Omega|_{\lambda=0}} \sim 1 + 10\left(\frac{\omega_0}{\omega}\right)^{2/3} \lambda \gamma^4. \quad (44) \]

By using the current bound on the parameter of noncommutativity $[21] \theta < 10^{-2} (\text{TeV})^{-2}$, one has that $\lambda = 2b\theta < 2n10^{-23}$, where $n$ is the value of $b$ in Tesla, and $1 \text{ Tesla} \sim 10^{-21} (\text{TeV})^2$. Thus for an electron synchrotron the correction is
\[ X < 1 + \left(\frac{\omega_0}{\omega}\right)^{2/3} n \times 10^{-21} \times \left(\frac{E(\text{MeV})}{\text{MeV}}\right)^4, \quad (45) \]
where $E$ is the energy of the electron, $\gamma_{\text{max}} \sim 2E(\text{MeV})/\text{MeV}$.

For instance, for the most energetic synchrotron (SPring-8, Japan) $E = 8 \text{ GeV}$, $b \sim 1 \text{ Tesla}$, and when $\omega/\omega_0 \sim \gamma^2$ we have\n\[ X < 1 + 10^{-10}. \]

Thus, there is an impressive "amplification" of the effects induced by a nonzero noncommutativity parameter (better, our $\lambda$). As a matter of fact one gains 13 orders of magnitude (from $10^{-23}$ to $10^{-10}$ for the case considered in this example), and this gain is independent from the actual input value for $\theta$.

5 Conclusions

We investigated synchrotron radiation in noncommutative classical electrodynamics. The spectrum of this radiation is considerably modified by noncommutativity of the coordinates. These departures from the standard spectrum work as an impressive "amplifier" of the noncommutative effects. On the other hand we cannot obtain a tight bound on $\theta$. This study indicates that the phenomenology of synchrotron radiation in noncommutative electrodynamics deserves further investigation.

We notice a partial analogy of the present theory, in the linearized approximation, with nonlinear optic. A comparison of the latter with the noncommutative case could give interesting results because the properties of the "medium" (as described by $\epsilon_{ij}$ and $\mu_{ij}$), depend in our framework on the external background magnetic field $\vec{b}$. It is then possible to separate the effects of noncommutativity, coupled to $\vec{b}$, from the other electrodynamical effects.

We also saw a peculiar acausal behaviour for the electric and magnetic fields as time-derivatives higher than two. This Abraham-Lorentz-like effect, which naturally arose even if only one space-space component of $\theta^{\mu\nu}$ is taken to be different from zero, is due to the two different speeds of light allowed in this theory. Moreover, it is in contrast with the general belief that such acausal effects should arise in this context only when $\theta^{0i} \neq 0$. We take this last result as a confirmation that the fascinating issues of the space-time properties in presence of noncommuting coordinates is far from being fully clarified.

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