Exact response of the non-relativistic harmonic oscillator†

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Abstract:

Using Green’s function and operator techniques we give a closed expression for the response of a non-relativistic system interacting through confining, harmonic forces. The expression for the incoherent part permits rapid evaluation of coefficients in a $1/q$ expansion. A comparison is made with standard approximation methods.

†This note, together with his best wishes are dedicated by ASR to his colleague and friend Klaus Dietrich at the occasion of his 60th birthday.
1. Introduction

Consider a system with mass $m_A$ composed of $A$ constituents with equal mass $m$. We are interested in its linear response or structure function, when the system is probed at momentum and energy transfer $(q, \omega)$

$$S(q, \omega) = A^{-1} \sum_n |F_{0n}(q)|^2 \delta(\omega - q^2/2m_A - E_{n0}),$$

where $F_{0n}(q) = \langle 0 | \rho^*_q | n \rangle$ is the inelastic form factor between the ground state $|0\rangle$ and excited states $|n\rangle$, and $\rho_q = \sum_j e^j e^{i\vec{q}\cdot \vec{r}'_j}$ the intrinsic charge density. $E_{n0}$ are intrinsic excitation energies without target recoil, and $\vec{r}'_j = \vec{r}_j - \vec{R}$ intrinsic coordinates with $\vec{R}$ the centre of mass coordinate.

A formal summation over $n$ yields an alternative expression

$$S(q, \omega) = (\pi A)^{-1} \sum_{i,j=1..A} \text{Im} \left( \langle 0 | e^{-i\vec{q}\cdot \vec{r}'_i} G(\omega + E_0 - q^2/2m_A - i\eta)e^{i\vec{q}\cdot \vec{r}'_j} | 0 \rangle \right)$$

$$G(z) = (z - K - V)^{-1}$$

with $G$, the exact Green’s function of the system in terms of the total kinetic and potential energy.

Over the years, general methods have been devised to approximately calculate the above response. Recently there has been interest in the high-$q$ response for systems with singular attractive interactions of the confining type\textsuperscript{1,2}. For those systems, perturbative methods in the confining interaction $V$ fail and non-perturbative methods are required. We first mention the $1/q$ expansion of the (reduced) response as discussed by Gersch, Rodriguez and Smith (GRS)\textsuperscript{3}

$$\phi(q, y) = (q/m)S(q, \omega) = \sum_k (m/q)^k F_k(y),$$

where the energy loss $\omega$ has been replaced by a suitable alternative variable

$$y = -q/2 + m\omega/q$$
Alternatively one may perform an actual summation of the series (1.1) \(^2\). A pre-requisite for such a treatment is the knowledge of all the states of the complete spectrum, which is purely discrete for confining interactions.

None of the above methods leads to a closed expression for \( S \) or \( \phi \). However, Eq. (1.2) shows that such an expression would result if the exact Green's function there were known. This is only the case for a few selected systems. An enumeration of cases, which can be solved by means of path integral methods have recently been given \(^4\). At this point we recall that the above method has been previously suggested and applied to the response of systems, as complicated as liquid \(^4\)He probed at high \( q \) for fixed \( y \)\(^5\). Evaluations in practice will always require approximations.

In what follows, we will limit ourselves to a system of particles interacting through harmonic confining forces, which can be exactly solved and for which approximations can be tested. Since for this system the Hamiltonian separates in Jacobi variables, it suffices to treat only a ‘di-quark’ and actually for the same reason, one in one dimension only. In the following, the mass \( m_A \) will be replaced by \( 2m \).

In Section 2 we will discuss two methods which lead to a closed expression for the reduced response for the above system. In Section 3 we will give its \( 1/q \) expansion and compare results with the outcome of the GRS theory as well as with previously suggested approximations.

2. The response for a harmonically confined ‘di-quark’.

2a. The Green’s function method.

Consider the Hamiltonian of the relative motion of a harmonically bound ‘di-quark’

\[
H = \tilde{p}^2/m + \beta^4 \tilde{x}^2/m,
\]

with \( \beta \), the inverse harmonic oscillator length. For instance the path-integral method leads to the following expression for the Greens’ function for the relative motion \(^6\).

\[
G(x, x', t) = \sqrt{\frac{\beta^2}{2\pi isin\alpha}} \exp \left\{ \frac{(i\beta^2/2sin\alpha)}{2} \left[ (x^2 + x'^2)cos\alpha - 2xx' \right] \right\},
\]

\[(2.2)\]
where $\alpha = 2 \beta^2 s / q$; $s = tq / m$. The ground state wave function is $\langle x | 0 \rangle = [\beta^2 / \pi]^{1/4} e^{-\beta^2 x^2 / 2}$ and the part of the charge density fluctuation dependent on the relative coordinate $\rho_q(x) = e_1 e^{iqx / 2} + e_2 e^{-iqx / 2}$. Replacing the relative coordinates $x, x'$ by

$$x = Z + z / 2, \quad x' = Z - z / 2, \quad (2.3)$$

one finds

$$\rho_q(x) \rho_q(x') = e_1 e^{-iqz / 2} + e_2 e^{iqz / 2} + e_1 e_2 (e^{iQ} + e^{-iQ}) \quad (2.4)$$

The first two terms are (for equal charges identical) incoherent contributions due to a transferred $q$, which is absorbed and emitted by the same particle. For the remaining coherent contributions those particles are different. The same transformation (2.3) (which has nothing to do with a CM transformation, and is typical for the harmonic oscillator case) permits an easy evaluation of the $Z, z$ integrals. Taking the Fourier transform of (2.2) for the argument $\omega + \beta^2 / m - q^2 / 4m$, as required in Eq. (1.2) and using (1.4) to eliminate $\omega$ in favour of $y$, one shows

$$\phi = \phi^{\text{incoh}} + \phi^{\text{coh}}$$

$$\phi(q, y)^{\text{incoh}} = \left( e_1^2 + e_2^2 \right) \int_{-\infty}^{\infty} \frac{ds}{2\pi} \exp[iys] \exp \left[ -\frac{\beta^2 s^2}{4} \left( \frac{\sin \alpha / 2}{\alpha / 2} \right)^2 \right] \exp \left[ -iqs / 4 \left( \frac{\sin \alpha - 1}{\alpha} \right) \right] \quad (2.5a)$$

$$\phi(q, y)^{\text{coh}} = e_1 e_2 \int_{-\infty}^{\infty} \frac{ds}{2\pi} \exp[iys] \exp \left[ -\frac{\beta^2 s^2}{4} \left( \frac{\cos \alpha / 2}{\alpha / 2} \right)^2 \right] \exp \left[ iqs / 4 \left( \frac{\sin \alpha}{\alpha} + 1 \right) \right] \quad (2.5b)$$

Except for the harmonic oscillator, we do not know of a system for which the response has been given in a simple closed form like (2.5). As is the case with the GRS theory, the incoherent part (2.5a) is seen to permit a series expansion in $\alpha \propto q^{-1}$. However, the GRS expression \cite{3} comprises three integrals and in addition an ordering operator which can only be realized in the expansion (1.3).
2b. Derivation using operator techniques

In what follows we will sketch a different approach to obtain the response $S(q, \omega)$, Eq. (1.1), for the case of a 'di-quark'. We do so, by directly evaluating the time-depending operators after using the Fourier transform of the $\delta$ function. Eq. (1.1) then becomes

$$S(q, \omega) = \frac{1}{A} \int \frac{dt}{2\pi} \langle 0 | \rho_q^{\dagger}(\hat{x}) \exp(-iHt) \rho_q(\hat{x}) \exp \left[ i(\omega - \frac{q^2}{2m_A} + H)t \right] |0\rangle$$ \hspace{1cm} (2.6)

Consider first the time-translated space coordinate $\hat{x}(t)$ i.e.

$$\hat{x}(t) = e^{-iHt} \hat{x} e^{iHt} = \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^n\hat{x}(t)}{dt^n} \bigg|_{t=0}$$ \hspace{1cm} (2.7)

where

$$\frac{d^n\hat{x}(t)}{dt^n} \bigg|_{t=0} = (-i)^n [H, [H, [H, ..., [H, \hat{x}]]]] \hspace{1cm} (2.8)$$

Using elementary commutation rules and the Hamiltonian, Eq. (2.1), for the relative motion, one obtains

$$\hat{x}(t) = \hat{x} \cos(2\beta^2 \frac{t}{m}) - \hat{p} \frac{\beta}{\beta^2} \sin(2\beta^2 \frac{t}{m}) \hspace{1cm} (2.9)$$

We now apply the Glauber formula $e^{A+B} = e^A e^{B e^{-\frac{1}{2}[A,B]}}$, which holds if $[A,B]$ is a c-number, to the operator $\exp(iq\hat{x}(t)/2)$

$$\exp \left( i \frac{q}{2} \hat{x}(t) \right) = \exp \left\{ i \frac{q}{2} \left[ \hat{x} \cos(2\beta^2 \frac{t}{m}) - \frac{\hat{p}}{\beta^2} \sin(2\beta^2 \frac{t}{m}) \right] \right\}$$

$$= \exp \left[ i \frac{q}{2} \cos(2\beta^2 \frac{t}{m}) \right] \exp \left[ -i \frac{q}{2\beta^2} \sin(2\beta^2 \frac{t}{m}) \right] \exp \left[ -i \frac{q^2}{8\beta^4} \cos(2\beta^2 \frac{t}{m}) \sin(2\beta^2 \frac{t}{m}) \right] \hspace{1cm} (2.10)$$

Inserting in Eq.(2.6) the unity operator $\int dk/(2\pi) |k\rangle \langle k| = I$, and replacing $m_A$ by $2m$, one obtains

$$S(\omega, q) = \frac{1}{A} \int \frac{dt}{2\pi} \exp \left[ i(\omega - \frac{q^2}{4m})t \right] \int \frac{dk}{2\pi} \langle 0 | \rho_q^{\dagger}(\hat{x}) \exp(-iHt) \rho_q(\hat{x}) \exp(iHt) |k\rangle \langle k|0\rangle =$$
\[
\int \frac{dt}{2\pi} \exp\left[i(\omega - \frac{q^2}{4m})t\right] \exp\left[-i \frac{q^2}{16\beta^2} \sin(4\beta^2 \frac{t}{m})\right] \int \frac{dk}{2\pi} \left[\frac{(e_1^2 + e_2^2)}{2} \langle 0|k + \frac{q}{2} \cos(2\beta^2 \frac{t}{m}) - 1 \rangle \right] + \\
+ e_1 e_2 \langle 0|k + \frac{q}{2} \cos(2\beta^2 \frac{t}{m}) + 1 \rangle \exp\left[-i \frac{qk}{2\beta^2} \sin(2\beta^2 \frac{t}{m})\right] \langle k|0 \rangle \tag{2.11}
\]

After Fourier transforming the ground state wave function one can easily perform the integration over \(k\). Replacing the variables \(t\) and \(\omega\) by \(s\) and \(y\), respectively, one obtains Eqs. (2.5a) and (2.5b).

The approach of this section can be extended to any Hamiltonian for which explicit expressions for (2.7) and the initial state are available. Examples are the harmonic oscillator in any excited initial state and many body-systems interacting through harmonic oscillator forces.

3. \(1/q\) expansion, standard approximations, conclusions.

Let us expand the integrand of the incoherent part, Eq. (2.5a), as a series in \(1/q\). In order to enable the integration term by term, Eq. (1.1) (and therefore Eq. (2.5)) has to be convoluted with some suitable smearing function, for instance the experimental resolution. This procedure results in a finite asymptotic limit for \(q \to \infty\). In contradistinction the coherent part can be shown to vanish exponentially in \(q^2\) in that limit and is totally negligible for finite, large \(q\).

One finds for the coefficient functions in the GRS expansion (1.3)

\[
F_0(y) = \frac{1}{\sqrt{\pi\beta^2}} \exp(-y^2/\beta^2)
\]

\[
(m/q)F_1(y) = -\left(\frac{2\beta}{q}\right) \left(\frac{y}{\beta}\right) \left[1 - \frac{2}{3} \left(\frac{y}{\beta}\right)^2\right] F_0(y)
\]

\[
((m/q)^2)F_2(y) = -\frac{1}{6} \left(\frac{2\beta}{q}\right)^2 \left[1 - 9 \left(\frac{y}{\beta}\right)^2 + 8 \left(\frac{y}{\beta}\right)^4 - \frac{4}{3} \left(\frac{y}{\beta}\right)^6\right] F_0(y)
\]

\[
((m/q)^3)F_3(y) = \frac{1}{2} \left(\frac{2\beta}{q}\right)^3 \left(\frac{y}{\beta}\right) \left[1 - \frac{47}{9} \left(\frac{y}{\beta}\right)^2 + \frac{74}{15} \left(\frac{y}{\beta}\right)^4 - \frac{4}{3} \left(\frac{y}{\beta}\right)^6 + \frac{8}{81} \left(\frac{y}{\beta}\right)^8\right] F_0(y)
\]

We now compare this result from (2.5) with expressions and approximations which have
routinely been used in the past. First comes the GRS expansion \((1.3)^3\). One checks that the first two coefficients above agree with those given before \(^8\). The expression for \(F_3\) is new and higher order coefficients \(F_k\) may be calculated as well. The following comments are in order:

i) \((m/q)^k F_k(y)/F_0(y)\) is a polynomial of order \(3k\) in \(y/\beta^1\),

ii) the relevant expansion parameter is \(2\beta/q\),

iii) For increasing \(q\) at fixed \(y\) the response acquires non-zero contributions from ever increasing values of the energy \(E_n\), with \(n \to \infty\) for \(q \to \infty\). Since for \(n \to \infty\) the harmonic oscillator wave function tends to a plane wave\(^9\), one obtains the standard asymptotic limit \(F_0\).

Next we mention the first cumulant approximation \(^{10}\) and an iteration of \(F_1\) \(^{5,11}\). The former makes sense only for at least three constituents and we therefore discuss only the \(F_1\) iterant. With \(\phi_0(x) = \langle x|0 \rangle\), the ground state wave function

\[
\phi(q,y) = (2\pi)^{-1} \int_{-\infty}^{\infty} dse^{isy} \int dx \phi_0(x-s) \phi_0(x) \exp \left\{ (-im/q) \int_0^s d\sigma [V(x-\sigma) - V(x)] \right\}
\]

\[
= (2\pi)^{-1} \int_{-\infty}^{\infty} dse^{isy} \exp \left[ -\frac{1}{4} \beta^2 s^2 \right] \exp \left[ \frac{i}{6} \beta^4 s^3 \right] \exp \left[ -\frac{1}{4} \beta^6 s^4 \frac{q}{q^2} \right] \exp \left[ -\frac{1}{4} \beta^6 s^4 \frac{q}{q^2} \right]
\]

(3.2)

Note that the coefficient of the term \(1/q\) is equal to the one obtained from Eq. (2.5a), whereas the other ones are quite different.

In conclusion we have applied the Green’s function and an operator technique to obtain the response of systems interacting through harmonic confining forces. For those systems, one can give a closed expression, Eq. (2.5), which can be tested against approximations\(^{5,10,11}\).

The method can also applied to other systems for which the Green’s function in closed form is available, as is the case for a \(\delta\)-shell potential and a particle in an infinitely deep square well\(^4\) which has been previously investigated, using the summation method\(^1\).

Finally let us note that the form (1.1) shows the response to be a sum of \(\delta\)-functions in the energy transfer \(\omega\). Our approach starts with the everywhere regular Green’s function in
the conjugate variable $t$. After performing the manipulations and integrations as prescribed by (1.2), one obtains the reduced response $\phi(q, y)$, Eq.(2.5), which is the Fourier transform of a regular function of $q$ and $t$. On the one hand all of the above equations are only valid in the realm of the distribution function theory. On the other hand the expansion (1.3) for the reduced response with the coefficients (3.1) is a regular function of $q$ and $y$. This apparently occurs when $\omega$ is replaced by $y$: the spacing between the roots of the arguments of the delta functions appearing in Eq.(1.1) vanishes for $q \to \infty$.

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References

1S.A. Gurvitz and A.S. Rinat, Phys. Rev. C47 (1993), to be published


4C. Grosche and F. Steiner, SISSA preprints 1/93/FM; 18/93/FM.


