Classical $v_{gr} \neq c$ solutions of Maxwell equations and the tunneling photon effect

Salvatore Esposito

Dipartimento di Scienze Fisiche, Università di Napoli, Mostra d’Oltremare Pad. 19, 80125 Napoli Italy
and
INFN, Sezione di Napoli, Mostra d’Oltremare Pad. 20, I-80125 Napoli Italy
e-mail: sesposito@na.infn.it

Abstract

We propose a very simple but general method to construct solutions of Maxwell equations propagating with a group velocity $v_{gr} \neq c$. Applications to wave guides and a possible description of the known experimental evidences on photonic tunneling are discussed.
1 Introduction

It is a well known tool that solutions of Maxwell equations in vacuum

\[
\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
\]

are electromagnetic waves:

\[
\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0
\]

\(\phi\) being any component of \(\mathbf{E}, \mathbf{B}\). From eq. (2) follows that the velocity of advancement of the related wave surface (front velocity) is \(c\) \([1]\), the speed of light in vacuum. For a wave packet, it is of particular relevance also the concept of “group velocity”, being the velocity \(v_{gr}\) with which the maximum of the wave packet propagates. To this regard, it is commonly believed that the group velocity of an electromagnetic wave in vacuum takes always the value \(c\). However, there is no proof \([2]\) that electromagnetic waves with \(v_{gr} = c\) are the only solutions of eq. (1) or (2) and, in particular, there are many works \([3, 4]\) dealing with both \(v_{gr} < c\) and \(v_{gr} > c\) solutions.

On the experimental side, many recent experiments on photonic tunneling conducted with different techniques and in different ranges of frequency \([5, 6, 7, 8, 9]\) have shown that, in peculiar conditions, (evanescent) electromagnetic waves travel a barrier with (group) velocity \(v_{gr} > c\). In these experiments, as remembered in \([10]\), only the group velocity can be determined, so that these results obviously do not violate Einstein causality because, according to Sommerfeld and Brillouin \([11]\), it is the front velocity (not group velocity) to be relevant for this and, as stated above, the Maxwell theory predicts that electromagnetic waves in vacuum have always a constant front velocity equal to \(c\).

From the theoretical point of view, the difficulty in the interpretation of the experimental results lies mainly in the fact that in the barrier traversal no group velocity can be defined, the wave number being imaginary (evanescent waves), so that the time required for the traversal (directly measured) is not univocally defineable \([12]\).

In this paper we want to give a simple but general method for building up solutions of Maxwell equations propagating with group velocity \(v_{gr} \neq c\) that generalizes previous calculations \([4]\) and to show how the theory could describe, almost qualitatively, the experimental evidence for \(v_{gr} > c\) electromagnetic propagation. To this purpose, in the following section the general formalism is outlined while in section 3 this formalism is applied to electromagnetic signals in a wave guide. In section 4 the experimental evidence on photonic tunneling is reviewed and discussed, and its possible description in our approach is shown together with future possible tests.

\(^1\) Incidentally, we quote the introduction of the concept of “phase time” which is a generalization of that of group velocity, and it is applicable in this case.
However, it is worth noting that the theory developed in section 2 can be applied as well to propagation of usual electromagnetic waves in a medium (making the substitution $c \rightarrow c/n$, $n$ being the refractive index); in this case our approach can be an alternative one to well known phenomena such as, for example, plasma oscillations, Cherenkov effect and so on.

2 General formalism

In the experiments performed on superluminal barrier traversal, the wavelength $\lambda$ of the incident electromagnetic signal is always greater than or, at least, of the same order of a typical length of the experimental apparatus [5, 6, 7, 8, 9]. So, we have to use an approximation just opposite to the eikonal approximation, and we search for solution of eq. (2) in the form

$$\phi(x, t) = \phi_0(x; k, \omega) e^{i(kx - \omega t)}$$

(3)

with an amplitude $\phi_0$ strongly dependent on 4-position $x, t$. Substituting in eq. (2) we find the equation satisfied by $\phi_0$:

$$\partial^2 \phi_0 + 2i \left( \frac{\omega}{c^2} \frac{\partial \phi_0}{\partial t} + k \cdot \nabla \phi_0 \right) + \left( \frac{\omega^2}{c^2} - k^2 \right) \phi_0 = 0$$

(4)

For illustrative purpose we consider one-dimensional problems; taking $\phi_0$ dependent only on $\zeta = z - vt$ (and $\omega, k$), where $v$ is a given parameter (as well as $\omega, k$) that we will show to be identified with the group velocity, equation (4) now assumes the form

$$\left( 1 - \frac{v^2}{c^2} \right) \frac{\partial^2 \phi_0}{\partial \zeta^2} + 2i \left( k - \frac{v \omega}{c^2} \right) \frac{\partial \phi_0}{\partial \zeta} + \left( \frac{\omega^2}{c^2} - k^2 \right) \phi_0 = 0$$

(5)

Here we observe that, if $\omega = ck$ (and then $v = c$), eq. (5) is automatically satisfied, whatever being the dependence of $\phi_0$ on 4-position; however, for $\phi_0 \zeta$-dependent, these solutions are not the only ones. As we will show now, there are also $v \neq c$ solutions; these can be obtained imposing a “quantification rule” (in the sense that it quantifies the effective dispersion relation) specifying how strong the eikonal approximation is violated.

Let us impose that $\phi_0$ in addition to eq. (5) satisfies also

$$\left( 1 - \frac{v^2}{c^2} \right) \frac{\partial^2 \phi_0}{\partial \zeta^2} = -\Omega^2 \phi_0$$

(6)

with a given constant $\Omega^2 > 0$. In other words, we constrain $\phi_0$ to satisfy a harmonic motion equation whose frequency $\gamma \Omega$ ($\gamma = 1/\sqrt{1 - v^2/c^2}$) quantifies the effective dispersion relation. In fact, substituting eq. (5) in (5) we find that

$$\phi_0 = \phi_0' e^{i \gamma \zeta}$$

(7)

is solution of eq. (5) with

$$\omega^2 = c^2 \left( k^2 + \Omega^2 \right)$$

(8)
and

\[ v = \frac{c^2 k}{\omega} = c \sqrt{1 - \frac{c^2 \Omega^2}{\omega^2}} < c \]  

(9)

For real \( v \) we see that, for fixed \( \omega \), \( \phi \) is a product of waves; in this sense, provided \( \gamma \Omega v \ll \omega \) and noted that

\[ v = \frac{\partial \omega}{\partial k}|_{\omega} \]  

(10)

\( v \) may be interpreted as a group velocity.

Notice that for very small \( \omega \) and finite \( \Omega \) both \( k \) and \( v \) are imaginary, so that in this case, in a common language, we are describing evanescent waves.

Now, let us consider a wave packet with a given spectrum in \( \omega \)

\[ \phi(z, t) = \int d\omega \phi_0'(\omega) e^{\pm i \gamma v(\omega)(z-v(\omega)t)} e^{i(\omega_{(\omega)}z-\omega t)} = \]

(11)

with \( k'(\omega) = k(\omega) \pm \gamma \Omega \), \( \omega'(\omega) = \omega \pm \gamma \Omega v(\omega) \). Suppose that in the spectrum \( \phi_0'(\omega) \) of the wave packet one has a pronounced maximum \( \omega_0 \); expanding the integrand in \( \omega \) around \( \omega_0 \) we get

\[ \phi(z, t) \approx \int d\omega \phi_0'(\omega) e^{i \left\{ \left[ k_0 \pm \gamma \Omega + \frac{\partial k}{\partial \omega}|_{\omega_0} (\omega - \omega_0) \right] z - \left[ \omega_0 \pm \gamma \Omega \frac{\partial k}{\partial \omega}|_{\omega_0} + (\omega - \omega_0) \right] t \} \]

(12)

\( (k(\omega_0) = k_0) \). From this, it is easy to obtain the group velocity of the wave packet:

\[ v_{gr} = \lim_{\omega \to \omega_0} \frac{\partial \omega}{\partial k}|_{\omega_0} = \frac{\partial \omega}{\partial k}|_{\omega_0} = v(\omega_0) \]  

(13)

Let us stress that for \( \Omega \neq 0 \) we have \( v(\omega_0) < c \) (see eq. \( (9) \)); we then obtain solution of Maxwell equations propagating with a group velocity lower than \( c \) depending on “construction” boundary condition (see below) through \( \Omega^2 \). In fact, the space-time evolution of wave amplitude, and then \( \Omega^2 \) in eq. \( (6) \), can be “constructed” experimentally and the dispersion relation \( (8) \) or the group velocity in eq. \( (9) \) can be further measured.

Now, instead of eq. \( (6) \), let us impose the following equation

\[ \left( \frac{v^2}{c^2} - 1 \right) \frac{\partial^2 \phi_0}{\partial \zeta^2} = -\Omega^2 \phi_0 \]  

(14)

again with \( \Omega^2 > 0 \). Substituting in eq. \( (5) \) we now easily find that \( \phi_0 \) is solution of eq. \( (14) \) with a dispersion relation

\[ \omega^2 = c^2 \left( k^2 - \Omega^2 \right) \]  

(15)
propagating with a superluminal group velocity

\[ v = \frac{c^2 k}{\omega} = c \sqrt{1 + \frac{c^2 \Omega^2}{\omega^2}} > c \]  

(16)

Note that even in this case \( \phi_0 \) satisfies a harmonic motion equation (14); now, however, both \( k \) and \( v \) are always real, so that we have no evanescent waves.

Incidentally, let us observe [4] that our subluminal solutions can be equivalently constructed by requiring that \( \phi_0 \) satisfies an Helmoltz wave equation on which solutions a Lorentz boost, for example in the \( z \) direction, is applied. However, we again point out the fact that a quantification rule can be realized only by construction of a given experiment.

Let us now discuss this last point and, in particular, the introduction of the parameter \( \Omega \) in the effective dispersion relations. For simplicity, we confine ourselves only to subluminal group velocities, but the same will remain valid also for the superluminal case.

Our method is based mainly on eq. (6) which admits, for given initial conditions, an univocal solution \( \phi_0 \) in (7) and then \( \phi \) in (3). This wave, product of waves, is what one can measure and is univocally determined by the experimental setup employed. In this sense, the parameter \( \Omega \) is given “by construction” once the experiment is given. For example, in experiments operating in the eikonal approximation regime (this is not the case of photonic tunneling experiments), namely when the wavelength \( \lambda \) of the electromagnetic signal is much lower than a typical length of the experimental apparatus, the parameter \( \Omega \) is zero. On the contrary, in experiments with wave guides one can experimentally determine \( \Omega \) by measuring the (effective) “cutoff” frequency (as we will show in the next section, the introduction of \( \Omega \) leads to a modification of the cutoff frequency of the wave guide).

Finally, we stress that the actual theory can be consistently tested only in problems in more than one dimension. In fact, let us consider, for example, an incident signal on a wave guide: it is well known that the signal effectively propagates (with a finite group velocity) only in the axial direction, while across the guide a non-physical signal propagates with an arbitrarily high group velocity. In the following section we will show how our approach can be applied to realistic problems by giving a specific example.

### 3 An application: wave packets in a wave guide

As an application let us consider the propagation of an electromagnetic wave packet in a hollow wave guide, placed along the \( z \) axis, of arbitrary (but constant) cross-sectional shape with boundary surfaces being perfect conductors (the development of this paragraph is a generalization of chapter 8 of [13]). If \( \omega \) is the frequency of the incident signal, let us write the spatial and temporal dependence of the electric and magnetic field inside the guide as

\[ E(x, y, z, t) = E_0(x, y, \zeta) e^{i(kz - \omega t)} \]  

(17)

\[ B(x, y, z, t) = B_0(x, y, \zeta) e^{i(kz - \omega t)} \]  

(18)
where \( k \) is an as yet unknown wave number and \( \zeta \) is given in the previous section. The fields \( E_0, B_0 \) satisfy the wave equation

\[
\left( \nabla_\perp^2 + \left( \frac{\omega^2}{c^2} - k^2 \right) + 2i \left( k - \frac{v\omega}{c^2} \right) \frac{\partial}{\partial \zeta} + \left( 1 - \frac{v^2}{c^2} \right) \frac{\partial^2}{\partial \zeta^2} \right) \begin{pmatrix} E_0 \\ B_0 \end{pmatrix} = 0 \tag{19}
\]

where \( \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).

Now we impose the \( \zeta \)-dependence of the fields to be given by one of the quantification rules eq. (6) or (14); then the \( x, y \) dependence is determined by the equation

\[
\left( \nabla_\perp^2 + \frac{\omega^2}{c^2} - k^2 \mp \Omega^2 \right) \begin{pmatrix} E_0 \\ B_0 \end{pmatrix} = 0 \tag{20}
\]

where the upper sign refers to propagation with \( v < c \) and the other one to \( v > c \).

From this follows that, considering for example subluminal group velocities, in terms of the transverse and parallel (respect to the \( z \) guide axis) components, Maxwell equations become

\[
(k + \gamma \Omega) E_\perp + \frac{1}{c} (\omega + \gamma \Omega v) \mathbf{e}_3 \times B_\perp = -i \nabla_\perp E_z \tag{21}
\]

\[
(k + \gamma \Omega) B_\perp - \frac{1}{c} (\omega + \gamma \Omega v) \mathbf{e}_3 \times E_\perp = -i \nabla_\perp B_z \tag{22}
\]

\[
\mathbf{e}_3 \cdot (\nabla_\perp \times E_\perp) = \frac{i}{c} (\omega + \gamma \Omega v) B_z \tag{23}
\]

\[
\mathbf{e}_3 \cdot (\nabla_\perp \times B_\perp) = -\frac{i}{c} (\omega + \gamma \Omega v) E_z \tag{24}
\]

\[
\nabla_\perp \cdot E_\perp = -i (k + \gamma \Omega) E_z \tag{25}
\]

\[
\nabla_\perp \cdot B_\perp = -i (k + \gamma \Omega) B_z \tag{26}
\]

\( \mathbf{e}_3 \) being a unit vector in the \( z \) direction (for \( v > c \) it suffices to replace \( \gamma = \frac{1 - v^2/c^2}{2} \) with \( \tilde{\gamma} = \frac{(v^2/c^2 - 1)^{-1/2}}{2} \)). We deduce that if \( E_z, B_z \) are known, then from the first two equations (21),(22) the transverse components of \( \mathbf{E}, \mathbf{B} \) are determined, once the \( \zeta \)-dependence is known from the quantification rules; this leads, with respect to the normal case in which \( v = c \), to an “effective” propagation frequency (and wave number) inside the guide given by \( \omega' = \omega + \gamma \Omega v \) (or \( \omega' = \omega + \tilde{\gamma} \Omega v \) for \( v > c \)), quantified just by the quantification rules.

In fact, for TM waves, specified by the conditions

\[
B_z = 0 \quad \text{everywhere} \tag{27}
\]

\[
E_z = 0 \quad \text{on the guide surface}
\]

\[\text{In general what follows remains valid if the substitution } \Omega \to -\Omega \text{ is performed. For clearness of notation we restrict only to one sign.}\]
we have
\[ E_\perp = i \frac{k + \gamma \Omega}{\Gamma^2} \nabla_\perp E_z \] (28)

while for TE waves, specified by the conditions
\[ E_z = 0 \quad \text{everywhere} \] (29)
\[ \frac{\partial B_z}{\partial n} = 0 \quad \text{on the guide surface} \]

the transverse components of the magnetic field is
\[ B_\perp = i \frac{k + \gamma \Omega}{\Gamma^2} \nabla_\perp B_z \] (30)

In every case
\[ B_\perp = Y e_3 \times E_\perp \] (31)

with
\[ Y = \frac{\omega + \gamma \Omega v}{c(k + \gamma \Omega)} \quad \text{TM waves} \] (32)
\[ Y = \frac{c(k + \gamma \Omega)}{\omega + \gamma \Omega v} \quad \text{TE waves} \] (33)

In eqs. (28), (30)
\[ \Gamma^2 = \frac{\omega^2}{c^2} - k^2 \mp \Omega^2 \] (33)

are the eigenvalues of the wave equation (20) with the boundary condition given respectively by eq. (27),(29). For example, for a rectangular wave guide with cross-section \( a \cdot b \), the eigenvalues for TE waves are \( \Gamma^2 \):
\[ \Gamma^2_{mn} = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \] (34)

with \( m, n \) positive integers.

The equation (33) is the dispersion relation for \( k \): for an incident wave with a given \( \omega \), the propagating modes are those with the wave number
\[ c^2 k^2 = \omega^2 - c^2 (\Gamma^2 \pm \Omega^2) \] (35)

First, let us consider the case in which \( v < c \); the propagation inside the guide effectively happens only for real values of \( k \), i.e. for frequencies \( \omega \) greater than the “cutoff” frequency
\[ \omega_c = c \sqrt{\Gamma^2 + \Omega^2} \] (36)
while for $\omega < \omega_c$ we are in the evanescent waves regime. Note that the effective cutoff frequency $\omega_c$ is greater than the usual value $\omega_{0c} = c \Gamma (\Omega = 0)$ and furthermore, for evanescent modes, the $z$-dependence does not show a pure exponential decay but is of the type $e^{i\Omega z} e^{-|k|z}$ (obviously we take $\Omega$ always real).

Now, let us consider the most interesting case for which $v > c$; we can easily prove that no cutoff arise. In fact, calculating the group velocity from eq. (35)
\[ v = c \sqrt{1 - \frac{\omega_{0c}^2}{\omega^2} + \frac{c^2 \Omega^2}{\omega^2}} \] (37)
we find that it is effectively $v > c$ only if the condition
\[ c^2 \Omega^2 > \omega_{0c}^2 \] (38)
is fulfilled. This condition makes that in eq. (35) the wave number is always real. Note that if $c^2 \Omega^2 < \omega_{0c}^2$ we again have propagation with $v < c$, with the same features as discussed above, but now the effective cutoff frequency $\omega_c = c \sqrt{\Gamma^2 - \Omega^2}$ is lower than the usual value $\omega_{0c}$.

The obtained results bring to a very important consequence: a group velocity (and then a traversal time) can always be coherently defined for $v > c$ propagation. Notice that the fact that $\Omega$ is a property of the experimental setup (see at the end of the previous section) guarantees the univocity of the identification $v_{gr} = v(\omega_0)$ ($v$ depending on $\Omega$) and so ambiguities in the definition of the traversal time are really not present.

4 Discussion on photonic tunneling

Recently Martin and Landauer [14] have shown that the propagation of electromagnetic evanescent waves in a wave guide can be viewed as a photonic tunneling process through a barrier; this facilitates the experimental study of tunneling phenomena, due to the charge neutrality of photons with respect to other particles such as electrons. From the theoretical point of view, an open question is that of the barrier traversal time, because in the barrier region the momentum of the tunneling particle is imaginary so that no velocity can be defined. There are several approaches to the problem [12] leading to different definitions of the traversal time. Instead, experimentally this time can be univocally measured, for example, from the coincidence of two photons, one travelling through the barrier and the other travelling in vacuum.

Enders and Nimtz [5, 6] have studied photonic tunneling by means of microwave transmission through undersized wave guides operating below their cutoff (of the order of 6 ÷ 9 GHz) and have obtained traversal times (for opaque barriers) from pulsed measurements in the time domain or, indirectly, in the frequency domain. Alternatively, Steinberg, Kwiat and Chiao [7] and later Spielmann, Szipőcs and Krausz [8] employed some 1D photonic band-gap material as barrier for measuring tunneling times, in the UV and optical region respectively, with the aid of a two-photon interferometer.
The experimental evidence can be summarized as follows. Both in microwave and in photonic band-gap experiments photonic tunneling is observed; if the barrier medium is non-dissipative, the traversal time in this opaque region is nearly independent of the barrier thickness (Hartman effect [15]), so that for particular values of this length superluminal group velocities have been inferred. Furthermore, two strange properties have been detected. First, in microwave experiments, dissipative tunneling studies have shown that the Hartman effect disappears with increasing dissipation [6]. Second, the measurements of the tunneling of optical pulses through photonic band-gaps reveal that the pulses transmitted through a particular sample are significantly shorter than the incident ones [8]; this effect disappears for increasing transmission coefficients. The latter two effects seem to indicate, in our opinion, that real propagation in the opaque region happens, and this stimulates to apply our theory to the present case. For example, we stress that a usual mass term (for \( v < c \)) or a tachionic mass [16] for the photon (for \( v > c \)) cannot take into account the observed superluminal tunneling, because of the dependence on the barrier thickness of the measured group velocity (i.e. the dependence on a characteristic parameter and not on an intrinsic one such as usual or tachionic mass). Instead, our quantification frequency \( \Omega \) is not an intrinsic property, but would just depend on the employed experimental setup, so that the dependence on it of the group velocity (see for example eq. (37)) seems to go in the right direction for taking into account the superluminal tunneling. Moreover, the simple fact that in our approach a group velocity can always be defined for superluminal propagation eliminates the ambiguities in the definition of the traversal time.

On the other hand, our theory can be tested independently from photonic tunneling, for example constructing “ad hoc” an electromagnetic apparatus which realizes eq. (6) or (14) in some regions and then measuring the dispersion relation between \( \omega \) and \( k \) or the group velocity of the propagating waves.

5 Conclusions

We have studied the propagation with group velocity \( v_{gr} \neq c \) of the solutions of Maxwell equations (in vacuum) and have shown that it is possible (and not violating Einstein causality) provided a peculiar space-temporal dependence of the wave amplitude is given through a definite quantification rule.

As an application of the presented formalism we have considered the propagation in a wave guide and obtained, for \( v_{gr} < c \), a different effective cutoff frequency respect to the normal case, while for \( v_{gr} > c \) no effective cutoff arises so that the wave number is always real and a group velocity can always be defined. In the evanescent regime for the \( v_{gr} < c \) case, moreover, the real propagating waves show not a pure exponential decay but are only damped waves.

Discussing photonic tunneling, we have pointed out how the presented approach can qualitatively describe the experimental evidences on this effect, even if definitive conclusions are not yet reached and further experimental and theoretical investigations are
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