Topological order in Josephson junction ladders with Mobius boundary conditions

Gerardo Cristofano\textsuperscript{1}, Vincenzo Marotta\textsuperscript{1}, Adele Naddeo\textsuperscript{2}

Abstract

We propose a CFT description for a closed one-dimensional fully frustrated ladder of quantum Josephson junctions with Mobius boundary conditions \cite{1}, in particular we show how such a system can develop topological order. Such a property is crucial for its implementation as a "protected" solid state qubit.

Keywords: Josephson Junction Ladder, Topological order, qubit
PACS: 11.25.Hf, 74.50.+r, 03.75.Lm
Work supported in part by the European Communities Human Potential Program under contract HPRN-CT-2000-00131 Quantum Spacetime

\textsuperscript{1}Dipartimento di Scienze Fisiche, Universit\`a di Napoli “Federico II” and INFN, Sezione di Napoli–Via Cintia - Compl. universitario M. Sant’Angelo - 80126 Napoli, Italy
\textsuperscript{2}Dipartimento di Scienze Fisiche, Universit\`a di Napoli “Federico II” and INFM, Unit\`a di Napoli–Via Cintia - Compl. universitario M. Sant’Angelo - 80126 Napoli, Italy
1 Introduction

The concept of topological order was first introduced to describe the ground state of a quantum Hall fluid [2]. Although today's interest in topological order mainly derives from the quest for exotic non-Fermi liquid states relevant for high \( T_c \) superconductors [3], such a concept is of much more general interest [4].

Two features of topological order are very striking: fractionally charged quasiparticles and a ground state degeneracy depending on the topology of the underlying manifold, which is lifted by quasiparticles tunnelling processes. For Laughlin fractional quantum Hall (FQH) states both these properties are well understood [5], but for superconducting devices the situation is less clear.

Josephson junctions networks appear to be good candidates for exhibiting topological order, as recently evidenced in Refs. [6] [7] by means of Chern-Simons gauge field theory. Such a property may allow for their use as “protected” qubits for quantum computation. In this paper we shall show that fully frustrated Josephson junction ladders (J JL) with non trivial geometry may support topological order, making use of conformal field theory techniques [1]. A simple experimental test of our predictions will be also proposed.

The paper is organized as follows.

In Section 2 we introduce the fully frustrated quantum Josephson junctions ladder (J JL) focusing on non trivial boundary conditions.

In Section 3 we recall some aspects of the \( m \)-reduction procedure [8], in particular we show how the \( m = 2, p = 0 \) case of our twisted model (TM) [9] well accounts for the symmetries of the model under study. In such a framework we give the whole primary fields content of the theory on the plane and exhibit the ground state wave function.

In Section 4, starting from our CFT results, we show that the ground state is degenerate, the different states being accessible by adiabatic flux change techniques. Such a degeneracy is shown to be strictly related to the presence in the spectrum of quasiparticles with non abelian statistics and can be lifted non perturbatively through vortices tunnelling.

In Section 5 some comments and outlooks are given.

In the Appendix we recall briefly the boundary states introduced in Ref. [10] in the framework of our TM.

2 Josephson junctions ladders with Mobius boundary conditions

In this Section we briefly describe the system we will study in the following, that is a closed ladder of Josephson junctions (see Fig.1) with Mobius boundary conditions. With each site \( i \) we associate a phase \( \varphi_i \) and a charge \( 2e n_i \), representing a superconducting grain coupled to its neighbours by Josephson couplings; \( n_i \) and \( \varphi_i \) are conjugate variables satisfying the usual phase-number commutation relation. The system is described by the quantum phase model
(QPM) Hamiltonian:

\[
H = -\frac{E_C}{2} \sum_i \left( \frac{\partial}{\partial \varphi_i} \right)^2 - \sum_{\langle ij \rangle} E_{ij} \cos \left( \varphi_i - \varphi_j - A_{ij} \right),
\]

where \( E_C = \frac{(2e)^2}{C} \) (\( C \) being the capacitance) is the charging energy at site \( i \), while the second term is the Josephson coupling energy between sites \( i \) and \( j \) and the sum is over nearest neighbours. \( A_{ij} = \frac{\Phi}{\Phi_0} \int_i^j A \cdot dl \) is the line integral of the vector potential associated to an external magnetic field \( B \) and \( \Phi_0 = \frac{hc}{2e} \) is the superconducting flux quantum. The gauge invariant sum around a plaquette \( \sum_p A_{ij} = 2\pi f \) with \( f = \frac{\Phi}{\Phi_0} \), where \( \Phi \) is the flux threading each plaquette of the ladder.

Let us label the phase fields on the two legs with \( \varphi^{(a)}_i \), \( a = 1, 2 \) and assume \( E_{ij} = E_x \) for horizontal links and \( E_{ij} = E_y \) for vertical ones. Let us also make the gauge choice \( A_{ij} = +\pi f \) for the upper links, \( A_{ij} = -\pi f \) for the lower ones and \( A_{ij} = 0 \) for the vertical ones, which corresponds to a vector potential parallel to the ladder and taking opposite values on upper and lower branches.

Thus the effective quantum Hamiltonian \( \mathcal{H} \) can be written as \( \mathcal{H} \):

\[
-H = \frac{E_C}{2} \sum_i \left[ \left( \frac{\partial}{\partial \varphi_i^{(1)}} \right)^2 + \left( \frac{\partial}{\partial \varphi_i^{(2)}} \right)^2 \right] + \\
\sum_i \left[ E_x \sum_{a=1,2} \cos \left( \varphi_{i+1}^{(a)} - \varphi_i^{(a)} + (-1)^a \pi f \right) + E_y \cos \left( \varphi_i^{(1)} - \varphi_i^{(2)} \right) \right].
\]

The correspondence between the effective quantum Hamiltonian \( \mathcal{H} \) and our TM model can be best traced performing the change of variables \( \varphi_i^{(1)} = \)

Figure 1: Josephson junction ladder with Mobius boundary conditions
\( X_i + \phi_i, \varphi_i^{(2)} = X_i - \phi_i \), so getting:

\[
H = -\frac{E_C}{2} \sum_i \left[ \left( \frac{\partial}{\partial X_i} \right)^2 + \left( \frac{\partial}{\partial \varphi_i} \right)^2 \right] - \sum_i [2E_x \cos (X_{i+1} - X_i) \cos (\phi_{i+1} - \phi_i - \pi f) + E_y \cos (2\phi_i)]
\]

(3)

where \( X_i, \phi_i \) (i.e. \( \varphi_i^{(1)}, \varphi_i^{(2)} \)) are only phase deviations of each order parameter from the commensurate phase and should not be identified with the phases of the superconducting grains [11].

When \( f = \frac{1}{2} \) and \( E_C = 0 \) (classical limit) the ground state of the 1D frustrated quantum XY (FQXY) model displays - in addition to the continuous \( U(1) \) symmetry of the phase variables - a discrete \( Z_2 \) symmetry associated with an antiferromagnetic pattern of plaquette chiralities \( \chi_{p} = \pm 1 \), measuring the two opposite directions of the supercurrent circulating in each plaquette. The evidence for a chiral phase in Josephson junction ladders has been investigated in Ref. [12] while a field theoretical description of chiral order is developed in [13].

Performing the continuum limit of the Hamiltonian (3):

\[
- H = \frac{E_C}{2} \int dx \left[ \left( \frac{\partial}{\partial X} \right)^2 + \left( \frac{\partial}{\partial \phi} \right)^2 \right] + \int dx \left[ E_x \left( \frac{\partial X}{\partial x} \right)^2 + E_x \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{E_y}{2} \cos (2\phi) \right]
\]

(4)

we see that the \( X \) and \( \phi \) fields are decoupled. In fact the \( X \) term of the above Hamiltonian is that of a free quantum field theory while the \( \phi \) one coincides with the quantum sine-Gordon model. Through an imaginary-time path-integral formulation of such a model it can be shown that the 1D quantum problem maps into a 2D classical statistical mechanics system, the 2D fully frustrated XY model, where the parameter \( \alpha = \left( \frac{E_x}{E_C} \right)^{\frac{1}{4}} \) plays the role of an inverse temperature [11]. For small \( E_C \) there is a gap for creation of kinks in the antiferromagnetic pattern of \( \chi_p \) and the ground state has quasi long range chiral order. We work in the regime \( E_x \gg E_y \) where the ladder is well described by a CFT with central charge \( c = 2 \).

We are now ready to introduce the modified ladder [1], see Fig. 1. In order to do so let us first require the \( \varphi^{(a)}, a = 1, 2 \), variables to recover the angular nature by compactification of both the up and down fields. In such a way the XY-vortices, causing the Kosterlitz-Thouless transition, are recovered. As a second step let us introduce at a point \( x = 0 \) a defect which couples the up and down edges through its interaction with the two legs, that is let us close the ladder and impose Mobius boundary conditions. In the limit of strong coupling such an interaction gives rise to non trivial boundary conditions for the fields [10]. In the following we give further details on such an issue, in particular we

4
adopt the \( m \)-reduction technique \cite{8} \cite{9}, which accounts for non-trivial boundary conditions \cite{10} for the Josephson ladder in the presence of a defect line. In the Appendix the relevant chiral fields \( \varphi_e^{(a)} \), \( a = 1, 2 \), which emerge from such conditions, are explicitly constructed, by using the folding procedure.

3 \( m \)-reduction technique

In this Section we focus on the \( m \)-reduction technique for the special \( m = 2 \) case and apply it to the system described by the Hamiltonian \cite{11}. In the Appendix each phase field \( \varphi^{(a)} \) is written as a sum of two fields of opposite chirality defined on an half-line, because of the presence of a defect at \( x = 0 \). Within a “bosonization” framework it is shown there how it is possible to reduce to a problem with two chiral fields \( \varphi_e^{(a)} \), \( a = 1, 2 \), each defined on the whole \( x \)-axis, and the corresponding dual fields. Now we identify in the continuum such chiral phase fields \( \varphi_e^{(a)} \), \( a = 1, 2 \), each defined on the corresponding leg, with the two chiral fields \( Q^{(a)} \), \( a = 1, 2 \) of our CFT, the TM, with central charge \( c = 2 \).

In order to construct such fields we start from a CFT with \( c = 1 \) described in terms of a scalar chiral field \( Q \) compactified on a circle with radius \( R^2 = 2 \), explicitly given by:

\[
Q(z) = q - i p \ln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \tag{5}
\]

with \( a_n \), \( q \) and \( p \) satisfying the commutation relations \([a_n, a_{n'}] = n \delta_{n,n'} \) and \([q, p] = i\); its primary fields are the vertex operators \( U^\alpha(z) = e^{i \alpha Q(z)} \). It is possible to give a plasma description through the relation

\[
|\psi|^2 = e^{-\beta H_{\text{eff}}} \quad \text{where} \quad \psi(z_1, ..., z_N) = \langle N| \prod_{i=1}^{N} U^\alpha(z_i)|0\rangle = \prod_{i<j}^{N} (z_i - z_j)^2 \quad \text{is the ground state wave function. It can be immediately seen that} \quad H_{\text{eff}} = -\sum_{i<j}^{N} \ln |z_i - z_j| \quad \text{and} \quad \beta = \frac{2}{R^2} = 1, \quad \text{that is only vorticity} \quad v = 1 \quad \text{vortices are present in the plasma.}
\]

Starting from such a CFT mother theory one can use the \( m \)-reduction procedure, which consists in considering the subalgebra generated only by the modes in eq. \cite{9} which are a multiple of an integer \( m \), so getting a \( c = m \) orbifold CFT (daughter theory, i.e. the twisted model (TM)) \cite{9}. With respect to the special \( m = 2 \) case, the fields in the mother CFT can be organized into components which have well defined transformation properties under the discrete \( Z_2 \) (twist) group, which is a symmetry of the TM. By using the mapping \( z \rightarrow z^{1/2} \) and by making the identifications \( a_{2n+1} \rightarrow \sqrt{2} a_{n+1/2} \), \( q \rightarrow \frac{1}{\sqrt{2}} q \) the \( c = 2 \) daughter CFT is obtained. It is interesting to notice that such a daughter CFT gives rise to a vortices plasma of half integer vorticity, that is to a fully frustrated XY model, as it will appear in the following.

Its primary fields content can be expressed in terms of a \( Z_2 \)-invariant scalar field \( X(z) \), given by

\[
X(z) = \frac{1}{2} (Q^{(1)}(z) + Q^{(2)}(z)), \tag{6}
\]
describing the continuous phase sector of the new theory, and a twisted field
\[ \phi(z) = \frac{1}{2} \left( Q^{(1)}(z) - Q^{(2)}(z) \right), \]
which satisfies the twisted boundary conditions \( \phi(e^{i\pi} z) = -\phi(z) \). Such fields coincide with the ones introduced in eq. (4).

The whole TM theory decomposes into a tensor product of two CFTs, a twisted invariant one with \( c = \frac{3}{2} \) and the remaining \( c = \frac{1}{2} \) one realized by a Majorana fermion in the twisted sector. In the \( c = \frac{3}{2} \) subtheory the primary fields are composite vertex operators \( V(z) = U_X(z) \psi(z) \) or \( V_{\text{qh}}(z) = U_X(z) \sigma(z) \), where \( U_X(z) = \frac{1}{\sqrt{z}} : e^{i\alpha \phi(z)} : \) is the vertex of the charged sector with \( \alpha^2 = 2 \) for the \( SU(2) \) Cooper pairing symmetry used here.

Regarding the other component, the highest weight state in the neutral sector can be classified by the two chiral operators:
\[ \psi(z) = \frac{1}{2\sqrt{z}} \left( : e^{i\alpha \phi(z)} : + : e^{i\alpha \phi(-z)} : \right), \]
\[ \bar{\psi}(z) = \frac{1}{2\sqrt{z}} \left( : e^{i\alpha \phi(z)} : - : e^{i\alpha \phi(-z)} : \right); \]
which correspond to two \( c = \frac{1}{2} \) Majorana fermions with Ramond (invariant under the \( Z_2 \) twist) or Neveu-Schwartz (\( Z_2 \) twisted) boundary conditions in a fermionized version of the theory. Let us point out that the energy-momentum tensor of the Ramond part of the neutral sector develops a cosine term:
\[ T_{\psi}(z) = -\frac{1}{4} (\partial \phi)^2 - \frac{1}{16z^2} \cos \left( \frac{\sqrt{2}}{2} \phi \right), \]
a clear signature of a tunneling phenomenon which selects a new stable vacuum, the linear superposition of the two ground states. The Ramond fields are the degrees of freedom which survive after the tunneling and the \( Z_2 \) (orbifold) symmetry, which exchanges the two Ising fermions, is broken.

So the whole energy-momentum tensor within the \( c = \frac{3}{2} \) subtheory is:
\[ T = T_X(z) + T_{\psi}(z) = -\frac{1}{2} (\partial X)^2 - \frac{1}{4} (\partial \phi)^2 - \frac{1}{16z^2} \cos \left( \frac{\sqrt{2}}{2} \phi \right). \]
The correspondence with the Hamiltonian in eq. (4) is more evident once we observe that the neutral current \( \partial \phi \) appearing above coincides with the term \( \partial \phi - \frac{\pi}{2} \) of eq. (4), since the \( \frac{\pi}{2} \)-term coming there from the frustration condition, here it appears in \( \partial \phi \) as a zero mode, i.e. a classical mode. Besides the fields appearing in eq. (4) there are the \( \sigma(z) \) fields, also called the twist fields, which appear in the primary fields \( V_{\text{qh}}(z) \) combined to a vertex with charge \( \frac{1}{2} \). The twist fields have non local properties and decide also for the non trivial properties of the vacuum state, which in fact can be twisted or not in our formalism. Such a property for the vacuum is more evident for the torus topology, where the \( \sigma \)-field is described by the conformal block \( \chi_{\frac{1}{16}} \) (see Section 4).
The evidence of a phase transition in ladder systems at $c = \frac{3}{2}$ has been investigated in [14] within a CFT framework. Within this framework the ground state wavefunction is described as a correlator of $N_2 e$ Cooper pairs:

$$< N_2 e | \prod_{i=1}^{N_2} V^{\frac{1}{2}}(z_i) | 0 > = \prod_{i<j=1}^{N_2} (z_i - z_j) Pf \left( \frac{1}{z_i - z_j} \right)$$

(11)

where $Pf \left( \frac{1}{z_i - z_j} \right) = A \left( \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \ldots \right)$ is the antisymmetrized product over pairs of Cooper pairs, so reproducing well known results [15]. In a similar way we also are able to evaluate correlators of $N_e$ Cooper pairs in the presence of (quasi-hole) excitation [15][9] with non Abelian statistics [16]. It is now interesting to notice that the charged contribution appearing in the correlator of $N_e$ electrons is just:

$$< N_e | \prod_{i=1}^{N_e} U_X^{1/2}(z_i) | 0 > = \prod_{i<j=1}^{N_e} (z_i - z_j)^{1/4}$$

giving rise to a vortices plasma with $H_{eff} = -\frac{1}{4} \sum_{i<j-1}^{N} \ln |z_i - z_j|$ at the corresponding "temperature" $\beta = 2$, that is it describes vortices with vorticity $v = \frac{1}{2}$!

On closed annulus geometries, as it is the discretized analogue of a torus, we must properly account for boundary conditions at the ends of the finite lattice since they determine in the continuum the pertinent conformal blocks yielding the statistics of quasiparticles as well as the ground state degeneracy. We have two possible boundary conditions which correspond to two different ways to close the double lattice, i.e. $\phi_a \rightarrow \phi_a$ or $\phi_{N+a} \rightarrow \phi_{a+1}$, $a = 1, 2$. It is not difficult to work out that, for the ladder case, the twisted boundary conditions can be implemented only on the odd lattice.

Indeed for $f = \frac{1}{2}$ the ladder is invariant under the shift of two sites, so that there are two topologically inequivalent boundary conditions for even or odd number of sites. In the even case the end sites are of the same kind of the starting one, while in the odd case a ferromagnetic line corresponds to an antiferromagnetic one. The even (odd) case corresponds to untwisted (twisted) boundary conditions, as depicted in Fig. 2. The odd case selects out two degenerate ground states which are in different topological sectors, so the system may develop topological order in the twisted sector of our theory. Let us notice that in the discrete case not all the vacua in the different sectors of our theory are connected. For instance the states in the untwisted sector, which correspond to the even ladder, are physically disconnected from those of the twisted one, which correspond to the odd ladder.
4 Topological order and “protected” qubits

The aim of this Section is to fully exploit the issue of topological order in a quantum JJL. In order to meet such a request let us use the results of the 2-reduction technique for the torus topology [9]. On the torus the TM primary fields are described in terms of the conformal blocks of the $c = \frac{3}{2}$ subtheory and an Ising model, so reflecting the decomposition on the plane outlined in the previous Section. The following characters, expressed in terms of the torus variable $w = \frac{1}{2\pi i} \ln z$,

$$\chi_{c=3/2}^{(0)}(w|\tau) = \chi_0(\tau)K_0(w|\tau) + \chi_{c=3/2}^{3/2}(w|\tau)K_2(w|\tau)$$ (12)

$$\chi_{c=3/2}^{(1)}(w|\tau) = \chi_{\frac{c}{2}}(\tau) (K_1(w|\tau) + K_3(w|\tau))$$ (13)

$$\chi_{c=3/2}^{(2)}(w|\tau) = \chi_{\frac{c}{2}}(\tau)K_0(w|\tau) + \chi_0(\tau)K_2(w|\tau)$$ (14)

represent the field content of the $Z_2$ invariant $c = \frac{3}{2}$ CFT with a “charged” component ($K_\alpha(w|\tau)$, see definition given below) and a “neutral” component ($\chi_\beta$, the conformal blocks of the Ising Model). In order to understand the physical significance of the $c = 2$ conformal blocks in terms of the charged low energy excitations of the system, let us evidence their electric charge (and magnetic flux contents in the dual theory, obtained by exchanging the compactification radius $R^2_c \rightarrow R^2_m$ in the charged sector of the CFT). Hence let us consider the “charged” sector conformal blocks appearing in eqs. (12-14):

$$K_{2l+i}(w|\tau) = \frac{1}{\eta(\tau)} \begin{bmatrix} 2l+i & 0 \\ 0 & 1 \end{bmatrix} (2w|4\tau), \quad \forall (l,i) \in (0,1)^2,$$ (15)

corresponding to primary fields with conformal dimensions $h_{2l+i} = \frac{1}{2} \alpha_{(l,i)}^2 = \frac{1}{2} (2l+i + 2\delta_{(l,i),0})^2$ and electric charges $2e^{\frac{\alpha_{(l,i)}}{R_X}}$ (magnetic charges in the dual theory $\frac{2e}{2\pi} (\alpha_{(l,i)}R_X)$), $R_X = 1$ being the compactification radius.

Now we turn to the whole $c = 2$ theory. The characters of the twisted sector are given by:

$$\chi_{(0)}(w|\tau) = \bar{\chi}_{(0)} \left( \chi_{c=3/2}^{=3/2}(w|\tau) + \chi_{c=3/2}^{3/2}(w|\tau) \right)$$ (16)

$$\chi_{(1)}(w|\tau) = \left( \bar{\chi}_0 + \bar{\chi}_{\frac{1}{2}} \right) \chi_{c=3/2}^{c=3/2}(w|\tau)$$ (17)

where $\bar{\chi}_\beta$ are the $c = \frac{1}{2}$ Ising characters. Such a factorization is a consequence of the parity selection rule (m-ality), which gives a gluing condition for the “charged” and “neutral” excitations. Furthermore the characters of the un-
twisted sector are [9]:

\[
\begin{align*}
\tilde{\chi}^{+}_{(0)}(w|\tau) &= \tilde{\chi}_{0}\chi^{c=3/2}_{(0)}(w|\tau) + \tilde{\chi}_{\frac{1}{2}}\chi^{c=3/2}_{(2)}(w|\tau) \\
&= \left(\tilde{\chi}_{0}\chi_{0} + \tilde{\chi}_{\frac{1}{2}}\chi_{\frac{1}{2}}\right)K_{0} + \left(\tilde{\chi}_{0}\chi_{0} + \tilde{\chi}_{\frac{1}{2}}\chi_{0}\right)K_{2} \\
\tilde{\chi}^{+}_{(1)}(w|\tau) &= \tilde{\chi}_{0}\chi^{c=3/2}_{(2)}(w|\tau) + \tilde{\chi}_{\frac{1}{2}}\chi^{c=3/2}_{(0)}(w|\tau) \\
&= \left(\tilde{\chi}_{0}\chi_{\frac{1}{2}} + \tilde{\chi}_{\frac{1}{2}}\chi_{0}\right)K_{0} + \left(\tilde{\chi}_{0}\chi_{0} + \tilde{\chi}_{\frac{1}{2}}\chi_{0}\right)K_{2} \\
\tilde{\chi}^{-}_{(0)}(w|\tau) &= \tilde{\chi}_{0}\chi^{c=3/2}_{(0)}(w|\tau) - \tilde{\chi}_{\frac{1}{2}}\chi^{c=3/2}_{(2)}(w|\tau) \\
&= \left(\tilde{\chi}_{0}\chi_{0} - \tilde{\chi}_{\frac{1}{2}}\chi_{\frac{1}{2}}\right)K_{0} + \left(\tilde{\chi}_{0}\chi_{0} - \tilde{\chi}_{\frac{1}{2}}\chi_{0}\right)K_{2} \\
\tilde{\chi}^{-}_{(1)}(w|\tau) &= \tilde{\chi}_{0}\chi^{c=3/2}_{(2)}(w|\tau) - \tilde{\chi}_{\frac{1}{2}}\chi^{c=3/2}_{(0)}(w|\tau) \\
&= \left(\tilde{\chi}_{0}\chi_{\frac{1}{2}} - \tilde{\chi}_{\frac{1}{2}}\chi_{0}\right)K_{0} + \left(\tilde{\chi}_{0}\chi_{0} - \tilde{\chi}_{\frac{1}{2}}\chi_{0}\right)K_{2} \\
\tilde{\chi}_{(0)}(w|\tau) &= \tilde{\chi}_{\frac{1}{2}}\chi^{c=3/2}_{(1)}(w|\tau) = \tilde{\chi}_{\frac{1}{2}}\chi^{c=3/2}_{(2)}(w|\tau) (K_{1} + K_{3}) .
\end{align*}
\]

The conformal blocks given above represent the collective states of highly correlated vortices, which appear to be incompressible.

For closed geometries the JJL with Mobius boundary conditions gives rise to a line defect in the bulk. So it becomes mandatory to use a folding procedure to map the problem with a defect line into a boundary one, where the defect line appears as a boundary state. The TM boundary states have been constructed in Ref. [10] together with the corresponding chiral partition functions and briefly recalled in the Appendix. In particular we get an "untwisted" sector and a "twisted" one, corresponding to periodic and Mobius boundary conditions respectively. That gives rise to an essential difference in the low energy spectrum for the system under study: in the first case only two-spinon excitations are possible (i.e. $SU(2)$ integer spin representations) while, in the last case, the presence of a topological defect provides a clear evidence of single-spinon excitations (i.e. $SU(2)$ half-integer spin representations). All that takes place in close analogy with spin-1/2 closed zigzag ladders, which are expected to map to fully frustrated Josephson ladders in the extremely quantum limit. In this way all previous results obtained by means of exact diagonalization techniques [17] are recovered.

In the following we discuss topological order referring to the characters which in turn are related to the different boundary states present in the system through such chiral partition functions [11]. We can build a topological invariant $\mathcal{P} = \prod \chi_{\gamma}$ where $\gamma$ is a closed contour that goes around the hole. Such a choice allows us to define two degenerate ground states with $\mathcal{P} = 1$ and $\mathcal{P} = -1$ respectively, labelled $|1\rangle$, $|2\rangle$, as in Fig. 3(a): there must be an odd number of plaquettes (odd ladder) to satisfy this condition which is imposed by the request of topological protection. The value $\mathcal{P} = -1$ selects twisted boundary conditions at the ends of the chain while $\mathcal{P} = 1$ selects periodic boundary conditions (see Fig. 3(b)).
Figure 3: degenerate ground states for: (a) odd ladder, $P = \pm 1$; (b) even ladder, $P = 1$.

The case of finite discrete systems has been discussed in detail in Ref. [18], our theory is its counterpart in the continuum.

It is now possible to identify the two degenerate ground states shown in Fig.3(a) with the characters (11), (12) in the twisted sector of our theory (see Appendix) and then to remove such a degeneracy through vortices tunneling: the last operation is also needed in order to prepare the qubit in a definite state.

We recognize such two degenerate ground states as the ones corresponding to the twisted characters (16-17):

$$|0\rangle \rightarrow \chi_{(0)}(w|\tau), |1\rangle \rightarrow \chi_{(1)}(w|\tau).$$

(23)

Upon performing an adiabatic change of local magnetic fields which drags one half vortex across the system, i.e. through the transport of a half flux quantum around the $B$-cycle of the torus, we are able to flip the state of the system, so lifting the degeneracy.

Such two states can be distinguished through the presence or absence respectively of a half flux quantum trapped inside the central hole which, in turn, is related to the presence or absence of the Ising character $\chi^{16}$ in the $c = \frac{3}{2}$ subtheory. The trapped half flux quantum can be experimentally detected, so giving a way to read out the state of the system. Furthermore it should be noticed that the presence of the twist operators $\sigma$, described here by the Ising character $\chi^{16}$, gives rise to non Abelian statistics, which can be evidenced by their fusion rules $\sigma \sigma = 1 + \psi$ [15] [9].

5 Conclusions and outlooks

In this paper we have shown that Josephson junctions ladders with non trivial geometry may develop topological order allowing for the implementation of “protected” qubits, a first step toward the realization of an ideal solid state quantum computer. Josephson junctions ladders with annular geometry have been fabricated within the trilayer $Nb/Al - AlO_x/Nb$ technology and experimentally investigated [19]. So in principle it could be simple to conceive an experimental setup in order to test our predictions.
6 Appendix

In such Appendix we recall briefly the TM boundary states (BS) recently constructed in [10]. For closed geometries, that is for the torus, the JtL with an impurity gives rise to a line defect in the bulk. In a theory with a defect line the interaction with the impurity gives rise to the following non trivial boundary conditions for the fields:

\[ \phi_L^{(a)} (x=0) = \mp \phi_R^{(a)} (x=0) - \phi_0, \quad a = 1, 2. \] (24)

In order to describe it we resort to the folding procedure. Such a procedure is used in the literature to map a problem with a defect line (as a bulk property) into a boundary one, where the defect line appears as a boundary state of a theory which is not anymore chiral and its fields are defined in a reduced region which is one half of the original one. Our approach, the TM, is a chiral description of that, where the chiral \( \phi \) field defined in \((-L/2, L/2)\) describes both the left moving component and the right moving one defined in \((-L/2, 0), (0, L/2)\) respectively, in the folded description [10]. Furthermore to make a connection with the TM we consider more general gluing conditions:

\[ \phi_L (x=0) = \mp \phi_R (x=0) - \phi_0 \]

the \((-+)\) sign staying for the twisted (untwisted) sector. We are then allowed to use the boundary states given in [20] for the \( c = 1 \) orbifold at the Ising\(^2 \) radius. The \( X \) field, which is even under the folding procedure, does not suffer any change in boundary conditions [10]. Let us now write each phase field as the sum \( \phi^{(a)} (x) = \phi_L^{(a)} (x) + \phi_R^{(a)} (x) \) of left and right moving fields defined on the half-line because of the defect located in \( x = 0 \). Then let us define for each leg the two chiral fields \( \phi_e^{(a)} (x) = \phi_L^{(a)} (x) \pm \phi_R^{(a)} (-x) \), each defined on the whole \( x \)-axis [21]. In such a framework the dual fields \( \phi_o^{(a)} (x) \) are fully decoupled because the corresponding boundary interaction term in the Hamiltonian does not involve them [22]; they are involved in the definition of the conjugate momenta \( \Pi^{(a)} = \left( \partial_x \phi_e^{(a)} \right) = \left( \frac{\partial}{\partial \phi_e^{(a)}} \right) \) present in the quantum Hamiltonian. Performing the change of variables \( \phi_e^{(1)} = X + \phi, \ \phi_e^{(2)} = X - \phi \) \( (\phi_o^{(1)} = X + \phi, \ \phi_o^{(2)} = X - \phi \) for the dual ones) we get the quantum Hamiltonian [11] but, now, all the fields are chiral ones.

It is interesting to notice that the condition (24) is naturally satisfied by the twisted field \( \phi (z) \) of our twisted model (TM) (see eq. 4).

The most convenient representation of such BS is the one in which they appear as a product of Ising and \( c = \frac{3}{2} \) BS. These last ones are given in terms of the BS \( |\alpha> \) for the charged boson and the Ising ones \( |f>, |\uparrow>, |\downarrow> \) according to (see ref. [23] for details):

\[ |\chi^{c=3/2}_{(0)}> = |0 > \otimes |\uparrow> + |2 > \otimes |\downarrow> \] (25)

\[ |\chi^{c=3/2}_{(1)}> = |1 > \otimes (|\uparrow> + |3 >) \otimes |f> \] (26)

\[ |\chi^{c=3/2}_{(2)}> = |0 > \otimes |\downarrow> + |2 > \otimes |\uparrow>. \] (27)
Such a factorization naturally arises already for the TM characters \[9\].

The vacuum state for the TM model corresponds to the \(\tilde{\chi}_0\) character which is the product of the vacuum state for the \(c = \frac{3}{2}\) subtheory and that of the Ising one. From eqs. (18,20) we can see that the lowest energy state appears in two characters, so a linear combination of them must be taken in order to define a unique vacuum state. The correct BS in the untwisted sector are:

\[
|\tilde{\chi}_{(0,0),0} > = \frac{1}{\sqrt{2}} \left( |\tilde{\chi}^+_0 > + |\tilde{\chi}^-_0 > \right) = \sqrt{2} (|0 > \otimes |\uparrow \uparrow > + |2 > \otimes |\downarrow \downarrow >)
\]

(28)

\[
|\tilde{\chi}_{(0,0),1} > = \frac{1}{\sqrt{2}} \left( |\tilde{\chi}^+_0 > - |\tilde{\chi}^-_0 > \right) = \sqrt{2} (|0 > \otimes |\downarrow \downarrow > + |2 > \otimes |\uparrow \uparrow >)
\]

(29)

\[
|\tilde{\chi}_{(1,0),0} > = \frac{1}{\sqrt{2}} \left( |\tilde{\chi}^+_1 > + |\tilde{\chi}^-_1 > \right) = \sqrt{2} (|0 > \otimes |\uparrow \uparrow > + |2 > \otimes |\downarrow \downarrow >)
\]

(30)

\[
|\tilde{\chi}_{(1,0),1} > = \frac{1}{\sqrt{2}} \left( |\tilde{\chi}^+_1 > - |\tilde{\chi}^-_1 > \right) = \sqrt{2} (|0 > \otimes |\downarrow \downarrow > + |2 > \otimes |\uparrow \uparrow >)
\]

(31)

\[
|\tilde{\chi}_0 (\varphi_0) > = \frac{1}{2^{1/4}} (|1 > + |3 >) \otimes |D_0 (\varphi_0) >
\]

(32)

where we also added the states \(|\tilde{\chi}_0 (\varphi_0) >\) in which \(|D_0 (\varphi_0) >\) is the continuous orbifold Dirichlet boundary state defined in ref. 20. For the special \(\varphi_0 = \pi/2\) value one obtains:

\[
|\tilde{\chi}_0 > = \frac{1}{2^{1/4}} (|1 > + |3 >) \otimes |f >.
\]

(33)

For the twisted sector we have:

\[
|\chi_{(0)} > = (|0 > + |2 >) \otimes (|\uparrow \bar{f} > + |\downarrow \bar{f} >)
\]

(34)

\[
|\chi_{(1)} > = \frac{1}{2^{1/4}} (|1 > + |3 >) \otimes (|f \uparrow > + |f \downarrow >).
\]

(35)

Now, by using as reference state \(|A >\) the vacuum state given in eq. (28), we compute the chiral partition functions \(Z_{AB}\) where \(|B >\) are all the BS just obtained [10]:

\[
Z_{<\tilde{\chi}_{(0,0),0} || \tilde{\chi}_{(0,0),0} >} = \tilde{\chi}_{(0,0),0}
\]

(36)

\[
Z_{<\tilde{\chi}_{(0,0),0} || \tilde{\chi}_{(1,0),0} >} = \tilde{\chi}_{(1,0),0}
\]

(37)

\[
Z_{<\tilde{\chi}_{(0,0),0} || \tilde{\chi}_{(0,0),1} >} = \tilde{\chi}_{(0,0),1}
\]

(38)

\[
Z_{<\tilde{\chi}_{(0,0),0} || \tilde{\chi}_{(1,0),1} >} = \tilde{\chi}_{(1,0),1}
\]

(39)

\[
Z_{<\tilde{\chi}_{(0,0),0} || \tilde{\chi}_{(0),0} >} = \tilde{\chi}_{(0),0}
\]

(40)

\[
Z_{<\tilde{\chi}_{(0,0),0} || \tilde{\chi}_{(0)}} > = \tilde{\chi}_{(0)}
\]

(41)

\[
Z_{<\tilde{\chi}_{(0,0),0} || \chi_{(0)}} > = \chi_{(0)}
\]

(42)
So we can discuss topological order referring to the characters with the implicit relation to the different boundary states present in the system. Furthermore these BS can be associated to different kinds of linear defects, which are compatible with conformal invariance \[10\].

References


