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Solvability of the Direct and Inverse Problems for the Nonlinear Schrödinger Equation

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Abstract. In this paper we study rigorous spectral theory and solvability for both the direct and inverse problems of the Dirac operator associated with the nonlinear Schrödinger equation. We review known results and techniques, as well as incorporating new ones, in a comprehensive, unified framework. We identify functional spaces in which both direct and inverse problems are well posed, have a unique solution and the corresponding direct and inverse maps are one to one.

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1. Overview

A number of physically and mathematically significant nonlinear evolution equations are associated with a pair of linear problems, a linear eigenvalue problem and an auxiliary problem, such that the given evolution equation results as the compatibility condition between them. The Cauchy problem for the nonlinear system is then solved via the so-called Inverse Scattering Transform (IST) technique.

The solution of the initial value problem of a nonlinear evolution equation by IST proceeds in three steps, as follows:

1. the direct problem – the transformation of the initial data from the original "physical" variables ($q(x, 0)$) to the transformed "scattering" variables ($S(k, 0)$);
2. time dependence – the evolution of the transformed data often according to simple, explicitly solvable evolution equations (i.e., finding $S(k, t)$);
3. the inverse problem – the recovery of the evolved solution in the original variables ($q(x, t)$) from the evolved solution in the transformed variables ($S(k, t)$).

However, even though this approach has been widely and extensively applied to a large number of nonlinear integrable equations (including the nonlinear Schrödinger systems considered in the present work), for many equations a rigorous analysis of both the scattering map $q \to S$ and the inverse map is still not totally developed.

$$\begin{tikzcd} q(x,0) \arrow[r, "\text{Forward Integral Equation}"] & S(k,0) = \{\rho(k,0), \quad \{k_j, C_j(0)\}_{j=1}^{J}\} \arrow[d, "\text{Simple ODE}"] \\ q(x,t) \arrow[r, "\text{Riemann–Hilbert or GLM eqs}"] & S(k,t) = \{\rho(k,t), \quad \{k_j, C_j(t)\}_{j=1}^{J}\} \end{tikzcd}$$

Figure 1. Scheme of the inverse scattering transform.

Therefore it is unclear under what conditions these integrable systems are really "solvable".

A satisfactory treatment of scattering and inverse scattering for a given spectral problem should aim for the following:

(i) to formulate a notion of scattering data S which is meaningful for (essentially) all reasonable potentials q in a given functional class, such as, for instance, $q \in L_1(\mathbb{R})$;
(ii) to show that the map $q \to S$ is injective;
(iii) to show that for (essentially) each set of data satisfying appropriate constraints there is a corresponding q (i.e., that the inverse map is well-defined).

In the setting of the classical Schrödinger operator these issues are discussed, for instance, in (Faddeev, 1963 and 1967; Agranovic and Marchenko, 1963; Levitan and Sargsjan, 1975; Chadan and Sabatier, 1977; Deift and Trubowitz, 1979; Levitan, 1980; Marchenko, 1986; Melin, 1985).

In the present paper we face these issues for the scalar nonlinear Schrödinger equation. The paper reviews most of the known results and techniques, as well as incorporating some new ones, in a comprehensive, unified framework. In a forthcoming publication we plan to address the same problems for the vector/matrix nonlinear Schrödinger equations and also for integrable discretizations of both the scalar and vector systems.

2. Introduction

The scalar nonlinear Schrödinger (NLS) equation

$$iq_t = q_{xx} \pm 2|q|^2 q, \tag{2.1}$$

where $q = q(x,t)$, results from the coupled pair of nonlinear evolution equations

$$iq_t = q_{xx} - 2rq^2, \tag{2.2a}$$
$$-ir_t = r_{xx} - 2qr^2 \tag{2.2b}$$

if we let $r = \mp q^*$, where * denotes complex conjugate.

The system (2.2a)–(2.2b) can be written as the compatibility condition between the following two linear problems (Lax pair):

$$v_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v, \tag{2.3a}$$

sometimes referred to as AKNS (or ZS) spectral problem, and

$$v_t = \begin{pmatrix} 2ik^2 + iqr & -2kq - iq_x \\ -2kr + ir_x & -2ik^2 - iqr \end{pmatrix} v, \tag{2.3b}$$

where v is a 2-component vector, $v(x,t) = (v^{(1)}(x,t), v^{(2)}(x,t))^{\mathrm{T}}$. In (2.3a)–(2.3b), $k \in \mathbb{C}$ is a (spectral) parameter and under the isospectral hypothesis (i.e., assuming k is time independent) the compatibility condition $v_{xt} = v_{tx}$ yields the nonlinear system of Equations (2.2a)–(2.2b).

It is convenient to write (2.3a) in the compact form

$$v_x = (ik\mathbf{J} + \tilde{\mathbf{Q}})v, \tag{2.4}$$

where

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\mathbf{Q}} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \tag{2.5}$$

In the forthcoming sections we shall see how one can construct a theory for both the direct and inverse problems of the operator (2.4). These issues have been addressed in, for example, the important work of (Beals and Coifman, 1984, 1985), who have considered general matrix one-dimensional problems. Their approach to the inverse problem employs singular integral equations that follow from Riemann–Hilbert problems. A somewhat inconvenient byproduct of this formulation is, however, that there is not quite an overlap between the conditions required for the inverse problem and those used in the inverse side, unless the potentials are in the Schwartz class. Here we study Gel'fand–Levitan–Marchenko equations of the inverse problem which allows one to proceed somewhat further.

In this paper we shall show how to any integrable potential of the system (2.4) corresponds data $F(x)$ and an eigenfunction $K(x,\cdot)$ which are in $L_1[x,\infty)$ for all $x \in \mathbb{R}$ (direct problem). Reciprocally, given data $F(x)$ in $L_1[x,\infty)$ for all $x \in \mathbb{R}$, there exists a solution $K(x,\cdot)$ to the equations of the inverse problem which is also in $L_1[x,\infty)$ for all $x \in \mathbb{R}$ (inverse problem); however, to guarantee that the potential recovered by this procedure is also in $L_1[x,\infty)$ for all $x \in \mathbb{R}$ somewhat more stringent conditions on the data are required (see Section 4.4). See, again, Faddeev (1963, 1967), Agranovich and Marchenko (1963), Levitan and Sargsjan (1975), Deift and Trubowitz (1979), Levitan (1980) and Marchenko (1986) regarding this controversial point for the classical Schrödinger operator.

3. Direct Problem

In the sequel we shall address the study of the spectral properies of the operator (2.4) under the assumption $q, r \in L_1(\mathbb{R})$. This is termed the direct problem.

3.1. SUMMARY OF THE RESULTS OF THE DIRECT PROBLEM

(a) If $q, r \in L_1(\mathbb{R})$, one can define eigenfunctions of (2.4) (cf. (3.2), (3.3))

$$\begin{pmatrix} \psi_{11}(x,k) \\ \psi_{21}(x,k) \end{pmatrix}, \begin{pmatrix} N_{11}(x,k) \\ N_{21}(x,k) \end{pmatrix}$$

which are analytic functions of the spectral parameter $k = k_{\mathrm{R}} + ik_{\mathrm{I}}$ for $k_{\mathrm{I}} < 0$ (and continuous for $k_{\mathrm{I}} \leqslant 0$) and

$$\begin{pmatrix} \psi_{12}(x,k) \\ \psi_{22}(x,k) \end{pmatrix}, \begin{pmatrix} N_{12}(x,k) \\ N_{22}(x,k) \end{pmatrix}$$

which are analytic for $k_{\mathrm{I}} > 0$ (and continuous for $k_{\mathrm{I}} \geqslant 0$) (these results are proved, with all details, for instance, in (Ablowitz *et al.*, 2004)). Also, such eigenfunctions are bounded by

$$|N_{11}(x,k)| \leqslant I_0\big(2\sqrt{Q(x)R(x)}\,\big), \qquad |N_{21}(x,k)| \leqslant Q(x)I_0\big(2\sqrt{Q(x)R(x)}\,\big),$$
$$|N_{12}(x,k)| \leqslant R(x)I_0\big(2\sqrt{Q(x)R(x)}\,\big), \qquad |N_{22}(x,k)| \leqslant I_0\big(2\sqrt{Q(x)R(x)}\,\big),$$

where

$$Q(x) = \int_x^\infty |q(y)|\,\mathrm{d}y, \qquad R(x) = \int_x^\infty |r(y)|\,\mathrm{d}y$$

and I_0 is the 0th order modified Bessel function (cf., for instance, Ablowitz and Segur, 1981).

(b) The eigenfunctions $\psi_{lj}(x,k)$ admit triangular representations with corresponding kernels $K_{lj}(x,z)$ (cf. (3.13)). If $q, r \in L_1(\mathbb{R})$ then $K_{lj}(x,z) \in L_\infty(\mathbb{R}_x) \otimes L_1(\mathbb{R}_z)$ (see (3.20) in Prop. 2) and if $q, r \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ then $K_{lj}(x,z) \in L_\infty(\mathbb{R}_x) \otimes L_1(\mathbb{R}_z)$ and also $K_{lj}(x,z) \in L_\infty(\mathbb{R}^2)$, i.e., they are bounded with respect to both variables (see (3.21), Prop. 2).

(c) One can introduce scattering data $a(k)$, $b(k)$, $\rho(k)$ etc. (see (3.35), (3.36), (3.37)) for all potentials in $L_1(\mathbb{R})$. Using the results of (b) (and a generalization of Wiener's theorem, cf. App. A), one can then show that if $q, r \in L_1(\mathbb{R})$, then:

  (c1) $b(k)$, for $k \in \mathbb{R}$, is the Fourier transform of an L_1-function (cf. Prop. 3);
  (c2) $(a(k) - 1)$, for $k \in \mathbb{R}$, is the Fourier transform of an L_1-function (cf. Prop. 4);

(c3) if, in addition, $|a(k)| > 0$ (which always holds for the defocusing NLS, Equation (2.1) with the $-$ sign, i.e. the system (2.2a)–(2.2b) with $r = q^*$ (cf. (3.43))), then $\rho(k)$, $k \in \mathbb{R}$, is the Fourier transform of an L_1-function (cf. Theorem 1);

(c4) for the focusing NLS (Equation (2.1) with the $+$ sign, corresponding to $r = -q^*$), a small norm condition on the potentials is required in order to get the same result as in (c3) (cf. Corollary 1).

3.2. EIGENFUNCTIONS

If the potentials q, r are decaying as $|x| \to \infty$, eigenfunctions of the scattering problem (2.4), i.e., solutions of the differential equations

$$(\partial_x - ikJ_l)\psi_{lj} - (\tilde{\mathbf{Q}}\psi)_{lj} = 0, \quad l, j = 1, 2 \tag{3.1}$$

with $J_1 = -1$, $J_2 = 1$, are defined through the integral equations

$$\psi_{lj}(x, k) = \delta_{lj}\mathrm{e}^{iJ_jkx} - \int_x^\infty \mathrm{e}^{iJ_lk(x-z)}(\tilde{\mathbf{Q}}\psi)_{lj}(z, k)\,\mathrm{d}z, \tag{3.2}$$

$$N_{lj}(x, k) = \delta_{lj} - \int_x^\infty \mathrm{e}^{i(J_l-J_j)k(x-z)}(\tilde{\mathbf{Q}}N)_{lj}(z, k)\,\mathrm{d}z, \tag{3.3}$$

where

$$\psi_{lj}(x, k) = N_{lj}(x, k)\mathrm{e}^{iJ_jkx}. \tag{3.4}$$

Column-wise, the vectors

$$\bar{\psi}(x, k) = \begin{pmatrix} \psi_{11}(x, k) \\ \psi_{21}(x, k) \end{pmatrix}, \qquad \psi(x, k) = \begin{pmatrix} \psi_{12}(x, k) \\ \psi_{22}(x, k) \end{pmatrix}$$

are such that

$$\bar{\psi}(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-ikx}, \qquad \psi(x, k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{ikx}, \quad x \to +\infty$$

and therefore they are a set of linearly independent solutions of the second-order system (2.4). Note that one can also introduce “left” eigenfunctions

$$\phi(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-ikx}, \qquad \bar{\phi}(x, k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{ikx}, \quad x \to -\infty$$

which constitute a second set of linearly independent solutions of the scattering problem and

$$\phi(x, k) = \begin{pmatrix} \phi_{11}(x, k) \\ \phi_{21}(x, k) \end{pmatrix}, \qquad \bar{\phi}(x, k) = \begin{pmatrix} \phi_{12}(x, k) \\ \phi_{22}(x, k) \end{pmatrix}$$

satisfy the integral equations

$$\phi_{lj}(x,k) = \delta_{lj}\mathrm{e}^{iJ_jkx} + \int_{-\infty}^{x} \mathrm{e}^{iJ_lk(x-z)}(\tilde{\mathbf{Q}}\phi)_{lj}(z,k)\,\mathrm{d}z, \tag{3.5}$$

$$M_{lj}(x,k) = \delta_{lj} + \int_{-\infty}^{x} \mathrm{e}^{i(J_l-J_j)k(x-z)}(\tilde{\mathbf{Q}}M)_{lj}(z,k)\,\mathrm{d}z, \tag{3.6}$$

where

$$\phi_{lj}(x,k) = M_{lj}(x,k)\mathrm{e}^{iJ_jkx}. \tag{3.7}$$

The study of the convergence of the Neumann series for the Volterra integral equations (3.3) (cf., for instance, Ablowitz *et al.*, 2004 for detailed calculations) yields that if the potentials $q, r \in L_1(\mathbb{R})$, then $N_{11}(x,k)$, $N_{21}(x,k)$ (and $\psi_{11}(x,k)$, $\psi_{21}(x,k)$) are analytic functions of k on the lower half k-plane and continuous up to the real axis, $N_{12}(x,k)$, $N_{22}(x,k)$ (and $\psi_{12}(x,k)$, $\psi_{22}(x,k)$) are analytic on the upper half k-plane and continuous up to the real axis.

Also, under the same assumption on the potentials, one can show by iteration that the eigenfunctions satisfy the bounds

$$\begin{aligned} &|N_{11}(x,k)| \leqslant I_0\big(2\sqrt{Q(x)R(x)}\,\big),\\ &|N_{21}(x,k)| \leqslant R(x)I_0\big(2\sqrt{Q(x)R(x)}\,\big), \quad k_{\mathrm{I}} \leqslant 0, \end{aligned} \tag{3.8}$$

where I_0 is the 0th order modified Bessel function

$$I_0(2\sqrt{x}) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

and

$$Q(x) = \int_x^{\infty} |q(y)|\,\mathrm{d}y, \qquad R(x) = \int_x^{\infty} |r(y)|\,\mathrm{d}y. \tag{3.9}$$

Indeed, from (3.3) it follows that for any k with $k_{\mathrm{I}} \leqslant 0$

$$|N_{11}(x,k)| \;\leqslant\; 1 + \int_x^{\infty} |q(z)||N_{21}(z,k)|\,\mathrm{d}z, \tag{3.10}$$

$$|N_{21}(x,k)| \;\leqslant\; \int_x^{\infty} |r(z)|\,|N_{11}(z,k)|\,\mathrm{d}z. \tag{3.11}$$

Hence, iteration yields

$$\begin{aligned} |N_{11}(x,k)| \;&\leqslant\; 1 + \int_x^{\infty}\mathrm{d}z\,|q(z)|\int_z^{\infty}\mathrm{d}y\,|r(y)||N_{11}(y,k)|\\ &\leqslant\; 1 + \int_x^{\infty}\mathrm{d}z\,|q(z)|\int_z^{\infty}\mathrm{d}y\,|r(y)| + \\ &\quad + \int_x^{\infty}\mathrm{d}z\,|q(z)|\int_z^{\infty}\mathrm{d}y\,|r(y)|\int_y^{\infty}\mathrm{d}z_1\,|q(z_1)|\int_{z_1}^{\infty}\mathrm{d}y_1\,|r(y_1)| + \cdots \end{aligned}$$

and, taking into account that each integral is a decreasing function of the lower limit of integration, we have

$$\begin{aligned}|N_{11}(x,k)| \leqslant\ & 1+\int_x^\infty \mathrm{d}z\,|q(z)|\int_x^\infty \mathrm{d}y\,|r(y)|+\\ &+\left[\int_x^\infty \mathrm{d}z\,|q(z)|\int_z^\infty \mathrm{d}z_1\,|q(z_1)|\right]\times\\ &\times\left[\int_x^\infty \mathrm{d}y\,|r(y)|\int_z^\infty \mathrm{d}y_1\,|r(y_1)|\right]+\cdots.\end{aligned}$$

One can then show by induction that for any $j\in\mathbb{N}$ and for any $f\in L_1(\mathbb{R})$

$$\begin{aligned}&\frac{1}{j!}\int_x^\infty |f(\xi)|\left[\int_\xi^\infty |f(\xi')|\,\mathrm{d}\xi'\right]^j \mathrm{d}\xi\\ &=-\frac{1}{(j+1)!}\int_x^\infty \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\int_\xi^\infty |f(\xi')|\,\mathrm{d}\xi'\right]^{j+1}\mathrm{d}\xi\\ &=\frac{1}{(j+1)!}\left[\int_x^\infty |f(\xi)|\,\mathrm{d}\xi\right]^{j+1}.\end{aligned}$$

Hence it follows that

$$|N_{11}(x,k)|\leqslant 1+Q(x)R(x)+\frac{(Q(x)R(x))^2}{(2!)^2}+\frac{(Q(x)R(x))^3}{(2!)^3}+\cdots$$

with Q, R defined in (3.9), i.e., one obtains the first of (3.8). Substituting this bound into (3.11) yields the second of (3.8).

One can obtain similar bounds for N_{12}, N_{22}, namely

$$\begin{aligned}&|N_{22}(x,k)|\leqslant I_0\big(2\sqrt{Q(x)R(x)}\big),\\ &|N_{12}(x,k)|\leqslant Q(x)I_0\big(2\sqrt{Q(x)R(x)}\big),\quad k_{\mathrm{I}}\geqslant 0.\end{aligned}\tag{3.12}$$

To study the direct problem in a rigorous way we shall need a new set of functions as follows. We introduce the 2×2 matrix $\mathbf{K}(x,z)=[K_{lj}(x,z)]_{l,j=1,2}$ via the "triangular" representation for the eigenfunctions ψ_{lj}

$$\psi_{lj}(x,k)=\delta_{lj}\mathrm{e}^{iJ_jkx}+\int_x^\infty \mathrm{e}^{iJ_jkz}K_{lj}(x,z)\,\mathrm{d}z,\quad z\geqslant x,\tag{3.13}$$

where $K_{lj}(x,z)$ is identically zero for $z<x$ and column-wise

$$\bar{K}(x,z)=\begin{pmatrix}K_{11}(x,z)\\ K_{21}(x,z)\end{pmatrix},\qquad K(x,z)=\begin{pmatrix}K_{12}(x,z)\\ K_{22}(x,z)\end{pmatrix}.\tag{3.14}$$

The importance of these objects is that all dependence on the spectral parameter has been encoded in terms of an exponential factor. This will be critical to establish integrability of the scattering data.

PROPOSITION 1. *The kernels K_{lj} of the triangular representations* (3.13) *satisfy the following integral equations*

$$K_{l\bar{l}}(x,z) = -\frac{1}{2}\tilde{Q}_{l\bar{l}}\left(\frac{x+z}{2}\right) - \int_x^{\frac{x+z}{2}} \tilde{Q}_{l\bar{l}}(y) K_{\bar{l}\bar{l}}(y, x-y+z)\,\mathrm{d}y, \qquad (3.15)$$

$$K_{ll}(x,z) = -\int_x^{\infty} \tilde{Q}_{l\bar{l}}(y) K_{\bar{l}l}(y, y-x+z)\,\mathrm{d}y, \qquad (3.16)$$

where $l = 1, 2$, $\bar{l} = l + 1 \bmod 2$ *and* $\tilde{Q}_{l\bar{l}}$ *are the off-diagonal elements of the matrix potential* $\tilde{\mathbf{Q}}$ *introduced in* (2.5)*, i.e., say,*

$$K_{11}(x,z) = -\int_x^{\infty} q(y) K_{21}(y, y-x+z)\,\mathrm{d}y, \qquad (3.17)$$

$$K_{21}(x,z) = -\frac{1}{2} r\left(\frac{x+z}{2}\right) - \int_x^{\frac{x+z}{2}} r(y) K_{11}(y, x-y+z)\,\mathrm{d}y. \qquad (3.18)$$

Proof. Comparing the integral equations (3.2) for the eigenfunctions and the triangular representations (3.13)

$$\int_x^{\infty} \mathrm{e}^{iJ_jkz} K_{lj}(x,z)\,\mathrm{d}z = -\int_x^{\infty} \mathrm{e}^{iJ_lk(x-z)} \tilde{Q}_{l\bar{l}}(z) \psi_{\bar{l}j}(z,k)\,\mathrm{d}z$$

and direct substitution of (3.13) in the right-hand side shows that

$$\begin{aligned} &\int_x^{\infty} \mathrm{e}^{iJ_jkz} K_{lj}(x,z)\,\mathrm{d}z \\ &\quad = -\int_x^{\infty} \mathrm{d}z\, \mathrm{e}^{iJ_lk(x-z)} \tilde{Q}_{l\bar{l}}(z) \left[\delta_{\bar{l}j} \mathrm{e}^{iJ_jkz} + \int_z^{\infty} \mathrm{e}^{iJ_jky} K_{\bar{l}j}(z,y)\,\mathrm{d}y\right]. \qquad (3.19) \end{aligned}$$

The right-hand side in (3.19) is the sum of two terms. As to the first term, taking into account that $J_l = -J_{\bar{l}}$, by changing variables one gets

$$I = -\int_x^{\infty} \mathrm{d}z\, \mathrm{e}^{iJ_lk(x-z)+iJ_jkz} \tilde{Q}_{l\bar{l}}(z) \delta_{\bar{l}j} = -\frac{1}{2} \int_x^{\infty} \mathrm{d}z\, \mathrm{e}^{iJ_jkz} \tilde{Q}_{l\bar{l}}\left(\frac{x+z}{2}\right) \delta_{\bar{l}j}.$$

For the second term in the square bracket:

- if $l \neq j$ (hence $J_l = -J_j$ and $j = \bar{l}$) one has, by first changing variables to $z' = z$, $y' = y - x + z$ and then exchanging the integrals

$$\begin{aligned} II &= -\int_x^{\infty} \mathrm{d}z \int_z^{\infty} \mathrm{d}y\, \mathrm{e}^{ikJ_j(y-x+z)} \tilde{Q}_{l\bar{l}}(z) K_{\bar{l}j}(z,y) \\ &= -\int_x^{\infty} \mathrm{d}z' \int_{2z'-x}^{\infty} \mathrm{d}y'\, \mathrm{e}^{iJ_jky'} \tilde{Q}_{l\bar{l}}(z') K_{\bar{l}j}(z', y'+x-z') \\ &= -\int_x^{\infty} \mathrm{d}y' \int_x^{\frac{y'+x}{2}} \mathrm{d}z'\, \mathrm{e}^{iJ_jky'} \tilde{Q}_{l\bar{l}}(z') K_{\bar{l}j}(z', y'+x-z'), \end{aligned}$$

- if $l = j$ (hence $J_l = J_j$ and also $\bar{l} \neq j$), setting $z' = z$ and $y' = x - z + y$

$$
\begin{aligned}
II &= -\int_x^\infty \mathrm{d}z \int_z^\infty \mathrm{d}y\, \mathrm{e}^{ikJ_j(x-z+y)} \tilde{Q}_{l\bar{l}}(z) K_{\bar{l}j}(z, y) \\
&= -\int_x^\infty \mathrm{d}z' \int_x^\infty \mathrm{d}y'\, \mathrm{e}^{iJ_jky'} \tilde{Q}_{l\bar{l}}(z') K_{\bar{l}j}(z', y' - x + z').
\end{aligned}
$$

Therefore, from (3.19) it follows that for $l \neq j$ ($\Rightarrow j = \bar{l}$)

$$
\int_x^\infty \mathrm{d}z\, \mathrm{e}^{iJ_jkz} \left[K_{l\bar{l}}(x, z) + \frac{1}{2} \tilde{Q}_{l\bar{l}}\left(\frac{x+z}{2} \right) + \int_x^{\frac{z+x}{2}} \mathrm{d}y\, \tilde{Q}_{l\bar{l}}(y) K_{\bar{l}\bar{l}}(y, z + x - y) \right] = 0
$$

and for $l = j$ ($\Rightarrow j \neq \bar{l}$)

$$
\int_x^\infty \mathrm{d}z\, \mathrm{e}^{iJ_jkz} \left[K_{ll}(x, z) + \int_x^\infty \mathrm{d}y\, \tilde{Q}_{l\bar{l}}(y) K_{\bar{l}l}(y, z - x + y) \right] = 0. \qquad \square
$$

PROPOSITION 2. *If the potentials $q, r \in L_1(\mathbb{R})$, then the integral equations* (3.15)–(3.16) *have a solution $K_{lj}(x, z)$ which is identically zero for $z < x$ and $K_{lj}(x, z) \in L_\infty(\mathbb{R}_x) \otimes L_1(\mathbb{R}_z)$, i.e., such that*

$$
\sup_{x \in \mathbb{R}} \int_{-\infty}^\infty |K_{lj}(x, z)|\, \mathrm{d}z \equiv \sup_{x \in \mathbb{R}} \int_x^\infty |K_{lj}(x, z)|\, \mathrm{d}z < \infty. \tag{3.20}
$$

If the potentials $q, r \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$, then the above integral equations have a solution $K_{lj}(x, z)$ in both $L_\infty(\mathbb{R}_x) \otimes L_1(\mathbb{R}_z)$ and $L_\infty(\mathbb{R}^2)$, i.e., (3.20) *holds, and also*

$$
\sup_{x, z \in \mathbb{R}} |K_{lj}(x, z)| = \sup_{x \in \mathbb{R}} \sup_{z \geqslant x} |K_{lj}(x, z)| < \infty. \tag{3.21}
$$

Proof. We use Picard's method. Define recursively $K_{ij}^{(n)}(x, z)$ to be identically zero for $z < x$ and for $z \geqslant x$ via

$$
K_{21}^{(1)}(x, z) = -\frac{1}{2} r\left(\frac{x+z}{2} \right), \tag{3.22}
$$

$$
K_{11}^{(1)}(x, z) = -\int_x^\infty q(y) K_{21}^{(1)}(y, y - x + z)\, \mathrm{d}y \tag{3.23}
$$

and

$$
K_{11}^{(n)}(x, z) = -\int_x^\infty q(y) K_{21}^{(n)}(y, y - x + z)\, \mathrm{d}y, \tag{3.24}
$$

$$
K_{21}^{(n+1)}(x, z) = -\frac{1}{2} r\left(\frac{x+z}{2} \right) - \int_x^{\frac{x+z}{2}} r(y) K_{11}^{(n)}(y, x - y + z)\, \mathrm{d}y. \tag{3.25}
$$

Let us introduce

$$
\varphi_{lj}^{(n)}(x) = \int_{-\infty}^\infty |K_{lj}^{(n)}(x, z)|\, \mathrm{d}z \equiv \int_x^\infty |K_{lj}^{(n)}(x, z)|\, \mathrm{d}z. \tag{3.26}
$$

Note that here and in the following, when the limits of integration are omitted, it is intended that the integral runs over the whole $\mathbb{R}$, i.e. from $-\infty$ to $+\infty$. From the integral equation (3.25) we have

$$\begin{aligned}
\varphi_{21}^{(n+1)}(x) &= \int |K_{21}^{(n+1)}(x,z)|\,\mathrm{d}z \\
&\leqslant \int |r(x)|\,\mathrm{d}x + \int \mathrm{d}z \int_x^{\frac{x+z}{2}} \mathrm{d}y\,|r(y)||K_{11}^{(n)}(y, x-y+z)| \\
&= \int |r(x)|\,\mathrm{d}x + \int_x^{\infty} \mathrm{d}y \int_{2y-x}^{\infty} \mathrm{d}z\,|r(y)||K_{11}^{(n)}(y, x-y+z)| \\
&= \int |r(x)|\,\mathrm{d}x + \int_x^{\infty} \mathrm{d}y \int_y^{\infty} \mathrm{d}z' \|r(y)||K_{11}^{(n)}(y, z')| \\
&\leqslant \int |r(x)|\,\mathrm{d}x + \int_x^{\infty} |r(y)|\varphi_{11}^{(n)}(y)\,\mathrm{d}y.
\end{aligned} \tag{3.27}$$

Note that in order to get the equality in the second line we exchanged the order of integration and took into account that for $z < x$ the kernels K_{lj} are identically zero; then we performed the change of variables $z' = x - y + z$.

Similarly, from (3.24) one obtains

$$\begin{aligned}
\varphi_{11}^{(n)}(x) &= \int |K_{11}^{(n)}(x,z)|\,\mathrm{d}z \leqslant \int \mathrm{d}z \int_x^{\infty} |q(y)||K_{21}^{(n)}(y, y-x+z)|\,\mathrm{d}y \\
&= \int_x^{\infty} \mathrm{d}y\,|q(y)| \int \mathrm{d}z\,|K_{21}^{(n)}(y, y-x+z)| \\
&= \int_x^{\infty} |q(y)|\varphi_{21}^{(n)}(y)\,\mathrm{d}y.
\end{aligned} \tag{3.28}$$

In their turn, (3.27) and (3.28) imply

$$\varphi_{21}^{(n+1)}(x) \leqslant \|r\|_1 + \int_x^{\infty} \mathrm{d}y\,|r(y)| \int_y^{\infty} \mathrm{d}z\,|q(z)|\varphi_{21}^{(n)}(z)$$

so that iteration yields

$$\varphi_{21}^{(\infty)}(x) \equiv \lim_{n\to\infty} \varphi_{21}^{(n)}(x) \leqslant \|r\|_1 I_0\bigl(2\sqrt{Q(x)R(x)}\bigr). \tag{3.29}$$

It follows that there exists a function $K_{21}^{(\infty)}(x,z)$ such that the sequence $K_{21}^{(n)}(x,z)$ is convergent towards $K_{21}^{(\infty)}(x,z)$ with respect to the norm (3.20) of $L_\infty(\mathbb{R}_x) \otimes L_1(\mathbb{R}_z)$:

$$\lim_{n\to\infty} \sup_x \int |K_{21}^{(n)}(x,z) - K_{21}^{(\infty)}(x,z)|\,\mathrm{d}z = 0,$$

$$\sup_x \int |K_{21}^{(\infty)}(x,z)|\,\mathrm{d}z \leqslant \|r\|_1 I_0\bigl(2\sqrt{\|r\|_1\|q\|_1}\bigr).$$

Likewise, by substituting into (3.28) one obtains

$$\varphi_{11}^{(\infty)}(x) \equiv \lim_{n\to\infty} \varphi_{11}^{(n)}(x) \leqslant \|r\|_1 Q(x) I_0\big(2\sqrt{Q(x)R(x)}\big). \tag{3.30}$$

One can prove similar results for the kernels K_{12}, K_{22}.

As to the second part of the proposition, we only need to prove that if, in addition, $q, r \in L_\infty(\mathbb{R})$, the kernels $K_{lj}(x, z)$ are bounded with respect to both variables. Define

$$\pi_{lj}(x) = \sup_{z\in\mathbb{R}} |K_{lj}(x, z)|. \tag{3.31}$$

Then from (3.17) it follows

$$\begin{aligned}
\pi_{11}^{(n)}(x) &\leqslant \sup_z \int_x^\infty |q(y)|K_{21}^{(n)}(y, y-x+z)|\,\mathrm{d}y \\
&\leqslant \int_x^\infty |q(y)| \sup_z |K_{21}^{(n)}(y, y-x+z)|\,\mathrm{d}y \\
&\leqslant \int_x^\infty |q(y)|\pi_{21}^{(n)}(y)\,\mathrm{d}y.
\end{aligned} \tag{3.32}$$

Similarly, from (3.18) it follows

$$\begin{aligned}
\pi_{21}^{(n+1)}(x) &\leqslant \frac{1}{2}\|r\|_\infty + \sup_z \int_x^{\frac{x+z}{2}} |r(y)||K_{11}^{(n)}(y, x-y+z)|\,\mathrm{d}y \\
&\leqslant \frac{1}{2}\|r\|_\infty + \sup_z \int_x^\infty |r(y)||K_{11}^{(n)}(y, x-y+z)|\,\mathrm{d}y \\
&\leqslant \frac{1}{2}\|r\|_\infty + \int_x^\infty |r(y)| \sup_z |K_{11}^{(n)}(y, x-y+z)|\,\mathrm{d}y \\
&\leqslant \frac{1}{2}\|r\|_\infty + \int_x^\infty |r(y)|\pi_{11}^{(n)}(y)\,\mathrm{d}y.
\end{aligned}$$

Thus, taking into account (3.32) we have

$$\pi_{21}^{(n+1)}(x) \leqslant \frac{1}{2}\|r\|_\infty + \int_x^\infty \mathrm{d}y\,|r(y)| \int_y^\infty |q(z)|\pi_{21}^{(n)}(z)\,\mathrm{d}z$$

and iteration yields

$$\pi_{21}^{(\infty)}(x) \leqslant \frac{\|r\|_\infty}{2} I_0\big(2\sqrt{Q(x)R(x)}\big)$$

so that the sequence $K_{21}^{(n)}(x, z)$ is convergent towards $K_{21}^{(\infty)}(x, z) \equiv K_{21}(x, z)$ and its norm satisfies

$$\sup_{x,z\in\mathbb{R}} |K_{21}(x, z)| \leqslant \frac{\|r\|_\infty}{2} I_0\big(2\sqrt{\|r\|_1\|q\|_1}\big). \qquad \square$$

3.3. SCATTERING DATA

The eigenfunctions with fixed boundary conditions as $x \to \pm\infty$ (ψ_{ij} and ϕ_{ij}, respectively) are two sets of linearly independent solutions of the second-orderscattering problem (see, for instance, Ablowitz *et al.*, 2004), hence the two sets are linearly dependent from each other. The coefficients of these linear combinations depend on k

$$\phi_{lj}(x,k) = \sum_{m=1,2} s_{mj}(k)\psi_{lm}(x,k), \quad l,j = 1,2 \tag{3.33}$$

or

$$(\phi,\bar{\phi}) = (\bar{\psi},\psi)\mathbf{S}$$

with $\mathbf{S}(k) = (s_{ij}(k))_{i,j=1,2}$ and

$$\begin{aligned} s_{11}(k) &\equiv a(k), \qquad & s_{12}(k) &\equiv \bar{b}(k), \\ s_{21}(k) &\equiv b(k), \qquad & s_{22}(k) &\equiv \bar{a}(k). \end{aligned}$$

The relations (3.33) hold for any k such that all four eigenfunctions exist. In particular, they hold on the real k-axis where the scattering matrix $\mathbf{S}$ is unimodular, i.e., the scattering coefficients satisfy the following unitarity relation

$$a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1. \tag{3.34}$$

From the integral equations (3.2) for the eigenfunctions one also obtains integral representations for the scattering coefficients (note that in Ablowitz *et al.*, 2004 we wrote down integral representations in terms of the ϕ's.) More precisely, one has

$$\begin{aligned} a(k) &= 1 - \int_{-\infty}^{\infty} \mathrm{e}^{-iky} r(y)\psi_{12}(y,k)\,\mathrm{d}y, \\ \bar{a}(k) &= 1 - \int_{-\infty}^{\infty} \mathrm{e}^{iky} q(y)\psi_{21}(y,k)\,\mathrm{d}y, \end{aligned} \tag{3.35}$$

$$\begin{aligned} b(k) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iky} r(y)\psi_{11}(y,k)\,\mathrm{d}y, \\ \bar{b}(k) &= \int_{-\infty}^{\infty} \mathrm{e}^{iky} q(y)\psi_{22}(y,k)\,\mathrm{d}y. \end{aligned} \tag{3.36}$$

As part of the scattering data, one then introduces the reflection coefficients via the relations

$$\rho(k) = \frac{b(k)}{a(k)}, \qquad \bar{\rho}(k) = \frac{\bar{b}(k)}{\bar{a}(k)}. \tag{3.37}$$

The scattering problem may include proper eigenvalues. A proper eigenvalue, $k_j = \xi_j + i\eta_j$, in the upper k-plane (i.e., $\eta_j > 0$) occurs precisely where $a(k_j) = 0$.

Because the eigenvalues, k_j, are the zeroes of $a(k)$, they correspond to the poles (in k) of $\mu_i(x,k) = M_{i1}(x,k)a^{-1}(k)$ (in the region $k_{\mathrm{I}} > 0$). For each simple pole we have

$$\mathrm{Res}\{\mu_i; k_j\} = C_j \mathrm{e}^{2ik_jx} N_{i2}(x,k_j), \tag{3.38}$$

where the last equality defines the "norming constant", C_j, corresponding to the eigenvalue k_j. Similarly, the eigenvalues in the region $k_{\mathrm{I}} < 0$, denoted $\bar{k}_j$, are the zeroes of $\bar{a}(k)$ and one has the analogue definition for the associated norming constants $\bar{C}_j$ as residues of $\bar{\mu}_i(x,k) = M_{i2}(x,k)\bar{a}^{-1}(k)$.

Finally, we observe that the symmetry in the potentials q, r

$$r(x) = \mp q^*(x) \tag{3.39}$$

induces a symmetry in the eigenfunctions

$$\begin{aligned} \psi_{11}(x,k) &= \psi_{22}^*(x,k^*), \qquad & \psi_{21}(x,k) &= \mp\psi_{12}^*(x,k^*), \\ \phi_{11}(x,k) &= \phi_{22}^*(x,k^*), \qquad & \phi_{21}(x,k) &= \mp\phi_{12}^*(x,k^*) \end{aligned}$$

which, in their turn, induce a symmetry in the scattering data, namely

$$\bar{a}(k) = a^*(k^*), \qquad \bar{b}(k) = \mp b^*(k^*) \tag{3.40}$$

and

$$\bar{\rho}(k) = \mp\rho^*(k). \tag{3.41}$$

As a consequence, the eigenvalues appear in complex-conjugate pairs k_j and $\bar{k}_j = k_j^*$ and one can show that the norming constants satisfy the condition

$$\bar{C}_j = \mp C_j^*.$$

Finally, we remark that (3.34) on the real axis becomes

$$|a(k)|^2 \pm |b(k)|^2 = 1.$$

Hence, in the focusing case ($r = -q^*$)

$$|a(k)|^2 = 1 - |b(k)|^2 \leqslant 1, \quad k \in \mathbb{R} \tag{3.42}$$

while for the de-focusing NLS ($r = q^*$)

$$|a(k)|^2 = 1 + |b(k)|^2, \quad k \in \mathbb{R}. \tag{3.43}$$

Besides, in this case the associated scattering problem is formally self-adjoint, hence its spectrum lies on the real axis, it follows that no eigenvalues exist for the de-focusing NLS when the potentials $q, r \in L_1(\mathbb{R})$.

PROPOSITION 3. *Suppose the potentials $q, r \in L_1(\mathbb{R})$. Then there exists a function $\hat{b}(x) \in L_1(\mathbb{R})$ such that the following representation holds*

$$b(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-iky}\hat{b}(y)\,\mathrm{d}y. \tag{3.44}$$

Proof. Inserting (3.13) into the first of (3.36) yields

$$\begin{aligned} b(k) &= \int_{-\infty}^{\infty} \mathrm{e}^{-2iky} r(y)\,\mathrm{d}y + \int_{-\infty}^{\infty}\mathrm{d}y \int_{y}^{\infty} \mathrm{d}z\,\mathrm{e}^{-ik(y+z)} r(y) K_{11}(y,z) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx} r\left(\frac{x}{2}\right)\mathrm{d}x + \int_{-\infty}^{\infty}\mathrm{d}y \int_{2y}^{\infty} \mathrm{d}x\,\mathrm{e}^{-ikx} r(y) K_{11}(y, x-y) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx} r\left(\frac{x}{2}\right)\mathrm{d}x + \int_{-\infty}^{\infty}\mathrm{d}x \int_{-\infty}^{\frac{x}{2}} \mathrm{d}y\,\mathrm{e}^{-ikx} r(y) K_{11}(y, x-y) \end{aligned}$$

hence the representation (3.44) "formally" follows with

$$\hat{b}(x) = \frac{1}{2} r\left(\frac{x}{2}\right) + \int_{-\infty}^{\frac{x}{2}} r(y) K_{11}(y, x-y)\,\mathrm{d}y.$$

Note next that

$$\begin{aligned} \int |\hat{b}(x)|\,\mathrm{d}x &\leqslant \int |r(x)|\,\mathrm{d}x + \int \mathrm{d}z \int_{-\infty}^{\frac{z}{2}} |r(y)||K_{11}(y, z-y)|\,\mathrm{d}y \\ &= \int |r(x)|\,\mathrm{d}x + \int \mathrm{d}y \int_{2y}^{\infty} \mathrm{d}z\,|r(y)||K_{11}(y, z-y)| \\ &= \int |r(x)|\,\mathrm{d}x + \int \mathrm{d}y\,|r(y)| \int_{y}^{\infty} \mathrm{d}z'\,|K_{11}(y, z')| \\ &\leqslant \int |r(x)|\,\mathrm{d}x + \int \mathrm{d}y\,|r(y)| \int \mathrm{d}z\,|K_{11}(y, z)| \\ &= \int |r(x)|\,\mathrm{d}x + \int |r(y)|\varphi_{11}(y)\,\mathrm{d}y \\ &\equiv \int |r(x)|(1 + \varphi_{11}(x))x, \end{aligned}$$

where $\varphi_{11}(x) \equiv \varphi_{11}^{(\infty)}(x)$. Then, from (3.30) we get

$$\int |\hat{b}(x)|\,\mathrm{d}x \leqslant \big(1 + \|r\|_1\|q\|_1 I_0\big(2\sqrt{\|r\|_1\|q\|_1}\,\big)\big)\|r\|_1$$

proving that $\hat{b} \in L_1(\mathbb{R})$. □

PROPOSITION 4. *Suppose the potentials $q, r \in L_1(\mathbb{R})$. Then there exists a function $\hat{a}(x) \in L_1(\mathbb{R})$ such that the following representation holds*

$$a(k) - 1 = -\int_{0}^{\infty} \mathrm{e}^{iky}\hat{a}(y)\,\mathrm{d}y. \tag{3.45}$$

Proof. Inserting (3.13) into (3.35) yields

$$\begin{aligned} a(k) - 1 &= -\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-iky} r(y) \int_{y}^{\infty} \mathrm{d}z\, \mathrm{e}^{ikz} K_{12}(y,z) \\ &= -\int_{-\infty}^{\infty} \mathrm{d}y \int_{y}^{\infty} \mathrm{d}z\, \mathrm{e}^{ik(z-y)} r(y) K_{12}(y,z) \\ &= -\int_{-\infty}^{\infty} \mathrm{d}y \int_{0}^{\infty} \mathrm{d}z'\, \mathrm{e}^{ikz'} r(y) K_{12}(y, z'+y) \end{aligned}$$

and then the representation (3.45) follows with

$$\hat{a}(x) = \int_{-\infty}^{\infty} r(y) K_{12}(y, x+y)\, \mathrm{d}y.$$

Note next that

$$\begin{aligned} \int |\hat{a}(x)|\, \mathrm{d}x &\leqslant \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} |r(y)||K_{12}(y, x+y)|\, \mathrm{d}y \\ &\leqslant \int_{-\infty}^{\infty} |r(y)| \varphi_{12}(y)\, \mathrm{d}y \leqslant \|r\|_1 \|q\|_1 I_0\big(2\sqrt{\|r\|_1 \|q\|_1}\,\big), \end{aligned}$$

where we used (3.26), $\varphi_{12}(x) \equiv \varphi_{12}^{(\infty)}(x)$ and the bound for φ_{12} analogous to (3.29). This result shows that indeed $\hat{a} \in L_1(\mathbb{R})$. □

THEOREM 1. *Suppose the potentials* $q, r \in L_1(\mathbb{R})$ *and that*

$$|a(k)| > 0, \quad \forall k \in \mathbb{R}. \tag{3.46}$$

Then the reflection coefficient introduced in (3.37) *is such that*

$$\rho(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-iky} \hat{R}(y)\, \mathrm{d}y, \tag{3.47}$$

where $\hat{R}(x) \in L_1(\mathbb{R})$. *Further, if the potentials are bounded, so is* $\hat{R}(x)$. *A similar result holds for* $\bar{\rho}(k)$.

Proof. Set

$$\rho(k) = \frac{b(k)}{a(k)} \equiv \frac{b(k)}{1 + \tilde{a}(k)}, \quad \tilde{a}(k) = a(k) - 1.$$

$\tilde{a}(k)$ is the Fourier transform of an integrable function (cf. Prop. 4) and (if $|a(k)| > 0$) then it maps $\mathbb{R}$ onto $D = \mathbb{C} - \{-1\}$, where $f(z) \equiv (1/1+z)$ is holomorphic. Hence, a modification of Wiener's theorem (see App. A) shows that there exists a function $\hat{h}(x) \in L_1(\mathbb{R})$ such that

$$\frac{1}{1 + \tilde{a}(k)} = f(\tilde{a}(k)) = \int_{-\infty}^{\infty} \mathrm{e}^{-iky} \hat{h}(y)\, \mathrm{d}y. \tag{3.48}$$

It also follows that

$$\hat{R}(x) \equiv \int \hat{b}(x-y)\hat{h}(y)\,\mathrm{d}y \in L_1(\mathbb{R}) \tag{3.49}$$

and hence (by Lemma 3 in the Appendix) that

$$R(k) \equiv \int \mathrm{e}^{-iky}\hat{R}(y)\,\mathrm{d}y = \frac{b(k)}{1+\tilde{a}(k)}. \tag{3.50}$$

Thus

$$\rho(k) = R(k) = \int \mathrm{e}^{-iky}\hat{R}(y)\,\mathrm{d}y, \tag{3.51}$$

where $\hat{R}(x) \in L_1(\mathbb{R})$. Similarly, one can show that $\hat{R}(x) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ if $q, r \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. □

COROLLARY 1. *Defocusing case. Suppose the potentials q, r are in $L_1(\mathbb{R})$ and that $r = q^*$ (defocusing case). Then*

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(\xi)\mathrm{e}^{i\xi x}\,\mathrm{d}\xi - i\sum_{j=1}^{J} C_j \mathrm{e}^{ik_j x} \tag{3.52}$$

is in $L_1(\mathbb{R})$ and

$$\rho(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-iky}F(y)\,\mathrm{d}y. \tag{3.53}$$

Proof. Indeed, as pointed out, this symmetry implies that $|a(k)|^2 = |b(k)|^2 + 1 \geqslant 1$ for any $k \in \mathbb{R}$ and hence (3.46) is satisfied. As commented earlier, there are no eigenvalues in the defocusing case for decaying potentials. Thus, the sum in (3.52) is void which implies

$$\rho(k) = \int \mathrm{e}^{-iky}F(y)\,\mathrm{d}y \tag{3.54}$$

and hence $\hat{R}(y) = F(y)$. □

COROLLARY 2. *Small norm case. Suppose the potentials q, r are in $L_1(\mathbb{R})$ with small norm*

$$\|r\|_1\|q\|_1 I_0\big(2\sqrt{\|r\|_1\|q\|_1}\big) < 1.$$

Then $F(x)$ (defined in (3.52)) is in $L_1(\mathbb{R})$ and (3.53) holds.

Proof. Using Prop. 4 we have

$$|1-|a(k)|| \leqslant |a(k)-1| \leqslant \int_0^\infty |\hat{a}(y)|\,\mathrm{d}y \leqslant \|r\|_1\|q\|_1 I_0\big(2\sqrt{\|r\|_1\|q\|_1}\,\big) < 1$$

which implies that $|a(k)| > 0$ for any $k \in \mathbb{C}$. This implies, first, that there are no eigenvalues and hence that $\hat{\rho}(y) = F(y)$. It also follows that condition (3.46) is satisfied and hence $\hat{\rho}(x) \equiv R(x)$ is in $L_1(\mathbb{R})$. □

Remark (General focusing case). When $r = -q^*$ and the potentials have small norm in the sense of the last corollary, then (3.47) holds. Otherwise, there is no guarantee that this is the case. In spite of this one has the following: $a(k)$ is analytic on the upper half plane and continuous on the real axis, and tends to 1 as $k \to \infty$. Hence, generically, zeros of $a(k)$ are denumerable and cannot accumulate towards the real axis.

4. Inverse Problem

When the potentials are integrable, and hence eigenfunctions exist, taking the Fourier transforms of Equations (3.33) and using the triangular representations (3.13) gives that the following relationship between $K(x, y)$ and $F(x)$ holds:

$$\bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^\infty K(x,s)F(s+y)\,\mathrm{d}s = 0, \qquad y \geqslant x, \tag{4.1}$$

$$K(x,y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x+y) + \int_x^\infty \bar{K}(x,s)\bar{F}(s+y)\,\mathrm{d}s = 0, \qquad y \geqslant x, \tag{4.2}$$

where K and $\bar{K}$ are two-component vectors. Alternatively, given data $F(x)$ these equations can be thought of as linear integral equations from which the 2×2 matrix in (3.14), and hence the eigenfunctions (3.13), are recovered. In this interpretation, (4.1)–(4.2) constitute the Gel'fand–Levitan–Marchenko (GLM) equations of the *inverse problem* whose aim is the reconstruction of eigenfunctions of the scattering problem in terms of the (spectral) data given by (3.52), i.e.

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(\xi)\mathrm{e}^{i\xi x}\,\mathrm{d}\xi - i\sum_{j=1}^{J} C_j \mathrm{e}^{ik_j x} \tag{4.3}$$

and

$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\rho}(\xi)\mathrm{e}^{-i\xi x}\,\mathrm{d}\xi + i\sum_{j=1}^{\bar{J}} \bar{C}_j \mathrm{e}^{-i\bar{k}_j x} \tag{4.4}$$

and, finally, of the potentials via

$$q(x) = -2K^{(1)}(x,x), \qquad r(x) = -2\bar{K}^{(2)}(x,x), \tag{4.5}$$

where $K^{(j)}$ and $\bar{K}^{(j)}$ for $j = 1, 2$ denote the jth component of the vectors K and $\bar{K}$ respectively.

In (4.3)–(4.4), $k_j, \bar{k}_j$ are the discrete eigenvalues (corresponding to the zeros of $a(k)$ and $\bar{a}(k)$, respectively, cf. (3.35)), $\{k_j = \xi_j + i\eta_j,\ \ \eta_j > 0\}_{j=1}^{J}$, $\{\bar{k}_j = \bar{\xi}_j + i\bar{\eta}_j, \bar{\eta}_j < 0\}_{j=1}^{\bar{J}}$, $\{C_j\}_{j=1}^{J}$, $\{\bar{C}_j\}_{j=1}^{\bar{J}}$ the associated norming constants, and ρ, $\bar{\rho}$ the reflection coefficients (cf. (3.37)).

In terms of the 2×2 matrix in (3.14)

$$\begin{aligned}\mathbf{K}(x, y) &\equiv [K_{ij}(x, y)]_{i,j=1,2} \equiv (\bar{K}(x, y), K(x, y)) \\ &\equiv \begin{pmatrix} \bar{K}^{(1)}(x, y) & K^{(1)}(x, y) \\ \bar{K}^{(2)}(x, y) & K^{(2)}(x, y) \end{pmatrix}\end{aligned}$$

the integral equations (4.1)–(4.2) of the inverse problem can be written as

$$K_{ij}(x, y) + \delta_{ij+1} F_j(x + y) + \int_x^{\infty} K_{ij+1}(x, s) F_j(y + s)\,\mathrm{d}s = 0, \tag{4.6}$$

where $i, j = 1, 2$ and $j + 1$ is intended mod 2,

$$F_1(x) \equiv F(x), \qquad F_2(x) \equiv \bar{F}(x). \tag{4.7}$$

Note that the symmetries in the scattering data

(i) $\bar{\rho}(k) = \mp\rho^*(k)$ for $k \in \mathbb{R}$ (cf. (3.41)),
(ii) $J = \bar{J}$ and $\bar{k}_j = k_j^*$, $\bar{C}_j = \mp C_j^*$ for $j = 1, \ldots, J$,
(iii) $\bar{F}(x) = \mp F^*(x)$, or

$$F_2(x) = \mp F_1^*(x) \tag{4.8}$$

correspond, from the direct side, to the reduction $r = \mp q^*$ in the potentials.

Moreover, the following characterization relation holds

$$1 \pm |\rho(k)|^2 = |a(k)|^{-2}, \quad k \in \mathbb{R} \tag{4.9}$$

$\rho(k)$ is the reflection coefficient, $1/a(k)$ is also called the transmission coefficient of the associated scattering problem.

4.1. SUMMARY OF THE RESULTS OF THE INVERSE PROBLEM

In this section we study the solvability of the GLM equations (4.1)–(4.2) given the data $\overrightarrow{F} = \binom{F_1}{F_2}$.

For a given $x \in \mathbb{R}$ and $p \geqslant 1$ we will consider the spaces $L_p[x, \infty)$

$$L_p[x, \infty) = \left\{ \overrightarrow{\Phi}(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix} : \|\overrightarrow{\Phi}\|_p < \infty \right\},$$

where we define the norm $\|\vec{\Phi}\|_p$ as

$$\|\vec{\Phi}\|_p^p \equiv \sum_{j=1,2} \int_x^\infty |\phi_j(s)|^p \, \mathrm{d}s. \tag{4.10}$$

Suppose that for all $x \in \mathbb{R}$, $\vec{F} = \binom{F_1}{F_2}$ is in $L_1[x, \infty)$ and, also, that $F_2(x) = \mp F_1^*(x)$ (which corresponds to potentials satisfying $q(x) = \mp r^*(x)$). Then we shall prove the following results.

(i) The integral operator Ω_x (cf. (4.11)) defining the GLM equations (4.6) is a compact operator from $L_1[x, \infty)$, $L_2[x, \infty)$ and $L_\infty[x, \infty)$ onto themselves (cf. Theorems 2, 3); more generally, for any $p \geqslant 1$, Ω_x is a compact operator on $L_p[x, \infty)$.

(ii) The homogeneous GLM equations admit no nontrivial solutions in either $L_1[x, \infty)$ or $L_2[x, \infty)$ for all $x \in \mathbb{R}$ (cf. Theorem 4).

(iii) The solution $K_{lj}(x, \cdot)$ to the GLM equations (4.6) exists and is unique in $L_1[x, \infty)$ for all $x \in \mathbb{R}$ (note that this is a direct consequence of the Fredholm alternative, cf. Theorem 5 and (i) and (ii)).

(iv) If, in addition, $\vec{F} \in L_2[x, \infty)$ for all $x \in \mathbb{R}$, then for all $x \in \mathbb{R}$ the solution $K_{lj}(x, \cdot)$ to the GLM equations exists and is unique in $L_1[x, \infty) \cup L_2[x, \infty)$ and, hence, it belongs to $L_1[x, \infty) \cap L_2[x, \infty)$ (cf. Theorem 5).

4.2. COMPACTNESS

For a given $x \in \mathbb{R}$ consider the following operator Ω_x, which associates to $\vec{\Phi} = \binom{\phi_1}{\phi_2}$ the 2-component vector $\Omega_x \vec{\Phi}$

$$\Omega_x \vec{\Phi}(z) \equiv \begin{pmatrix} (\Omega_x \vec{\Phi}(z))_1 \\ (\Omega_x \vec{\Phi}(z))_2 \end{pmatrix} = \begin{pmatrix} \int_x^\infty F_1(z+y)\, \phi_2(y)\, \mathrm{d}y \\ \int_x^\infty F_2(z+y)\, \phi_1(y)\, \mathrm{d}y \end{pmatrix}, \quad z \geqslant x, \tag{4.11}$$

$$\Omega_x \vec{\Phi}(z) = 0, \quad z < x.$$

THEOREM 2. *Assume* $\vec{F} = \binom{F_1}{F_2}$ *is in* $L_1[x, \infty)\ \forall x \in \mathbb{R}$. *Then* Ω_x *maps* $L_2[x, \infty)$ *onto itself and is a compact operator on this space.*

Proof. Let $R \geqslant x$. Consider

$$\sum_{j=1,2} \int_R^\infty \left|(\Omega_x \vec{\Phi}(z))_j\right|^2 \mathrm{d}z$$

$$\equiv \sum_{j=1,2} \int_R^\infty \left| \int_x^\infty F_{j+1}(z+y)\phi_j(y)\, \mathrm{d}y \right|^2 \mathrm{d}z, \tag{4.12}$$

where, as before, $j + 1$ is intended mod 2. For each of the two terms in the sum in the right-hand side we have

$$\int_R^\infty \mathrm{d}z \left| \int_x^\infty F_{j+1}(z+y)\phi_j(y)\, \mathrm{d}y \right|^2$$

$$
\begin{aligned}
&\leqslant \int_R^\infty \mathrm{d}z \int_x^\infty \mathrm{d}y\, |F_{j+1}(z+y)\phi_j(y)| \\
&= \int_R^\infty \mathrm{d}z \int_x^\infty \mathrm{d}y\, \sqrt{|F_{j+1}(z+y)|}\big[\sqrt{|F_{j+1}(z+y)|}|\phi_j(y)|\big]
\end{aligned}
$$

and using Schwartz inequality we get

$$
\begin{aligned}
&\int_R^\infty \mathrm{d}z \left|\int_x^\infty F_{j+1}(z+y)\phi_j(y)\,\mathrm{d}y\right|^2 \\
&\quad\leqslant \int_R^\infty \mathrm{d}z \left\{\int_x^\infty |F_{j+1}(z+y)|\,\mathrm{d}y \int_x^\infty |F_{j+1}(z+s)||\phi_j(s)|^2\,\mathrm{d}s\right\} \\
&\quad\leqslant \int_R^\infty \mathrm{d}z \left(\sup_{z\geqslant R}\int_{x+z}^\infty |F_{j+1}(y)|\,\mathrm{d}y\right)\left(\int_x^\infty |F_{j+1}(z+s)||\phi_j(s)|^2\,\mathrm{d}s\right).
\end{aligned}
$$

Since the integral is a monotonically decreasing function of the lower limit of integration

$$
\begin{aligned}
&\int_R^\infty \mathrm{d}z \left|\int_x^\infty F_{j+1}(z+y)\phi_j(y)\,\mathrm{d}y\right|^2 \\
&\quad\leqslant \int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y \int_R^\infty \mathrm{d}z \int_x^\infty |F_{j+1}(z+s)||\phi_j(s)|^2\,\mathrm{d}s \\
&\quad= \int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y \int_x^\infty \mathrm{d}s\, |\phi_j(s)|^2\left(\int_R^\infty |F_{j+1}(z+s)|\,\mathrm{d}z\right) \\
&\quad\leqslant \int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y \int_x^\infty \mathrm{d}s\, |\phi_j(s)|^2 \int_{R+s}^\infty |F_{j+1}(z)|\,\mathrm{d}z \\
&\quad\leqslant \int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y \int_x^\infty \mathrm{d}s\, |\phi_j(s)|^2\left(\sup_{s\geqslant x}\int_{R+s}^\infty |F_{j+1}(z)|\,\mathrm{d}z\right) \\
&\quad= \left(\int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y\right)^2 \int_x^\infty |\phi_j(s)|^2\,\mathrm{d}s. \qquad (4.13)
\end{aligned}
$$

Hence, from (4.12) and (4.13) it follows

$$
\begin{aligned}
&\sum_{j=1,2}\int_R^\infty \left|(\Omega_x \overrightarrow{\Phi}(z))_j\right|^2 \mathrm{d}z \\
&\quad\leqslant \sum_{j=1,2}\left(\int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y\right)^2\left[\int_x^\infty |\phi_j(s)|^2\,\mathrm{d}s\right] \\
&\quad\leqslant \sum_{j=1,2}\left[\left(\int_{R+x}^\infty |F_{j+1}(y)|\,\mathrm{d}y\right)^2\right]\|\overrightarrow{\Phi}\|_2^2. \qquad (4.14)
\end{aligned}
$$

First, let us take $R = x$. Then

$$
\|\Omega_x \overrightarrow{\Phi}\|_2^2 \leqslant \|\overrightarrow{\Phi}\|_2^2 \sum_{j=1,2}\left(\int_{2x}^\infty |F_j(y)|\,\mathrm{d}y\right)^2 < \infty \qquad (4.15)
$$

if $\overrightarrow{\Phi} \in L_2[x,\infty)$ under the assumption that $\overrightarrow{F} \in L_1[x,\infty)$ for all $x \in \mathbb{R}$. This means that Ω_x maps $L_2[x,\infty)$ onto itself.

Let now $\mathcal{B}_x$ be the "unit ball" in $L_2[x,\infty)$, i.e.,

$$\mathcal{B}_x = \left\{\overrightarrow{\Phi} \in L_2[x,\infty) : \|\overrightarrow{\Phi}\|_2 \leqslant 1\right\}.$$

Then using the bound (4.14)

$$\begin{aligned}
&\lim_{R\to\infty} \sup_{\overrightarrow{\Phi}\in\mathcal{B}_x} \sum_{j=1,2} \int_R^\infty |(\Omega_x \overrightarrow{\Phi}(z))_j|^2 \,\mathrm{d}z \\
&\quad\leqslant \lim_{R\to\infty} \left(\sup_{\overrightarrow{\Phi}\in\mathcal{B}_x} \|\overrightarrow{\Phi}\|_2^2 \sum_{j=1,2} \int_{R+x}^\infty |F_j(y)|\,\mathrm{d}y \right)^2 \\
&\quad\leqslant \sum_{j=1,2} \lim_{R\to\infty} \left(\int_{R+x}^\infty |F_j(y)|\,\mathrm{d}y \right)^2 = 0, \qquad (4.16)
\end{aligned}$$

where in the last line we used Lemma 1 proved in the Appendix. This proves (A.2).

Let us consider next

$$\left(\Omega_x \overrightarrow{\Phi}(z+h)\right)_j - \left(\Omega_x \overrightarrow{\Phi}(z)\right)_j = \int_x^\infty (\mathbf{\Delta}_h(z+y))_{j+1}\phi_j(y)\,\mathrm{d}y, \qquad (4.17)$$

where

$$(\mathbf{\Delta}_h(x+y))_j = F_j(x+y+h) - F_j(x+y).$$

Then

$$\begin{aligned}
&\sum_{j=1,2} \int_x^\infty \left|(\Omega_x \overrightarrow{\Phi}(z+h) - \Omega_x \overrightarrow{\Phi}(z))_j\right|^2 \mathrm{d}z \\
&\quad= \sum_{j=1,2} \int_x^\infty \left| \int_x^\infty (\mathbf{\Delta}_h(z+y)\overrightarrow{\Phi}(y))_j \,\mathrm{d}y \right|^2 \mathrm{d}z \\
&\quad\leqslant \|\overrightarrow{\Phi}\|_2^2 \sum_{j=1,2} \left(\int_{2x}^\infty |(\mathbf{\Delta}_h)_j(y)|\,\mathrm{d}y \right)^2 \\
&\quad= \|\overrightarrow{\Phi}\|_2^2 \sum_{j=1,2} \left(\int_{2x}^\infty |F_{j+1}(z+h) - F_{j+1}(z)|\,\mathrm{d}z \right)^2
\end{aligned}$$

and consequently

$$\begin{aligned}
&\lim_{h\to0} \Big(\sup_{\overrightarrow{\Phi}\in\mathcal{B}_x} \|\Omega_x \overrightarrow{\Phi}(z+h) - \Omega_x \overrightarrow{\Phi}(z)\|_2^2 \Big) \\
&\quad\leqslant \lim_{h\to0} \Big(\sup_{\overrightarrow{\Phi}\in\mathcal{B}_x} \|\overrightarrow{\Phi}\|_2^2 \Big) \sum_{j=1,2} \left(\int_{2x}^\infty |F_j(z+h) - F_j(z)|\,\mathrm{d}z \right)^2 \\
&\quad\leqslant \sum_{j=1,2} \lim_{h\to0} \left(\int_{2x}^\infty |F_j(z+h) - F_j(z)|\,\mathrm{d}z \right)^2 = 0, \qquad (4.18)
\end{aligned}$$

where in the last line we used the result in Lemma 2 stated in the Appendix, i.e., (A.1).

We note that the natural definition $f(\infty) = 0$ allows one to extend any square integrable function defined on $[x, \infty)$ to $[x, \infty]$. Hence, all the previous results apply to the latter case $f \in L_2[x, \infty]$. We have proven that if $\overrightarrow{F} \in L_1[x, \infty)$ for all $x \in \mathbb{R}$, Ω_x maps $L_2[x, \infty]$ onto itself

$$\Omega_x\colon\ L_2[x, \infty] \to L_2[x, \infty]$$

and, if $\mathcal{B}_x$ is the unit ball in $L_2[x, \infty]$, Ω_x maps $\mathcal{B}_x$ into $\mathcal{E}_x \equiv \{f \in L_2[x, \infty]\colon f$ is L_2-continuous$\}$ in a uniform way, i.e., the set $\mathcal{H}_x = \{\Omega_x\Phi,\ \Phi \in \mathcal{B}_x\}$ is uniformly L_2-equicontinuous (cf. Def. 1 in the Appendix and (4.18), (4.16)).

By a result in functional analysis (which follows from Kolmogorov's theorem by using the fact that $[x, \infty]$ is compact on $\bar{\mathbb{R}}$), the set $\mathcal{H}_x$ is a compact set of functions in $L_2[x, \infty]$. By the Ascoli–Arzela's theorem it then follows that the operator Ω_x is a compact operator on $L_2[x, \infty]$ for any $x \in \mathbb{R}$. □

THEOREM 3. *Assume* $\overrightarrow{F} = \binom{F_1}{F_2}$ *is in* $L_1[x, \infty)$ $\forall x \in \mathbb{R}$. *Then* Ω_x *also maps* $L_1[x, \infty)$ *onto itself and is a compact operator on this space.*

Proof. Let $R \geqslant x$. Consider

$$\begin{aligned}
&\int_R^\infty \left|(\Omega_x \overrightarrow{\Phi}(z))_j\right| \mathrm{d}z \\
&\quad\equiv \int_R^\infty \left| \int_x^\infty F_j(z+y)\phi_{j+1}(y)\,\mathrm{d}y \right| \mathrm{d}z \\
&\quad\leqslant \int_x^\infty \mathrm{d}y\, |\phi_{j+1}(y)| \int_R^\infty |F_j(z+y)|\,\mathrm{d}z \\
&\quad\leqslant \int_x^\infty \mathrm{d}y\, |\phi_{j+1}(y)| \sup_{y \geqslant x} \int_{R+y}^\infty |F_j(z)|\,\mathrm{d}z \\
&\quad= \left(\int_x^\infty |\phi_{j+1}(y)|\,\mathrm{d}y\right)\left(\int_{R+x}^\infty |F_j(z)|\,\mathrm{d}z\right).
\end{aligned}$$

Taking $R = x$ it follows that

$$\left\|\Omega_x \overrightarrow{\Phi}\right\|_1 = \sum_{j=1,2} \int_x^\infty \left|(\Omega_x \overrightarrow{\Phi}(z))_j\right| \mathrm{d}z \leqslant \|\overrightarrow{\Phi}\|_1 \sum_{j=1,2} \int_{2x}^\infty |F_j(z)|\,\mathrm{d}z, \quad (4.19)$$

hence $\Omega_x\colon\ L_1[x, \infty) \to L_1[x, \infty)$.

Also, using Lemmas 1 and 2 proved in the Appendix, one can show that given any family $\mathcal{B}_x$ of bounded functions of $L_1[x, \infty)$

$$\lim_{R \to \infty} \sup_{\overrightarrow{\Phi} \in \mathcal{B}_x} \int_R^\infty |\Omega_x \overrightarrow{\Phi}(z)|_j\,\mathrm{d}z = 0 \quad (4.20)$$

(cf. (A.1)) and also that

$$\lim_{h\to 0}\Big(\sup_{\overrightarrow{\Phi}\in\mathcal{B}_x}\|(\Omega_x\overrightarrow{\Phi}(z+h)-\Omega_x\overrightarrow{\Phi}(z))_j\|_1\Big)=0 \tag{4.21}$$

(i.e., (A.2)) which completes the proof. □

4.3. FREDHOLM ALTERNATIVE

The question of existence and uniqueness of solutions of linear integral equations is usually examined by the use of the Fredholm alternative. Consider the homogeneous equations corresponding to any one of the components of (4.1) and (4.2) $(y > x)$

$$h_1(y)+\int_x^{\infty}h_2(s)F_1(s+y)\,\mathrm{d}s=0, \tag{4.22}$$

$$h_2(y)+\int_x^{+\infty}h_1(s)F_2(s+y)\,\mathrm{d}s=0. \tag{4.23}$$

We first consider these equations on $L_2[x,\infty)$. Suppose there exists an L_2-solution $h(y) = (h_1(y), h_2(y))$ of (4.22)–(4.23) which vanishes identically for $y < x$. Multiply (4.22) by $h_1^*(y)$, (4.23) by $h_2^*(y)$, integrate with respect to y and use

$$\int_x^{\infty}|h_j(y)|^2\,\mathrm{d}y=\int_{-\infty}^{\infty}|h_j(y)|^2\,\mathrm{d}y$$

to obtain

$$\int_{-\infty}^{\infty}\Bigg\{|h_1(y)|^2+|h_2(y)|^2+ \\ +\int_{-\infty}^{\infty}[h_2(s)h_1^*(y)F_1(s+y)+h_1(s)h_2^*(y)F_2(s+y)]\,\mathrm{d}s\Bigg\}\,\mathrm{d}y=0. \tag{4.24}$$

If $r = -q^*$, then the symmetry condition (4.8), i.e., $F_2(x) = -F_1^*(x)$, allows the latter equation to be written as

$$\int_{-\infty}^{\infty}\Bigg\{|h_1(y)|^2+|h_2(y)|^2+2i\,\mathrm{Im}\int_{-\infty}^{\infty}h_2(s)h_1^*(y)F_1(s+y)\,\mathrm{d}s\Bigg\}\,\mathrm{d}y=0.$$

The real and imaginary parts must both vanish, whereupon it follows that $h(y) \equiv 0$.

Consider next the defocusing case, corresponding to the reduction $r(x) = q^*(x)$; then $F_2(x) = F_1^*(x)$ and Equation (4.24) becomes

$$\int_{-\infty}^{\infty}\Bigg\{|h_1(y)|^2+|h_2(y)|^2+2\,\mathrm{Re}\int_{-\infty}^{\infty}h_2(s)h_1^*(y)F_1(s+y)\,\mathrm{d}s\Bigg\}\,\mathrm{d}y=0. \tag{4.25}$$

Moreover, in this case the scattering problem is formally self-adjoint and there are no discrete eigenvalues, therefore from (4.3)–(4.4) and (4.7) we have

$$F_2^*(x) = F_1(x) = \frac{1}{2\pi}\int_{\infty}^{\infty} \rho(\xi)\mathrm{e}^{i\xi x}\,\mathrm{d}\xi \tag{4.26}$$

with $|\rho(k)|^2 = 1 - |a(k)|^{-2} < 1$ (cf. (4.9)). We next use Parseval's identity

$$\int_{-\infty}^{\infty} |h_j(y)|^2\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{h}_j(\xi)|^2\,\mathrm{d}\xi \tag{4.27}$$

for square integrable functions, where

$$\hat{h}_j(\xi) = \int_{-\infty}^{\infty} h_j(y)\mathrm{e}^{-i\xi y}\,\mathrm{d}y \tag{4.28}$$

is the Fourier transform of $h_j(y)$. Substituting these results into (4.25) and reversing the order of integration yields

$$\int_{-\infty}^{\infty} \left\{|\hat{h}_1(-\xi)|^2 + |\hat{h}_2(\xi)|^2 + 2\,\mathrm{Re}\left[\rho(\xi)\hat{h}_1(-\xi)\hat{h}_2^*(\xi)\right]\right\}\mathrm{d}\xi = 0. \tag{4.29}$$

From the characterization equation (4.9) in the defocusing case, i.e., when $r = q^*$, it follows that $|\rho(\xi)| < 1$, hence we have

$$\left|2\mathrm{Re}\left[\rho(\xi)\hat{h}_1(-\xi)\hat{h}_2^*(\xi)\right]\right| < 2|\hat{h}_1(-\xi)||\hat{h}_2(\xi)| \leqslant |\hat{h}_1(-\xi)|^2 + |\hat{h}_2(\xi)|^2$$

hence

$$h(y) \equiv 0,$$

i.e., the homogeneous integral equation admits no nontrivial L_2 solutions.

We next take up the L_1 case. Suppose that there exists a homogeneous solution $h_j(x, y) \in L_1$ and vanishing for $y < x$. Then its Fourier transform (4.28) is in L_∞ and, operating with $\int_x^\infty \mathrm{d}y\,\mathrm{e}^{i\xi y}$ on Equations (4.22)–(4.23) and using (4.26) shows that it satisfies

$$\hat{h}_j(\xi) + \hat{F}_j(\xi)\hat{h}_{j+1}(-\xi) = 0,$$

where $j = 1, 2$ and $j + 1$ is intended mod 2. This corresponds to the following homogeneous system of equations of 4 equations in the 4 unknowns $\hat{h}_j(\xi)$, $\hat{h}_j(-\xi)$ for $j = 1, 2$:

$$\begin{pmatrix} 1 & 0 & 0 & \hat{F}_2(\xi) \\ 0 & 1 & \hat{F}_1(\xi) & 0 \\ 0 & \hat{F}_2(-\xi) & 1 & 0 \\ \hat{F}_1(-\xi) & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{h}_1(\xi) \\ \hat{h}_2(\xi) \\ \hat{h}_1(-\xi) \\ \hat{h}_2(-\xi) \end{pmatrix} \equiv \mathbf{\Delta}\begin{pmatrix} \hat{h}_1(\xi) \\ \hat{h}_2(\xi) \\ \hat{h}_1(-\xi) \\ \hat{h}_2(-\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Existence of nontrivial solutions requires $\det \mathbf{\Delta} = 0$, i.e.,

$$\left[1 - \hat{F}_1(\xi)\hat{F}_2(-\xi)\right]\left[1 - \hat{F}_1(-\xi)\hat{F}_2(\xi)\right] = 0. \tag{4.30}$$

This, however, cannot happen in the "physical case" $r = \mp q^*$. Indeed, recall that one has $F_2(x) = \mp F_1^*(x)$ and $\hat{F}_2(\xi) = \mp \hat{F}_1^*(-\xi) \equiv \hat{F}^*(-\xi)$ so that the previous equation yields

$$\det \mathbf{\Delta} = \left[1 \pm |\hat{F}_1(\xi)|^2\right]\left[1 \pm |\hat{F}_1(-\xi)|^2\right] > 0.$$

For the upper sign it is trivial, for the lower sign (defocusing case, $r = q^*$) one has to take into account that $|\hat{F}(\xi)| = |\rho(\xi)| < 1$ for any real p.

Thus, we conclude that $\hat{h}_1(\xi) = \hat{h}_2(\xi) = 0$. Next, $h_j(y)$ can be recovered uniquely from Fourier's inversion theorem for L_1-functions:

$$h_j(y) = \frac{1}{2\pi} \lim_{n\to\infty} \int_{-\infty}^{\infty} \hat{h}_j(\xi) \mathrm{e}^{-i\xi y - |\xi|/n}\, \mathrm{d}\xi = 0.$$

In conclusion, we have shown the following results.

THEOREM 4. *If* $F_2(x) = \mp F_1^*(x)$, *and* $\overrightarrow{F} \in L_1[x, \infty)$ *for all* $x \in \mathbb{R}$, *the integral equations* (4.1), (4.2) *admit no homogenous solutions in either* $L_1[x, \infty)$ *or* $L_2[x, \infty)$ *but the trivial one.*

In addition, we have

THEOREM 5. *Suppose* $F_2(x) = \mp F_1^*(x)$, *and* $\overrightarrow{F} \in L_1[x, \infty)$ *for all* $x \in \mathbb{R}$. *Then*

(i) *The solution* $K_{lj}(x, \cdot)$ *to the GLM equations* (4.6) *exists and is unique in* $L_1[x, \infty)$ *for all* $x \in \mathbb{R}$.
(ii) *If, in addition,* $\overrightarrow{F} \in L_2[x, \infty)$ *for all* $x \in \mathbb{R}$, *then the solution* $K_{lj}(x, \cdot)$ *to the GLM equations exists and is unique in* $L_1[x, \infty) \cup L_2[x, \infty)$ *and, hence,* $K_{lj}(x, \cdot)$ *belongs to* $L_1[x, \infty) \cap L_2[x, \infty)$ *for all* $x \in \mathbb{R}$.

Proof. (i) follows from Theorems 2 to 4 and the classical Fredholm alternative theorem.

(ii) If $\overrightarrow{F} \in L_1[x, \infty) \cap L_2[x, \infty)$, one first proves existence and uniqueness of solution for the GLM equations in $L_1[x, \infty)$, then, using again the Fredholm alternative, in $L_2[x, \infty)$. It follows that (a) there is a unique solution in the whole $L_1[x, \infty) \cup L_2[x, \infty)$ and (b) the solutions coincide in $L_1[x, \infty) \cap L_2[x, \infty)$; consequently, the common solution lives in $L_1[x, \infty) \cap L_2[x, \infty)$. □

4.4. PROPERTIES OF POTENTIALS

We have seen in the previous section that the conditions $\overrightarrow{F} \in L_1[x, \infty)$ for all $x \in \mathbb{R}$ and $F_2(x) = \mp F_1^*(x)$ guarantees existence and uniqueness of a solution

$K_{lj}(x,\cdot)$ to the GLM equations (4.1)–(4.2) in $L_1[x,\infty)$. A natural question arises as to whether, under these conditions, the potentials q, r also belong to $L_1[x,\infty)$ for all $x \in \mathbb{R}$ (i.e., $\int_x^\infty |q(s)|\,\mathrm{d}s < \infty$, $\int_x^\infty |r(s)|\,\mathrm{d}s < \infty$). As it turns out, there is no guarantee that this is the case unless somewhat more stringent conditions are required.

In the first part of this section, we shall follow the ideas of Marchenko (1963) – relative to the classical Schrödinger operator – and show that if the data satisfy for all $a \in \mathbb{R}$

$$\int_a^\infty (1+|x|)|F_j'(x)|\,\mathrm{d}x < \infty \tag{4.31}$$

then the potentials are integrable on (x,∞) for all $x \in \mathbb{R}$. The given condition is, however, quite severe and hence this setting might not be general enough. Consider, for example, the data

$$F(x) = \chi_{[0,1]}(x) \equiv \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$F(x) = \mathrm{e}^{-|x|}, \qquad F(x) = \frac{\sin(1/x)}{1+x^2}$$

or

$$F(x) = \frac{\sin(x)}{1+x^2}$$

which are natural integrable and bounded data, but fail to satisfy the condition (4.31) and hence are beyond the scope of the theory considered here.

In the next section we shall show how to get around this difficulty and prove the integrability of the potential under less stringent conditions on the data.

4.4.1. *Integrability of Potentials. I*

Let us consider the GLM equations (4.6) for $i = 1$ and $j = 1, 2$ ($j+1$, as usual, is intended mod 2). Introduce, for convenience,

$$\Phi_1(x,y) = K_{11}(x,y), \qquad \Phi_2(x,y) = K_{12}(x,y) \tag{4.32}$$

so that the equations can be written in vector form

$$\begin{aligned} &\overrightarrow{\Phi}(x,y) + \begin{pmatrix} 0 \\ F_2(x+y) \end{pmatrix} + \\ &\quad + \int_x^\infty \begin{pmatrix} \Phi_2(x,s)F_1(y+s) \\ \Phi_1(x,s)F_2(y+s) \end{pmatrix} \mathrm{d}s = 0, \quad y \geqslant x \end{aligned} \tag{4.33}$$

or

$$\begin{aligned} &\overrightarrow{\Phi}(x,x+y) + \begin{pmatrix} 0 \\ F_2(2x+y) \end{pmatrix} + \\ &\quad + \int_0^\infty \begin{pmatrix} \Phi_2(x,x+s)F_1(y+s+2x) \\ \Phi_1(x,x+s)F_2(y+s+2x) \end{pmatrix} \mathrm{d}s = 0, \quad y \geqslant 0 \end{aligned} \tag{4.34}$$

and

$$q(x) = -2K_{12}(x, x) \equiv -2\Phi_2(x, x). \tag{4.35}$$

The results of Sections 4.2 and 4.3 show that if $F_j \in L_1[x, \infty)$ for all $x \in \mathbb{R}$, then the operator

$$\Omega_x\colon\ \overrightarrow{\Phi}(x, x+y) \to -\int_0^\infty \begin{pmatrix} \Phi_2(x, x+s)F_1(y+s+2x) \\ \Phi_1(x, x+s)F_2(y+s+2x) \end{pmatrix} \mathrm{d}s \tag{4.36}$$

is compact both as an operator from $L_1[0, \infty)$ to itself and $L_2[0, \infty)$ to itself and Equation (4.34) has a unique solution in $L_1[0, \infty)$ (and $L_2[0, \infty)$ hence in $L_1[0, \infty) \cup L_2[0, \infty)$).

In other words, since we know that for any $x \in \mathbb{R}$, there is a unique solution to (4.33) with

$$\int_x^\infty |\Phi_j(x, y)|\, \mathrm{d}y < \infty, \qquad \int_x^\infty |\Phi_j(x, y)|^2\, \mathrm{d}y < \infty, \quad j = 1, 2$$

then there is a unique solution to (4.34) such that

$$\int_0^\infty |\Phi_j(x, x+y)|\, \mathrm{d}y < \infty, \qquad \int_0^\infty |\Phi_j(x, x+y)|^2\, \mathrm{d}y < \infty, \quad j = 1, 2.$$

Equation (4.34) can be written in a compact form as

$$(I + \Omega_x)\overrightarrow{\Phi}(x, x+y) = \overrightarrow{h}(x, y), \quad \overrightarrow{h}(x, y) = -\begin{pmatrix} 0 \\ F_2(2x+y) \end{pmatrix} \tag{4.37}$$

existence and uniqueness of solution to (4.37) in, say, $L_2[0, \infty)$, for all $x \in \mathbb{R}$, implies that $(I + \Omega_x)^{-1}$ exists for all $x \in \mathbb{R}$.

From Equations (4.17) and (4.21) it follows that

$$\frac{\|\Omega_x \overrightarrow{\Phi}\|_2}{\|\overrightarrow{\Phi}\|_2} \leqslant \left(\sum_{j=1,2} \left(\int_{2x}^\infty |F_j(s)|\, \mathrm{d}s \right)^2 \right)^{1/2}, \qquad \frac{\|\Omega_x \overrightarrow{\Phi}\|_1}{\|\overrightarrow{\Phi}\|_1} \leqslant \sum_{j=1,2} \int_{2x}^\infty |F_j(s)|\, \mathrm{d}s$$

one also has by the triangular inequality

$$\begin{aligned} \frac{\|(I + \Omega_x)\overrightarrow{\Phi}\|_2}{\|\overrightarrow{\Phi}\|_2} &\leqslant 1 + \left(\sum_{j=1,2} \left(\int_{2x}^\infty |F_j(s)|\, \mathrm{d}s \right)^2 \right)^{1/2}, \\ \frac{\|(I + \Omega_x)\overrightarrow{\Phi}\|_1}{\|\overrightarrow{\Phi}\|_1} &\leqslant 1 + \sum_{j=1,2} \int_{2x}^\infty |F_j(s)|\, \mathrm{d}s \end{aligned}$$

hence

$$\lim_{x \to \infty} \|I + \Omega_x\|_{1,2} \leqslant 1.$$

From the other side,

$$\|(I+\Omega_x)^{-1}\|_{1,2} \leqslant \frac{1}{1-\|\Omega_x\|_{1,2}} \quad \text{if } \|\Omega_x\|_{1,2} < 1$$

and since $\|\Omega_x\|_j \leqslant \|\overrightarrow{F}\|_{j,[2x,\infty)}$ for both $j = 1, 2$, for the L_1-norm one has in particular that

$$\lim_{x\to\infty} \|(I+\Omega_x)^{-1}\|_1 \leqslant \lim_{x\to\infty} \frac{1}{1-\|\overrightarrow{F}\|_{1,[2x,\infty)}} = 1.$$

Since the L_1-norm of the operator $(I+\Omega_x)^{-1}$ is finite for all finite x and bounded as $x \to \infty$, for every $a \in \mathbb{R}$, one has

$$\sup_{x\geqslant a} \|(I+\Omega_x)^{-1}\|_1 = C(a) < \infty. \tag{4.38}$$

Now, from the Equation (4.37)

$$\begin{aligned} &\|\overrightarrow{\Phi}(x, x+y)\|_1 \\ &\quad = \sum_{j=1,2} \int_0^\infty |\Phi_j(x, x+y)|\,\mathrm{d}y \leqslant \|(I+\Omega_x)^{-1}\| \int_0^\infty |F_2(y+2x)|\,\mathrm{d}y. \end{aligned}$$

In particular, for both $j = 1, 2$

$$\int_0^\infty |\Phi_j(x, x+y)|\,\mathrm{d}y \leqslant \|(I+\Omega_x)^{-1}\|_1 \int_0^\infty |F_2(y+2x)|\,\mathrm{d}y. \tag{4.39}$$

Let us introduce

$$\tau_j(x) = \int_x^\infty |F_j'(s)|\,\mathrm{d}s, \qquad \tilde{\tau}_j(x) = \int_x^\infty \tau_j(s)\,\mathrm{d}s. \tag{4.40}$$

Note that if $F_j' \in L_{1,1}[x,\infty)$ for all $x \in \mathbb{R}$, the previous functions are well defined and they are both decreasing functions of their argument. Indeed, an integration by parts yields

$$\begin{aligned} \tilde{\tau}_j(x) &= s\tau_j(s)|_{s=x}^{s=\infty} - \int_x^\infty s\tau_j'(s)\,\mathrm{d}s \\ &= \lim_{x\to\infty} x\tau_j(x) - x\tau_j(x) + \int_x^\infty s|F_j'(s)|\,\mathrm{d}s \end{aligned}$$

and all terms are finite for $F_j' \in L_{1,1}[x,\infty)$.

Also, one has the following

$$|F_j(x)| = \left|\int_x^\infty F_j'(s)\,\mathrm{d}s\right| \leqslant \int_x^\infty |F_j'(s)|\,\mathrm{d}s \equiv \tau_j(x) \tag{4.41}$$

and consequently

$$\int_x^\infty |F_j(s)|\,\mathrm{d}s \leqslant \int_x^\infty \tau_j(s)\,\mathrm{d}s \equiv \tilde{\tau}_j(x). \tag{4.42}$$

Substituting the GLM equation for the second component into the first one, we get

$$\begin{aligned}&\Phi_2(x, x+y) + F_2(2x+y) - \\ &\quad - \int_0^\infty \mathrm{d}s'\, \Phi_2(x, x+s') \int_0^\infty \mathrm{d}s\, F_2(s+y+2x) F_1(s'+s+2x) = 0\end{aligned}$$

hence

$$\begin{aligned}&|\Phi_2(x, x+y)| \\ &\quad \leqslant |F_2(2x+y)| + \\ &\qquad + \int_0^\infty \mathrm{d}s'\, |\Phi_2(x, x+s')| \int_0^\infty \mathrm{d}s\, |F_2(s+y+2x)| |F_1(s'+s+2x)| \\ &\quad \leqslant |F_2(2x+y)| + \int_0^\infty \mathrm{d}s'\, |\Phi_2(x, x+s')| \int_0^\infty \mathrm{d}s\, \tau_2(y+2x) \tau_1(s+2x),\end{aligned}$$

where we used (4.41) and the fact that s, s' are positive and τ_j is decreasing. Consequently,

$$\begin{aligned}|\Phi_2(x, x+y)| &\leqslant \tau_2(2x+y)\left[1 + \tilde{\tau}_1(2x) \int_0^\infty \mathrm{d}s'\, |\Phi_2(x, x+s')|\right] \\ &\leqslant \tau_2(2x+y)[1 + \tilde{\tau}_1(2x) C(x) \tilde{\tau}_2(2x)],\end{aligned} \tag{4.43}$$

where we used (4.39) to get

$$\int_0^\infty |\Phi_2(x, x+s')|\,\mathrm{d}s' \leqslant C(x)\tilde{\tau}_2(2x)$$

being $C(x) \equiv \|(I + \Omega_x)^{-1}\|_1$. Then we have

$$\begin{aligned}\int_a^\infty |\Phi_2(x, x)|\,\mathrm{d}x &\leqslant \int_a^\infty \tau_2(2x)[1 + C(x)\tilde{\tau}_1(2x)\tilde{\tau}_2(2x)]\,\mathrm{d}x \\ &\leqslant \left[1 + \Big(\sup_{x \geqslant a} C(x)\Big) \tilde{\tau}_1(2a)\tilde{\tau}_2(2a)\right] \int_a^\infty \tau_2(2x)\,\mathrm{d}x \\ &= \frac{1}{2}\left[1 + \Big(\sup_{x \geqslant a} C(x)\Big) \tilde{\tau}_1(2a)\tilde{\tau}_2(2a)\right] \tilde{\tau}_2(2a)\end{aligned}$$

which, taking into account (4.38), then proves that $q(x) \in L_1[a, \infty)$ for any $a \in \mathbb{R}$.

Note that if we use “right” data, we get a potential in $L_1(-\infty, a]$ for all $a \in \mathbb{R}$, and then we can use the fact that “right” and “left” data are uniquely determined one in terms of the others to show that the potentials indeed coincide and to get a solution which is $L_1(\mathbb{R})$.

4.4.2. *Integrability of Potentials. II*

As we shall now show, integrability of the potential follows under much weaker conditions on the data. To this end, given $f\colon \mathbb{R} \to \mathbb{R}$, we define the function $\breve{f}$ as

$$\breve{f}(y) = \sup_{x \geqslant 0} |f(x+y)|. \tag{4.44}$$

PROPOSITION 5. *The operation* $\breve{}$ *has the following properties*

(i) $|f| \leqslant \breve{f}$, $\|f\|_p \leqslant \|\breve{f}\|_p$, $1 \leqslant p \leqslant \infty$. *Further* $\|f\|_\infty = \|\breve{f}\|_\infty$. *It follows, in particular, that*

$$f \in \breve{L}_1(\mathbb{R}) \Rightarrow \|f\|_\infty, \|f\|_1 < \infty;\ \lim_{x\to\infty} f(x) = 0,$$

(ii) $\breve{f} \geqslant 0$,
(iii) *if f is decreasing then* $\breve{f} = |f|$,
(iv) $\breve{f}$ *is decreasing,*
(v) $(\breve{\breve{f}}) = \breve{f}$,
(vi) $\breve{f} \leqslant \|f\|_\infty$,
(vii) $|f| \leqslant |g| \Rightarrow \breve{f} \leqslant \breve{g}$.

Proof. (i)–(iii) are straightforward.

(iv) One can write $\breve{f}(y) \equiv \sup_{x\geqslant 0} |f(x+y)| = \sup_{x\in A_y} |f(x)|$ where $A_y \equiv [y,\infty)$. As y increases the sequence of intervals A_y decreases: $y < z \Rightarrow A_z \subset A_y$ and hence $\sup_{x\in A_z} |f(x)| \leqslant \sup_{x\in A_y} |f(x)|$.

(v) Indeed one can write $(\breve{\breve{f}}) = \breve{\psi}$ where $\psi(y) \equiv \breve{f}(y)$ is decreasing. Thus by (iii) the result follows.

(vi) and (vii) are also straightforward. □

Consider now Equations (4.6) with the notation (4.32). To adapt the ideas to this vector case we use, as above, $|\vec{\Phi}(x,s)| := |\Phi_1(x,s)| + |\Phi_2(x,s)|$ and the same for $\vec{F}(x)$. Besides $|\breve{\vec{F}}(x)| \equiv \sup_{x\geqslant 0} |\vec{F}(x+y)|$. The following result holds.

THEOREM 6. *Suppose the data satisfies:* $\breve{F}_j \in L_{1,[a,\infty)}, \forall a \in \mathbb{R}$ *and* $j = 1, 2$. *Then* q, $\breve{q} \in L_{1,[a,\infty)}, \forall a \in \mathbb{R}$, *and the following estimates hold:*

$$\|q\|_{1,[a,\infty)} \leqslant \|\breve{q}\|_{1,[a,\infty)} \leqslant (1+S_a)\int_{2a}^{\infty} |\breve{\vec{F}}(y)|\,\mathrm{d}y < \infty, \tag{4.45}$$

where

$$S_a = \sup_{x\geqslant a} \int_x^\infty |\vec{\Phi}(x,s)|\,\mathrm{d}s < \infty.$$

Proof. We first prove the weaker claim $q \in L_{1,[a,\infty)}$, $\forall a \in \mathbb{R}$. Note

$$|q(x)| \leqslant 2|\overrightarrow{F}(2x)| + 2\int_x^\infty |\overrightarrow{\Phi}(x,s)||\overrightarrow{F}(x+s)|\,\mathrm{d}s$$

and

$$\begin{aligned}
&\int_x^\infty |\overrightarrow{\Phi}(x,s)||\overrightarrow{F}(x+s)|\,\mathrm{d}s \\
&\quad\leqslant \int_x^\infty |\overrightarrow{\Phi}(x,s)|\sup_{s\geqslant 0}|\overrightarrow{F}(2x+s)|\,\mathrm{d}s = \int_x^\infty |\overrightarrow{\Phi}(x,s)||\overrightarrow{\overrightarrow{F}}(2x)|\,\mathrm{d}s \\
&\quad\equiv C(x)|\overrightarrow{\overrightarrow{F}}(2x)|,
\end{aligned}$$

where

$$C(x) \equiv \|\overrightarrow{\Phi}\|_{1,[x,\infty)} = \int_x^\infty |\overrightarrow{\Phi}(x,s)|\,\mathrm{d}s.$$

Therefore

$$|q(x)| \leqslant 2|\overrightarrow{F}(2x)| + 2C(x)|\overrightarrow{\overrightarrow{F}}(2x)|.$$

$C(x)$ is finite for all finite x since $\overrightarrow{\Phi}(x,y) = -(I+\Omega_x)^{-1}\overrightarrow{h}(x+y)$ with $\overrightarrow{h}(x) = (0, F_2(x))^{\mathrm{T}}$ is the only solution of GLM in $L_{1,[x,\infty)}$ and

$$\|\overrightarrow{\Phi}\|_{1,[x,\infty)} \leqslant \|(I+\Omega_x)^{-1}\|_1 M(x), \tag{4.46}$$

where

$$M(x) \equiv \int_{2x} |\overrightarrow{F}(y)|\,\mathrm{d}y. \tag{4.47}$$

Still,

$$S_a = \sup_{x\geqslant a} C(x) \equiv \sup_{x\geqslant a}\int_x^\infty |\overrightarrow{\Phi}(x,s)\,\mathrm{d}s$$

could, in principle, blow up as $x \to \infty$. To show this is not the case, note

$$\|(I+\Omega_x)^{-1}\|_1 \leqslant \frac{1}{1-\|\Omega_x\|_1} \quad \text{if } \|\Omega_x\|_1 < 1. \tag{4.48}$$

By the estimate (4.21) we have, also

$$\|\Omega_x\|_1 \leqslant M(x) \tag{4.49}$$

and since $\lim_{x\to\infty} M(x) = 0$, the condition $\|\Omega_x\|_1 < 1$ is satisfied simply taking x long enough. Therefore, taking into account (4.46)–(4.49)

$$\lim_{x\to\infty} \|\overrightarrow{\Phi}\|_{1,[x,\infty)} \leqslant \lim_{x\to\infty} \frac{M(x)}{1-M(x)} = 0.$$

Thus $S_a < \infty$. Next,

$$\begin{aligned}
&\int_a^\infty |q(x)|\,\mathrm{d}x \\
&\quad \leqslant \int_{2a}^\infty |\vec{F}(x)|\,\mathrm{d}x + 2\int_a^\infty \mathrm{d}x\,|\vec{F}(2x)| \int_x^\infty |\vec{\Phi}(x,s)|\,\mathrm{d}s \\
&\quad \leqslant \int_{2a}^\infty |\vec{F}(x)|\,\mathrm{d}x + 2\int_a^\infty \mathrm{d}x\,|\vec{F}(2x)| \int_x^\infty |\vec{\Phi}(x,s)|\,\mathrm{d}s \\
&\quad \leqslant (1+S_a)\int_{2a}^\infty |\vec{F}(x)|\,\mathrm{d}x.
\end{aligned}$$

Thus $q \in L_{1,(a,\infty)}, \forall a \in \mathbb{R}$. We next prove the stronger claim $\breve{q} \in L_1[a,\infty)$. Note

$$|q(x)| \leqslant 2|\vec{F}(2x)| + 2|\vec{F}(2x)|C(x) \leqslant 2(1+S_a)|\vec{F}(2x)|$$

hence

$$|\breve{q}(x)| \leqslant 2(1+S_a)|\breve{\breve{\vec{F}}}(2x)| = 2(1+S_a)|\breve{\vec{F}}(2x)|,$$

where we used (v) of Prop. 5. Consequently, for any $a \in \mathbb{R}$,

$$\int_a^\infty \breve{q}(x)\,\mathrm{d}x \leqslant (1+S_a)\int_a^\infty |\breve{\vec{F}}(2x)|\,\mathrm{d}x < \infty. \qquad \square$$

We shall consider the following class of functions $\breve{L}_{1,(a,\infty)} \equiv \{f\colon \mathbb{R} \to \mathbb{R} \mid \breve{f} \in L_{1,(a,\infty)}\}$. We can reformulate the former result as follows.

THEOREM 7 (Integrability of NLS potentials). *Suppose the data satisfies*: $|\vec{F}| \in \breve{L}_{1,(a,\infty)}, \forall a \in \mathbb{R}$. *Then* $q \in \breve{L}_1(a,\infty), \forall a \in \mathbb{R}$ *and the bound* (4.45) *applies.*

We proved in Section 4.4.1 that the condition $F_j \in L'_{1,1}[a,\infty), \forall a \in \mathbb{R}$, where $L'_{1,1}[a,\infty) \equiv \{f : \int_a^\infty (1+|x|)|f'(x)|\,\mathrm{d}x < \infty\}$, guarantees integrability of potentials. This condition is, however, quite "severe". As we shall see now, condition $F_j \in \breve{L}_1[a,\infty)$ considerably "broadens" the class of admissible data.

PROPOSITION 6. *Suppose* $f \in L'_{1,1}[a,\infty)$ *for some* $a \in \mathbb{R}$. *Then it is also* $f \in \breve{L}_1[a,\infty)$, *viz.* $L'_{1,1}[a,\infty) \subset \breve{L}_1[a,\infty)$ *where* $L'_{1,1}[a,\infty) \equiv \{f : \int_a^\infty (1+|x|)|f'(x)|\,\mathrm{d}x < \infty\}$.

Proof.

$$\begin{aligned}
&|f(x)| \leqslant \int_x^\infty |f'(z)|\,\mathrm{d}z \\
&\quad \Rightarrow \sup_{x\geqslant 0}|f(x+y)| \leqslant \sup_{x\geqslant 0}\int_{x+y}^\infty |f'(z)|\,\mathrm{d}z = \int_y^\infty |f'(z)|\,\mathrm{d}z, \\
&\int_a^\infty \breve{f}(y)\,\mathrm{d}y \leqslant \int_a^\infty \mathrm{d}y \int_y^\infty |f'(z)|\,\mathrm{d}z = \int_a^\infty (z-a)||f'(z)|\,\mathrm{d}z < \infty. \qquad \square
\end{aligned}$$

PROPOSITION 7. *Suppose that f satisfies*

(i) $f \in L_1[a,\infty) \cap L_\infty[a,\infty)$,
(ii) *there exists* $M \in [a,\infty)$ *such that* $|f|$ *decreases for* $x \geqslant M$.

Then $f \in \breve{L}_1[a,\infty)$.

Proof. Note that $|f(y+x)|$ is decreasing whenever $y \geqslant M$ for all $x \geqslant 0$. Then we have

$$\breve{f}(y) = |f(y)|, \quad y \geqslant M$$

and therefore, using (vi) of Prop. 5

$$\begin{aligned}\int_a^\infty \breve{f}(y)\,\mathrm{d}y &= \int_a^M \breve{f}(y)\,\mathrm{d}y + \int_M^\infty \breve{f}(y)\,\mathrm{d}y \\ &\leqslant \int_a^M \|f\|_{1,[a,\infty)}\,\mathrm{d}y + \int_M^\infty |f(y)|\,\mathrm{d}y \\ &\leqslant (M-a)\|f\|_{\infty,[a,\infty)} + \|f\|_{1,[a,\infty)} < \infty. \qquad \square\end{aligned}$$

PROPOSITION 8. *Suppose there exists a function $g \geqslant 0$ such that f satisfies*

(i) $g \in L_1[a,\infty) \cap L_\infty[a,\infty)$,
(ii) *there exists* $M \in [a,\infty)$ *such that g decreases for* $x \geqslant M$,
(iii) $|f| \leqslant g$.

Then $f \in \breve{L}_1[a,\infty)$.

Proof. As in the proof of the previous proposition,

$$\begin{aligned}&\breve{f} \leqslant \|f\|_{\infty,[a,\infty)} \leqslant \|g\|_{\infty,[a,\infty)}, \quad y \leqslant M,\\ &\breve{f}(y) \leqslant \breve{g}(y) \equiv g(y), \quad y \geqslant M\end{aligned}$$

since $g(y)$ is decreasing whenever $y \geqslant M$. Hence

$$\begin{aligned}\int_a^\infty \breve{f}(y)\,\mathrm{d}y &= \int_a^M \breve{f}(y)\,\mathrm{d}y + \int_M^\infty \breve{f}(y)\,\mathrm{d}y \\ &\leqslant (M-a)\|g\|_{\infty,[a,\infty)} + \|g\|_{1,[a,\infty)} < \infty. \qquad \square\end{aligned}$$

Remark. With $g(x) = \frac{1}{1+x^2}$, we obtain that the function $f(x) = \frac{\sin x}{1+x^2}$ is in $\breve{L}_1[a,\infty)$ and hence is a bona fide data in the GLM equations. Note that $f \notin L'_{1,1}[a,\infty)$.

A convenient workable statement of our results is the following.

COROLLARY 3 (Characterization of data). *Suppose the data $F_j(x)$ for $j = 1,2$ satisfies*

(i) *either* (4.31),

(ii) *or, there exists a function g such that*

(iia) $g \in L_1[a, \infty) \cap L_\infty[a, \infty)\ \forall a \in \mathbb{R}$;
(iib) *g decreases for sufficiently large x;*
(iic) $|F_j| \leqslant g$.

Then $F_j \in \breve{L}_1[a, \infty)\ \forall a \in \mathbb{R}$ *and then it is also* $q, r, \breve{r}, \breve{q} \in L_{1,[a,\infty)}, \forall a \in \mathbb{R}$.

Appendix A

THEOREM 8 (Wiener). *Let* $l(x) \in L_1(\mathbb{R})$, $\hat{l}(k)$ *the corresponding Fourier transform. Let* $f\colon C \to \mathbb{R}$ *a given function, holomorphic in a set* $H \subset C$. *Suppose that*

$$\operatorname{Im} \hat{l}(k) \subset H,$$

where $\operatorname{Im} \hat{l}(k)$ *is the set of values, or range, that this function may take. Then there exists a function* $h \in L_1(\mathbb{R})$ *such that for all* $k \in \mathbb{R}$

$$f(\hat{l}(k)) = \hat{h}(k).$$

For a proof of this result, see (Chandrasekharan, 1989).

LEMMA 1. *For any* $F \in L_1[x, \infty)$

$$\lim_{R\to\infty} \int_{R+x}^{\infty} |F(y)|\, \mathrm{d}y = 0.$$

Proof. Let $g_R(y) = F(y)\theta(y - R - x)$. Then $\lim_{R\to\infty} g_R(y)$ exists and $\sup_{R\geqslant x} |g_R(y)| \leqslant |F(y)|$ with $F \in L_1[x, \infty)$. By Lebesgue's theorem it follows

$$\lim_{R\to\infty} \int_{R+x}^{\infty} |F(y)|\, \mathrm{d}y = \int \lim_{R\to\infty} |g_R(y)|\, \mathrm{d}y = 0. \qquad \square$$

LEMMA 2. *Any* $F \in L_1[x, \infty)$ *is* L_1*-continuous, i.e.,*

$$\lim_{h\to 0} \int_{x}^{\infty} |F(x+h) - F(x)|\, \mathrm{d}x = 0.$$

Consider now the closed compact interval $[x, \infty]$ and let

$$B \equiv \{f : [x, \infty] \to \mathbb{C} \mid f(y) = 0 \text{ for } y < x \text{ and } \|f\|_2 < \infty\}$$

with

$$B \subset L_2[x, \infty] \equiv \{f : [x, \infty] \to \mathbb{C} \mid \|f\|_2 < \infty\}.$$

Let us recall a relevant definition.

DEFINITION 1. For a given $x \in \mathbb{R}$ and $p \geqslant 1$, a function Ψ defined on $L_p[x, \infty]$ is L_p-continuous if

$$\text{(i)} \quad \lim_{h\to 0} \|\Psi(x+h) - \Psi(x)\|_p = 0, \tag{A.1}$$

$$\text{(ii)} \quad \lim_{R\to\infty} \int_R^\infty |\Psi(y)|^p \,\mathrm{d}y = 0. \tag{A.2}$$

LEMMA 3. *Assume both* $h, F \in L_1(\mathbb{R})$. *Then*

$$\int \mathrm{d}x\, \mathrm{e}^{i\xi x}\left\{\int h(y)F(x+y)\,\mathrm{d}y\right\} = \hat{F}(\xi)\hat{h}(-\xi),$$

where $\hat{h}, \hat{F}$ *are the Fourier transforms of* h, F *and both* $\hat{h}, \hat{F} \in L_\infty(\mathbb{R})$.

Proof.

$$\begin{aligned}
&\int \mathrm{d}y\left\{\int \mathrm{d}x\left|\mathrm{e}^{i\xi x}h(y)F(x+y)\right|\right\} \\
&\quad\leqslant \int \mathrm{d}y\,|h(y)| \int \mathrm{d}x\,|F(x+y)| \\
&\quad\equiv \int \mathrm{d}y\,|h(y)| \int \mathrm{d}z\,|F(z)| \equiv \|h\|_1\|F\|_1 < \infty.
\end{aligned}$$

Thus Fubini's theorem yields

$$\begin{aligned}
&\int \mathrm{d}y\left\{\int \mathrm{d}x\,\mathrm{e}^{i\xi x}h(y)F(x+y)\right\} \\
&\quad\equiv \int \mathrm{d}y\,h(y)\mathrm{e}^{-i\xi y}\int \mathrm{d}x\,\mathrm{e}^{i\xi(x+y)}F(x+y) \\
&\quad= \int \mathrm{d}y\,h(y)\mathrm{e}^{i\xi y}\int \mathrm{d}z\,\mathrm{e}^{-i\xi z}F(z) = \hat{F}(\xi)\int h(y)\,\mathrm{e}^{i\xi y}\,\mathrm{d}y \\
&\quad= \hat{F}(\xi)\hat{h}(-\xi).
\end{aligned}$$

□

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References

Ablowitz, M. J. and Segur, H. (1981) *Solitons and the Inverse Scattering Transform 4*, SIAM.

Ablowitz, M. J. and Clarkson, P. A. (1991) *Solitons Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc. Lecture Notes Series 149, Cambridge University Press.

Ablowitz, M. J., Prinari, B. and Trubatch, A. D. (2004) *Discrete and Continuous Nonlinear Schrödinger Systems*, London Math. Soc. Lecture Notes Series 302, Cambridge University Press.

Agranovic, A. K. and Marchenko, V. A. (1963) *The Inverse Problem of Scattering Theory*, Gordon and Breach, London.

Beals, R. and Coifman, R. R. (1984) Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.* **37**, 39.

Beals, R. and Coifman, R. R. (1985) Inverse scattering and evolution equations, *Comm. Pure Appl. Math.* **38**, 29.

Chadan, K. and Sabatier, P. C. (1977) *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag.

Chandrasekharan K. (1989) *Classical Fourier Transforms*, Springer-Verlag.

Deift, P. and Trubowitz, E. (1979) Inverse Scattering on the line, *Comm. Pure Appl. Math.* **32**, 121.

Faddeev, L. (1963) The inverse problem in the quantum theory of scattering, *J. Math. Phys.* **4**, 72.

Faddeev, L. (1967) Properties of the S-matrix of the one-dimensional Schrödinger equation, *Proc. Amer. Math. Soc.* **65**(2), 139.

Levitan, B. M. and Sargsjan, I. S. (1975) *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators*, Amer. Math. Soc., Providence, RI.

Levitan, B. M. (1980) Sufficient conditions for solvability of the inverse problem of scattering theory in the entire line, *Math. USSR Sb.* **36**, 323.

Marchenko, V. A. (1986) *Sturm–Liouville Operators and Applications*, Birkhäuser-Verlag.

Melin, V. A. (1985) Operator methods for inverse scattering on the real line, *Comm. Partial Differential Equations* **10**, 677.