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RECENT RESULTS FOR GENERALIZED EXPONENTIAL INTEGRALS

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Abstract—Basic properties of the exponential-integral function of real order, $E_\nu(x)$, and relevant expressions for evaluating this special function are presented. The mathematical results have been essentially obtained by generalizing known formulae valid for the usual exponential-integral, $E_n(x)$.

1. INTRODUCTION

The generalized exponential-integral function, $E_\nu(x)$, expressed as follows [1, 2]:

$$E_\nu(x) = \int_1^\infty e^{-tx} t^{-\nu}\, dt = \int_0^1 e^{-x/t} t^{\nu-2}\, dt \quad (x > 0, \nu \in R), \tag{1}$$

is a generalization to the real order, ν, of the usual exponential integral defined by Schloemilch as

$$E_n(x) = \int_1^\infty e^{-tx} t^{-n}\, dt \quad (x > 0, n \in N). \tag{2}$$

Function (1), whose behaviour is shown in Fig. 1, is a positive function (whenever $\nu \in R$), which is frequently used in astrophysics, neutron physics, quantum chemistry and other applied sciences. As far as its numerical evaluation is concerned, recently a constructive method has been developed for this special function, mainly based on suitable expansions and some basic properties of $E_\nu(x)$ [3, 4]. Some mathematical relations have been previously deduced in Ref. [1] within the framework of an analytical treatment concerning more general transcendental functions.†

Due to the practical importance of $E_\nu(x)$ function, it deserves some interest to obtain further properties and relations for this transcendental function and its evaluation, in addition to the significant results recently obtained, which are here collected in a unified context.

The results are essentially derived by generalizing known relations valid for the usual exponential integral, $E_n(x)$ [2, 5, 6].

The paper is organized as follows: in Section 2 we present some basic properties and in Section 3 the Taylor series expansion, while Section 4 is concerned with some special functions related to $E_\nu(x)$.

Moreover, in Section 5 we derive a few series representations valid for the region $(0 < x < 1)$, in Section 6 we present a polynomial expansion and then we obtain a formal representation valid in the region $(x > 0, \frac{1}{2} < \nu < 2)$. Finally, Section 7 is concerned with a continued fraction, while Section 8 presents asymptotic expansions.

†It is worth recalling that the notation, $E_\nu(x)$, used in the present paper for the generalized exponential integral corresponds to Milgram's function $E_\nu^0(x)$ [1].

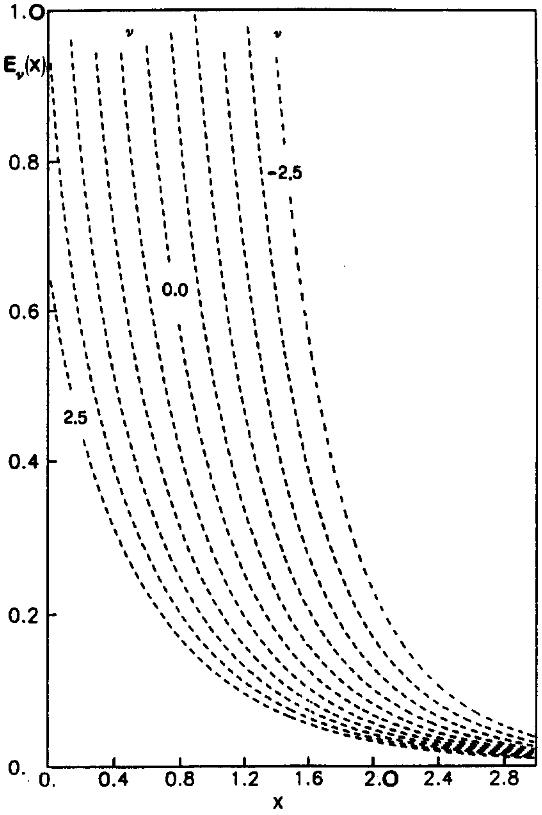

Fig. 1. Generalized exponential-integral function, $E_\nu(x)$, vs order, ν, and argument, x; $\nu = -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5$.

2. BASIC RELATIONS AND INEQUALITIES

As for the properties of $E_\nu(x)$, the following differential formula holds from equation (1)

$$\frac{d}{dx} E_\nu(x) = -E_{\nu-1}(x) \tag{3}$$

and, more generally,

$$\frac{d^m}{dx^m} E_\nu(x) = (-1)^m E_{\nu-m}(x). \tag{4}$$

The recurrence relation, deduced from equation (1) by means of a suitable integration by parts,

$$\nu E_{\nu+1}(x) = e^{-x} - x E_\nu(x) = e^{-x} + x \frac{d}{dx} E_{\nu+1}(x), \tag{5}$$

generalizes the well-known result when ν is integer.

From formula (5), it easily follows that

$$E_0(x) = \frac{e^{-x}}{x}. \tag{6}$$

Equation (6) can be used as starting point for recursive evaluation of $E_\nu(x)$, $(\nu = 0, -1, -2, \ldots)$.

Other special values of particular interest are the following:

$$E_\nu(0) = \begin{cases} \dfrac{1}{\nu - 1}, & (\nu > 1), \\ \infty, & (-\infty < \nu \leqslant 1). \end{cases} \tag{7}$$

As for inequalities, from definition (1) it immediately results that

$$E_\nu(x) > E_{\nu+1}(x) \quad (x > 0, \nu \in R). \tag{8}$$

Considering the recurrence (5) twice and the previous inequality (8), it follows that

$$\left(\frac{\nu - 1}{\nu}\right) E_\nu(x) < E_{\nu+1}(x) \quad (x > 0, \nu \in R^+). \tag{9}$$

Again, making use of the Cauchy–Bunyakovsky inequalities for the quantities $\alpha(t) = \exp[-tx/2]t^{-(\nu-1)/2}$ and $\beta(t) = \exp[-tx/2]t^{-(\nu+1)/2}$, one gets

$$E_{\nu+1}(x)E_{\nu-1}(x) > E_\nu^2(x) \quad (x > 0, \nu \in R^+). \tag{10}$$

Moreover, by differentiating the ratio $[E_\nu(x)/E_{\nu-1}(x)]$ and assuming formulae (3) and (10), another relation holds:

$$\frac{d}{dx}\left[\frac{E_\nu(x)}{E_{\nu-1}(x)}\right] > 0 \quad (x > 0, \nu \in R^+). \tag{11}$$

The following inequality

$$e^x E_\nu(x) \geqslant \frac{1}{\nu + x} \quad (x > 0, \nu \in R^+), \tag{12}$$

derives from recurrence (5), whose use combined with formula (8) implies moreover:

$$e^x E_\nu(x) \leqslant \frac{1}{\nu + x - 1} \quad (x > 0, \nu \geqslant 1). \tag{13}$$

Formulae (8)–(13) generalize known results valid for ν integer [5].

3. TAYLOR EXPANSION

By considering the following Taylor series:

$$E_\nu(x - y) = \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \frac{d^k}{dx^k} E_\nu(x) \quad (x > 0, \nu \in R), \tag{14}$$

and making use of equation (4), one obtains

$$E_\nu(x - y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} E_{\nu-k}(x) \quad (x > 0, \nu \in R), \tag{15}$$

which can be successfully used [3, 4] in the evaluation of $E_\nu(x)$, taking into account that the $E_{\nu-k}(x)$ values are obtainable, for both positive and negative values of the index, from recurrence (5).

4. CONNECTION WITH OTHER SPECIAL FUNCTIONS

From definition (1) of the generalized exponential integral, it follows that $\mathrm{E}_\nu(x)$ can also be expressed in terms of the incomplete gamma function, $\Gamma(a, x)$ as follows [1]:

$$\mathrm{E}_\nu(x) = x^{\nu-1}\Gamma(1-\nu, x), \tag{16}$$

and in terms of the incomplete gamma function, $\gamma(a, x)$, as

$$\mathrm{E}_\nu(x) = x^{\nu-1}[\Gamma(1-\nu) - \gamma(1-\nu, x)] \quad (\nu \neq 1, 2, 3, \dots), \tag{17}$$

where $\Gamma(a)$ is the usual Euler gamma function.

Further relations with other special functions can be easily derived from equations (16) and (17). More precisely, taking into account that the incomplete gamma function, $\Gamma(a, x)$, can be expressed as follows [7, pp. 337 and 267]

$$\Gamma(a, x) = \mathrm{e}^{-x}x^a\Psi(1, 1+a; x) = \mathrm{e}^{-x}\Psi(1-a, 1-a; x), \tag{18}$$

where $\Psi(b, c; x)$ is a confluent hypergeometric function of the second kind, from equation (16) one gets [1]

$$\mathrm{E}_\nu(x) = \mathrm{e}^{-x}x^{\nu-1}\Psi(\nu, \nu; x) = \mathrm{e}^{-x}\Psi(1, 2-\nu; x). \tag{19}$$

Analogously, since the incomplete gamma function, $\gamma(a, x)$, is a special case of the Kummer function, $\Phi(b, c; x)$, [8, p. 160]

$$\gamma(a, x) = \frac{x^a}{a}\,\mathrm{e}^{-x}\Phi(1, a+1; x), \tag{20}$$

by introducing the Tricomi version [8], $\Phi^*(a, c; x) = \Phi(a, c; x)/\Gamma(c)$, of the Kummer function, which is entire for every value of each argument, one obtains [4]:

$$\mathrm{E}_\nu(x) = \Gamma(1-\nu)[x^{\nu-1} - \mathrm{e}^{-x}\Phi^*(1, 2-\nu; x)] \quad (\nu \neq 1, 2, 3, \dots). \tag{21}$$

Formula (21) is of interest for computational purposes, because the function $\Phi^*(b, c; x)$ can be expanded in a fast converging series [8, p. 41]

$$\Phi^*(b, c; x) = \mathrm{e}^{x/2}\sum_{m=0}^{\infty} a_m(x/2)^m T_{c+m-1}(kx) \quad (k = c/2 - b), \tag{22}$$

and therefore it results [4] that

$$\mathrm{E}_\nu(x) = \Gamma(1-\nu)\left[x^{\nu-1} - \mathrm{e}^{-x/2}\sum_{m=0}^{\infty} a_m(x/2)^m T_{m+1-\nu}(-\nu x/2)\right] \quad (\nu \neq 1, 2, 3, \dots). \tag{23}$$

Here, the coefficients, a_m, are defined by recursion as follows [8, 3]:

$$(n+1)a_{n+1} = (n-\nu+1)a_{n-1} + \nu a_{n-2} \quad (n = 2, 3, \dots), \quad a_0 = 1,\ a_1 = 0,\ a_2 = 1 - \nu/2, \tag{24}$$

and $T_r(t)$ are the Tricomi functions [8, 3], which are entire for every value of r and are related to the Bessel functions of the first kind, $J_r(t)$, [8]:

$$T_r(t) \equiv t^{-r/2}J_r(2\sqrt{t}). \tag{25}$$

By inserting in equation (23), the known expansion [8, 4] for the Tricomi functions

$$T_r(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(r+k+1)}\frac{t^k}{k!}, \tag{26}$$

one gets

$$\mathrm{E}_\nu(x) = x^{\nu-1}\Gamma(1-\nu) - \mathrm{e}^{-x/2} \sum_{m,k=0}^{\infty} a_m (x/2)^{m+k} \frac{\nu^k}{k!(1-\nu)_{m+1+k}} \quad (\nu \neq 1, 2, 3, \ldots), \tag{27}$$

where $(z)_h$ is the usual Pochhammer symbol.

In the next section we discuss the relevant formulation for $\mathrm{E}_\nu(x)$, to be used for practical evaluation of the generalized exponential integral in the range $(0 < x < 1)$.

5. SERIES EXPANSIONS FOR THE REGION $(0 < x < 1)$

In the case $(0 < x < 1)$, the involved formulation for $\mathrm{E}_\nu(x)$ $(\nu \in R)$ is essentially derived [3, 4] from the use of equation (5) combined with proper series expansions for a suitable starting value, $\mathrm{E}_{\nu_0}(\nu_0)$, with $0 < \nu_0 \leqslant 1$.

More precisely, apart from the case $\mathrm{E}_\nu(x), (\nu = 0, -1, \ldots)$ considered above, the starting element, $\mathrm{E}_{\nu_0}(\nu_0)$, ($\nu_0 = 1 - H(\nu) + \nu - [\nu]$ for $\nu \neq [\nu]$ and $\nu_0 = 1$, otherwise)† can be expressed by different series expansions according as $0 < \nu_0 \leqslant d$ $(d = 0.9)$ or otherwise.

In particular, when $0 < \nu_0 \leqslant d$, $\mathrm{E}_{\nu_0}(\nu_0)$ can be defined by means of expression (27), since in the considered region the relevant series is positive and converges faster than outside.

In the remaining domain $(d < \nu_0 \leqslant 1)$ the following formulation can be used for $\mathrm{E}_{\nu_0}(\nu_0)$, based on Gautschi's recipes [9], valid in the considered region:

$$\mathrm{E}_\nu(x) = -x^{\nu-1}\left[\frac{g_1(1-\nu)}{1+(1-\nu)g_1(1-\nu)} + g_2(x, 1-\nu)\right] - \sum_{m=1}^{\infty} \frac{(-1)^m x^m}{(1-\nu+m)m!}. \tag{28}$$

Here,

$$g_1(\mu) = \frac{1}{\mu}\left[\frac{1}{\mu\Gamma(\mu)} - 1\right] \quad (\mu < 1), \tag{29}$$

and

$$g_2(x, \mu) = \frac{x^\mu - 1}{\mu} = \begin{cases} \dfrac{\mathrm{e}^{\mu \ln x} - 1}{\mu \ln x} \ln x, & \text{for } |\mu \ln x| \geqslant 1, \\[2ex] \left[1 + \displaystyle\sum_{m=1}^{\infty} \frac{(\mu \ln x)^m}{(m+1)!}\right] \ln x, & \text{otherwise.} \end{cases} \tag{30}$$

In practical evaluation, $g_1(\mu)$ can be suitably calculated by making use of the power series expansion for $1/\Gamma(\mu)$, whose coefficients are tabulated in Ref. [10].

At this point, once one has expressed $\mathrm{E}_{\nu_0}(\nu_0)$, in the case $\nu \neq \nu_0$, the relevant formulation for $\mathrm{E}_\nu(x)$ can be obtained making use of recurrence (5).

In particular, one obtains different formulations according as $\nu \in N_0$, $\nu \in (R^+ - N)$, or $\nu \in R^-$.

In the former case, the use of recursion (5) yields the following relation [6]:

$$(n-1)!\mathrm{E}_n(x) = (-x)^{n-1}\mathrm{E}_1(x) + \mathrm{e}^{-x} \sum_{s=0}^{n-2} (n-2-s)!(-x)^s. \tag{31}$$

According to this formula, one can get the series representation for $\mathrm{E}_n(x)$, $(n \in N_0)$, from a series expansion valid for $\mathrm{E}_1(x)$. In particular, equation (28) can be successfully used, as done in Refs [3, 4].

†Here, $H(w)$ denotes Heaviside's step function and $[w]$ the truncated part of w.

Analogously, in the case of positive non-integer values of ν, $\nu = n + \alpha, n \in N, 0 < \alpha < 1$, by repeated application of equation (5), one obtains the following expression:

$$\Gamma(n+\alpha)\bar{\mathrm{E}}_{n+\alpha}(x) = \sum_{i=0}^{n-1} (-x)^{n-1-i}\Gamma(\alpha+i) + (-x)^n\Gamma(\alpha)\bar{\mathrm{E}}_\alpha(x), \tag{32}$$

where $\bar{\mathrm{E}}_r(x) = \mathrm{e}^x\mathrm{E}_r(x)$.

Equation (32) can also be written as

$$\mathrm{E}_{n+\alpha}(x) = \frac{(-x)^n}{(\alpha)_n}\mathrm{E}_\alpha(x) + \mathrm{e}^{-x}\sum_{j=1}^{n}\frac{(-x)^{n-j}}{(\alpha+j-1)_{n-j-1}}. \tag{33}$$

By means of this expression, we can find the series expansion of $\mathrm{E}_\nu(x)$, $(\nu > 0)$ from that given for $\mathrm{E}_\alpha(x)$ in equations (27) or (28), according as $0 < \alpha \leqslant d$ or $d < \alpha < 1$.

Finally, in the case of negative values of the order, $\nu = -n - \alpha, n \in N, 0 < \alpha \leqslant 1$, the repeated application of recurrence (5) backwards yields the expression

$$\bar{\mathrm{E}}_{-n-\alpha}(x) = \frac{1}{x}\left[\sum_{k=0}^{n}(n+\alpha-k+1)_k x^{-k} + (\alpha)_{n+1}x^{-n}\bar{\mathrm{E}}_{1-\alpha}(x)\right]. \tag{34}$$

Apart from the trivial case $\alpha = 1$, from this relation one can get the series representation for $\mathrm{E}_\nu(x)$ $(\nu = -n - \alpha, n \in N)$ from that obtained for $\mathrm{E}_{1-\alpha}(x)$ in equations (27) or (28), respectively, when $0 < 1 - \alpha \leqslant d$ or $d < 1 - \alpha < 1$.

In practice, following Gautschi's results [11], recursive evaluation of $\mathrm{E}_\nu(x)$, $(\nu \in R)$, starting from a suitable initial value, can be performed also in the range $(x \geqslant 1)$, as described in Sections 7 and 8.

6. POLYNOMIAL EXPANSION AND REPRESENTATION FOR THE REGION $(x > 0, \frac{1}{2} < \nu < 2)$

As for polynomial expansions, we mention the important relation [12]

$$\mathrm{E}_\nu(x) = \mathrm{e}^{-x}\sum_{n=0}^{\infty}\frac{L_n^{(1-\nu)}(x)}{(n+1)} \quad (1/2 < \nu < 2, x > 0), \tag{35}$$

in terms of the generalized Laguerre polynomials, $L_n^{(a)}(x)$ [13, p. 1038].

Expression (35) can be used as in Ref. [12] to obtain a significant series representation for $\mathrm{E}_\nu(x)$.

In fact, since the $L_n^{(a)}(x)$ are particular cases of the Kummer function, $\Phi(b, c; x)$, [8, p. 35]

$$L_n^{(a)}(x) = \binom{n+a}{n}\Phi(-n, a+1; x) = \frac{(a+1)_n}{n!}\Phi(-n, a+1; x) \quad (a > -1, x > 0), \tag{36}$$

considering that the corresponding $\Phi^*(-n, a+1; x)$ can be expanded [8] in terms of the Tricomi expansion, equation (22), and taking into account equation (26), from equations (35) and (36) it results [12]:

$$\begin{aligned}\mathrm{E}_\nu(x) &= \mathrm{e}^{-x/2}\Gamma(2-\nu)\sum_{n=0}^{\infty}\frac{(2-\nu)_n}{(n+1)!}\sum_{m=0}^{\infty}a_{n,m}\left(\frac{x}{2}\right)^m T_{m+1-\nu}((n+1-\nu/2)x)\\ &= \mathrm{e}^{-x/2}\sum_{n=0}^{\infty}\frac{(2-\nu)_n}{(n+1)!}\sum_{m,k=0}^{\infty}a_{n,m}\left(\frac{x}{2}\right)^{m+k}\frac{(\nu-2-2n)^k}{k!(2-\nu)_{m+k}} \quad (1/2 < \nu < 2, x > 0),\end{aligned} \tag{37}$$

where the coefficients, $a_{n,m}$, are defined by the recurrence

$$\begin{gathered}(i+1)a_{n,i+1} = (i+1-\nu)a_{n,i-1} - (2n+2-\nu)a_{n,i-2} \quad (i = 2, 3, \ldots),\\ a_{n,0} = 1,\ a_{n,1} = 0,\ a_{n,2} = 1 - \nu/2.\end{gathered} \tag{38}$$

7. CONTINUED FRACTION

We consider the important fraction

$$\mathrm{E}_\nu(x) = \mathrm{e}^{-x}\left(\frac{1}{x+}\,\frac{\nu}{1+}\,\frac{1}{x+}\,\frac{\nu+1}{1+}\,\frac{2}{x+}\,\frac{\nu+2}{1+}\,\frac{3}{x+}\cdots\right), \tag{39}$$

which can be derived considering the Legendre continued fraction expansion of $\Gamma(a, x)$, (see Ref. [14, p. 136]) and taking into account equation (16).

From equation (39), even and odd contractions can be obtained as in Refs [2, 15], where equation (39) has been successfully applied to the evaluation of $\mathrm{E}_\nu(x)$ for ν positive integer.

Likewise to the result found in Ref. [2, p. 157] for the case of ν integer, the expansion (39), which is convergent (see Ref. [2, p. 102]) for all $\nu \in R$ and $x > 0$, converges better with increasing x. Thus, in the region of sufficiently large x values, one can use equation (39) for evaluating the generalized exponential integral. In particular, one can make use of the recursive procedure of Ref. [4] (see Fig. 2) to evaluate a required $\mathrm{E}_\nu(x)$, whenever $\nu \in R$.

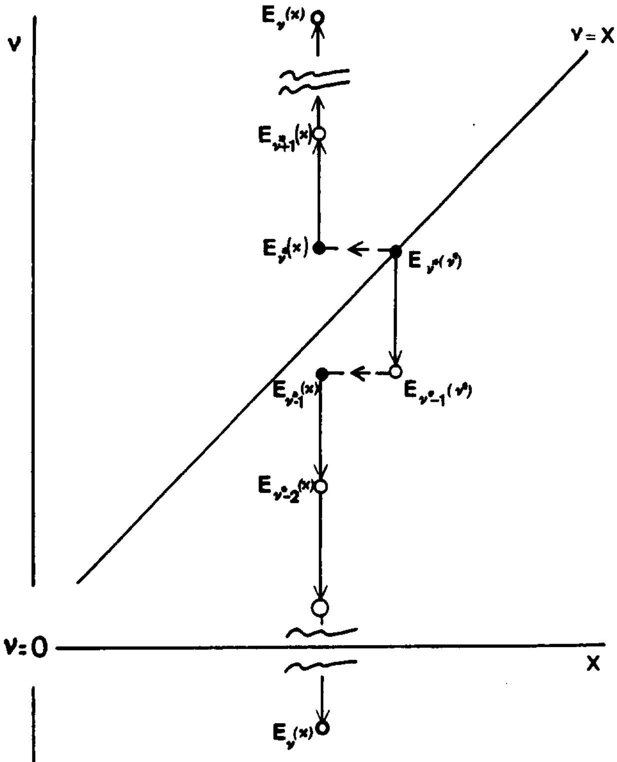

Fig. 2. Scheme of computational procedure for $\mathrm{E}_\nu(x)$ in the region of large x values and $\nu \in R$; the starting value, $E_{\nu^*}(\nu^*)$, is calculated by means of equation (44), the following values from equations (5) [○] and (15) [●], respectively, as far as the required $\mathrm{E}_\nu(x)$ is attained; $\nu^* = [x] + D + 1 - H(\nu) + \nu - [\nu]$, with $[w]$ truncated part of w, $H(\nu)$ Heaviside step function and $D = 1$ when $(1 - H(\nu) + \nu - [\nu]) < (x - [x])$ or $D = 0$ otherwise.

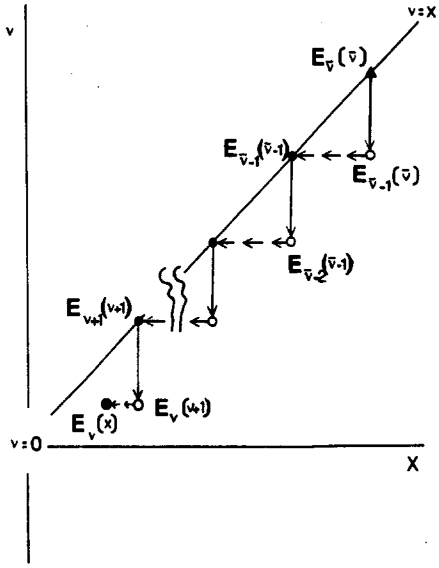

Fig. 3. Scheme of computational procedure for $\mathrm{E}_\nu(x)$ when $1 \leqslant x < x_a$, whenever $\nu \in R$, $\nu < x$; ▲, starting asymptotic value, equation (44) [○], computed by backward recursion, eq. (5) [●], computed by Taylor expansions, equation (15); $\bar{\nu} = \nu^* + k$, with ν^* defined as in Fig. 2 and $k = [x_a] - [\nu^*]$.

8. ASYMPTOTIC REPRESENTATIONS

As for asymptotic expansions, it is to be outlined that one can obtain from a well-known (see, e.g. Ref. [7, p. 341]) expansion for $\Gamma(a, x)$, (valid for large values of x) and from equation (16), the following expression:

$$\mathrm{E}_\nu(x) \sim \frac{\mathrm{e}^{-x}}{x}\left[1 - \frac{\nu}{x} + \frac{\nu(\nu+1)}{x^2} - \frac{\nu(\nu+1)(\nu+2)}{x^3} + \cdots\right]$$

$$= \frac{\mathrm{e}^{-x}}{x}\left[\sum_{m=0}^{N} \frac{(-1)^m}{x^m}\frac{\Gamma(\nu+m)}{\Gamma(\nu)} + O(x^{-N-1})\right] \quad (x \gg 1 \text{ and } \nu > 0). \tag{40}$$

In this expansion ν is fixed, i.e. it is not supposed that ν grows with x.

Equation (40) can also be obtained directly from formula (1) after introducing the variable $u = t - 1$ and by repeated integration by parts of $\mathrm{e}^x\mathrm{E}_\nu(x)$.

A similar result has been obtained in Ref. [1], in a more general context. In particular, equation (40) implies that

$$\mathrm{E}_\nu(x) \sim \frac{\mathrm{e}^{-x}}{x} \quad \text{as } x \to \infty. \tag{41}$$

Finally, just as outlined in Ref. [2] in the case of ν integer, the remainder term can be expressed in terms of the exponential integral $\mathrm{E}_{\nu+N+1}(x)$.

Furthermore, in the region of large positive integer ν and arbitrary positive x values, the following expansion is valid [5, 16]:

$$\mathrm{E}_\nu(x) = \frac{\mathrm{e}^{-x}}{x+\nu}\left[1 + \frac{\nu}{(x+\nu)^2} + \frac{\nu(\nu-2x)}{(x+\nu)^4} + \frac{\nu(6x^2 - 8\nu x + \nu^2)}{(x+\nu)^6} + \bar{R}(x,\nu)\right], \tag{42}$$

where [5]

$$-0.36\nu^{-4} \leqslant \bar{R}(x,\nu) \leqslant \left(1 + \frac{1}{x+\nu-1}\right)\nu^{-4}, \tag{43}$$

while, more generally [16], for ν real:

$$\mathrm{E}_\nu(x) = \frac{\mathrm{e}^{-x}}{x+\nu}\left[\sum_{n=0}^{k-1} \nu^{-n}(1 + x/\nu)^{-2n} h_n(x/\nu) + R_k(x,\nu)\right], \tag{44}$$

with [3]

$$\alpha_k \nu^{-k} \leqslant R_k(x,\nu) \leqslant \beta_k\left(1 + \frac{1}{x+\nu-1}\right)\nu^{-k}, \tag{45}$$

where the $h_n(t)$ are polynomials defined as follows [3]

$$h_n(t) = \sum_{j=0}^{n} c_{j,n} t^j \quad (n = 0, 1, 2, \ldots), \tag{46}$$

with $c_{0,0} = 1$ and $c_{j,n}$ generated columnwise in an upper triangular matrix by the formulae

$$\begin{aligned} &c_{0,n} = 1, \quad c_{n,n} = 0 && (n = 1, 2, \ldots), \\ &c_{m,n+1} = (m+1)c_{m,n} + (m - 2n - 1)c_{m-1,n} && (m = 1, 2, \ldots, n). \end{aligned} \tag{47}$$

In equation (45) the coefficients $\{\alpha_k, \beta_k\}$ are, respectively, the lower and upper bounds of $[h_k(t)/(1+t)^{2k}]$ in the interval $t \geqslant 0$.

Choosing $\nu = x$ in equation (44), this expansion results particularly suitable for calculations of $\mathrm{E}_\nu(x)$ (see Refs [3, 4]) in the asymptotic region $x \geqslant x_a$ ($x_a \sim 20$).

More precisely, starting from a suitable $\mathrm{E}_{\nu^*}(\nu^*)$, calculated via equation (44) and proceeding as illustrated in Fig. 2, with a combined use of Taylor expansions, equation (15), and recurrences, equation (5), one can evaluate any $\mathrm{E}_\nu(x)$, $\nu \in R$, in the region $x \geqslant x_a$.

Moreover, the above formulation, equation (44), can be used also when the required $E_\nu(x)$ lies in the "non-asymptotic" region $(1 \leqslant x < x_a)$.

In fact, according to the procedure described in Refs [3, 4] (see Fig. 3), starting from a suitable asymptotic value, $E_{\bar\nu}(\bar\nu)$, calculated via equation (44), by a repeated use of recurrences, equation (5), and Taylor expansions, equation (15), (to shift, respectively, the order and the argument), one can reach any required $E_\nu(x)$, $(\nu \in R)$ in $(1 \leqslant x < x_a)$.

We have thus discussed basic properties and some procedures for evaluating the generalized exponential integrals, $E_\nu(x)$, over the whole domain $(x > 0, \nu \in R)$.

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