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Solutions of a Burgers–Stefan problem

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Abstract

A method to solve a one-phase Stefan problem associated to the Burgers equation is outlined. It is shown that the problem admits an exact solution which is a shock wave. The shock wave travels with the appropriate free boundary velocity and is found to be stable. © 2000 Published by Elsevier Science B.V.

Stefan problems for the linear heat equation are physically important and there is an extensive literature (see Ref. [1] and references therein, and Refs. [2,3] associated with them. Indeed, they arise in very simple physical situations such as the process of evaporation and condensation of drops, the dissolution of gas bubbles in liquid and the melting of ice when heated at the boundary with a prescribed temperature.

In certain cases a method of solution has been found and existence theorems have been proven [2,4].

More recently, a class of Stefan problems in nonlinear heat conduction was considered in [5] and the exact solution was constructed in parametric form.

In the following we outline a method for solving a one-phase Stefan problem for the Burgers equation (Burgers–Stefan problem). We reduce the problem to a nonlinear integral equation in one variable (time) and show that the problem admits an explicit shock wave solution which is moreover stable in the free boundary configuration.

The Burgers–Stefan problem is a natural extension to nonlinear diffusive systems of the previous studies associated with the linear heat equation. In fact, the Burgers equation is the simplest evolution equation which combines together the effects of diffusion and nonlinearity.

The Burgers–Stefan problem is characterized as follows: the equation

$$u_t = u_{xx} - 2uu_x, \quad u = u(x,t), \quad t > 0 \tag{1}$$

is considered on the semi-infinite domain $x \in (-\infty, s(t))$, with the initial datum

$$u(x,0) = u_0(x) > 0, \quad -\infty < x < b \tag{2a}$$

$$u_0(b) = 0, \quad b > 0 \tag{2b}$$

subject to the boundary conditions

$$u(-\infty,t) = u_1 > 0, \quad t \geqslant 0 \tag{3a}$$

$$u(s(t),t) = 0, \quad t \geqslant 0 \quad \text{with } s(0) = b \tag{3b}$$

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$$u_x(s(t),t) = \dot{s}(t), \quad t \geqslant 0. \tag{3c}$$

In the above relations u_1, b are positive constants; $s(t)$ is an unknown function which describes the motion of the free boundary and has to be determined together with $u(x,t)$. (3c) is a condition on the flux at the free boundary, arising from energy considerations.

We start our analysis by introducing the generalized Hopf–Cole transformation [6]

$$u(x,t) = v(x,t) \Big/ \left[C(t) - \int_{s(t)}^{x} \mathrm{d}x' v(x',t) \right] \tag{4a}$$

$$v(x,t) = C(t) u(x,t) \exp\left[- \int_{s(t)}^{x} \mathrm{d}x' u(x',t) \right] \tag{4b}$$

with the initial condition

$$C(0) = 1 \tag{4c}$$

Under this transformation Eq. (1) is mapped into the linear heat equation

$$v_t = v_{xx} \tag{5}$$

with the compatibility condition

$$\dot{C}(t) = -v_x(s(t),t). \tag{7c}$$

Moreover from (2a) and (3a,b) we obtain the following set of initial and boundary data for Eq. (5):

$$v(x,0) = v_0(x) = u_0(x) \exp\left[- \int_{b}^{x} \mathrm{d}x' u_0(x') \right] \tag{6a}$$

with

$$v_0(x) > 0, \quad v_0(b) = 0 \tag{6b}$$

and

$$v(s(t),t) = C(t) u(s(t),t) = 0, \tag{7a}$$

$$v_x(s(t),t) = C(t) u_x(s(t),t) = -C(t)\dot{s}(t). \tag{7b}$$

The one-phase Burgers–Stefan problem is then reduced to the one-phase Stefan problem for the linear heat Eq. (5) with initial datum (6a,b) and with boundary conditions at the free boundary given by (7a–c).

In order to solve this problem, we first observe that (7b) and (7c) imply

$$C(t) = \exp[s(t) - b], \tag{8a}$$

which can be inverted as

$$s(t) = b + \ln\left[1 - \int_0^t \mathrm{d}t' v_x(s(t'),t') \right]. \tag{8b}$$

Next, we introduce the fundamental kernel of the heat equation

$$K(x,t) = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t} \right) \tag{9}$$

and integrate Green's identity

$$\frac{\partial}{\partial \xi}\left(K \frac{\partial v}{\partial \xi} - v \frac{\partial K}{\partial \xi} \right) - \frac{\partial}{\partial \tau}(Kv) = 0 \tag{10}$$

over the domain $-\infty < \xi < s(\tau)$, $0 < \varepsilon < \tau < t - \varepsilon$ and let $\varepsilon \to 0$. Using $v(s(\tau),\tau) = 0$ and $K(x-\xi,0) = \delta(x-\xi)$, we get the solution of the Stefan problem (5), (6a,b), (7a–c) as

$$v(x,t) = \int_{-\infty}^{b} K(x-\xi,t) v_0(\xi)\, \mathrm{d}\xi + \int_0^t \mathrm{d}\tau K \times (x - s(\tau), t-\tau) v_x(s(\tau),\tau) \tag{11}$$

with $s(t)$ given by (8b).

From Eq. (11) we see that $v(x,t)$ is known once $v_x(s(t),t)$ is known. We then take the derivative of both sides in (11) and evaluate it as $x \to s(t)^-$.

By putting $v_x(s(t),t) = z(t)$ and using cf. [2]

$$\lim_{x \to s(t)^-} \frac{\partial}{\partial x} \int_0 \mathrm{d}t' K(x - s(t'), t-t') z(t') = \tfrac{1}{2} z(t) + \int_0^t \mathrm{d}t' K_x(s(t) - s(t'), t-t') z(t'),$$

we obtain

$$z(t) = 2\int_{-\infty}^{b} \mathrm{d}\xi K(s(t) - \xi, t) v_0'(\xi) + 2\int_0^t \mathrm{d}t' K_x(s(t) - s(t'), t-t') z(t') \tag{12a}$$

with

$$s(t) = b + \ln\left(1 - \int_0^t \mathrm{d}t' z(t') \right). \tag{12b}$$

The Stefan problem (5), (6a,b), (7a–c) has then been reduced to the nonlinear integral Eq. (12a,b) in one independent variable. We have been able to establish existence and uniqueness of the solution $z(t)$ for small times [7] and will give a detailed report on this elsewhere. We point out that once $z(t)$ is shown to exist and to be unique, from (11) existence and uniqueness of $v(x,t)$ there follows. Hence the solution $u(x,t)$ of the original Burgers–Stefan problem exists and is unique (for small times), due to (4a) with $C(t)$ given by (8a) and (12b).

We now turn our attention to a particular solution of the Burgers–Stefan problem. Namely, we show that there is a shock solution which travels with the same velocity as that of the free boundary. We consider the usual shock solution of Eq. (1):

$$v(x,t) = u_1 + \frac{(u_2 - u_1)}{[1 + \exp((u_2 - u_1)(x - Vt - x_0))]} \tag{13a}$$

with

$$V = (u_1 + u_2), \quad u_2 < u_1 \tag{13b}$$

and u_2 a constant to be determined. (13a,b) describes a shock which is travelling to the right with velocity V, and is compatible with the initial condition (2a) and with the boundary condition (3a).

If we now impose the conditions at the free boundary (3b) and (3c), we get

$$s(t) = x_0 + \frac{s_0}{(u_1 - u_2)} + Vt \tag{14a}$$

and

$$V = -u_1 u_2 = -\dot{s}(t) \tag{14b}$$

(14a) and (14b) imply that the shock and the boundary are both moving to the right with the same velocity.

The same relations also imply that the initial position of the boundary, the velocity V and the value of the constant u_2 are given by

$$u_2 = -u_1 / (1 + u_1), \tag{15a}$$

$$V = u_1^2 / (1 + u_1), \tag{15b}$$

$$e^{s_0} = u_1 / |u_2|. \tag{15c}$$

In order to sketch the stability of such solution we consider a small perturbation affecting both the shock and the motion of the free boundary. In this context we set

$$u = \hat{u} + u' \tag{16a}$$

$$s(t) = \hat{s}(t) + s'(t) \tag{16b}$$

where $\hat{u}$ is the shock solution satisfying $\hat{u}(\hat{s}(t),t) = 0$ and u', s' are small perturbations. By linearizing Eq. (1) around $\hat{u}$, we get

$$\varphi_t = \varphi_{xx} - 2\hat{u}\varphi_x, \tag{17a}$$

where the position $u' = \varphi_x$ has been made.

The boundary conditions (3b,c) together with (16a,b) give the condition for φ at the free boundary

$$\left. \frac{\partial}{\partial t}\varphi_x + V\varphi_{xx} + V^2\varphi_x \right|_{x = s(t)} = 0, \tag{17b}$$

where (13b) and (14b) have also been used.

The change of variables

$$\varphi(x,t) = \varphi(X,t), \quad X = x - Vt \tag{18}$$

maps (17a) and (17b) into

$$\varphi_t = \varphi_{XX} - (2\hat{u} - V)\varphi_X \tag{19a}$$

and

$$\left. \frac{\partial}{\partial t}\varphi_X + V^2\varphi_X \right|_{X=0} = 0 \tag{19b}$$

respectively.

We now solve (19a) with the initial condition

$$\varphi(X,0) = f(X) \tag{19c}$$

and the asymptotically vanishing condition $\varphi \to 0$ as $X \to -\infty$.

In terms of the Laplace transform

$$\hat{\varphi}(X,q) = \int_0^\infty \mathrm{d}t e^{-qt}\varphi(X,t), \tag{20}$$

from (19a–c) we get

$$\hat{\varphi}(X,q) = \exp(-p(X)) \times \left[c_1 e^{kX} + \int_0^X d\xi \frac{e^{k(X-\xi)}}{2k} F(\xi) - \int_{-\infty}^X d\xi \frac{e^{-k(X-\xi)}}{2k} F(\xi) \right] \tag{21a}$$

with

$$F(X) = -f(X)\exp(+P(X)) \tag{21b}$$

$$P(X) = \int_0^X \left(\frac{V}{2} - \hat{u}(X') \right) dX' \tag{21c}$$

$$k = \left(q + V + \frac{V^2}{4} \right)^{1/2} \tag{21d}$$

and the constant c_1 given by

$$c_1 = f'(0) \Big/ \left[(V^2+q)\left(k - \frac{V}{2} \right) \right] - \frac{G(k)}{(k - V/2)}, \tag{21e}$$

where

$$G(k) = \int_{-\infty}^0 \frac{e^{k\xi}}{2k} F(\xi)\, d\xi. \tag{21f}$$

The small perturbation $u'(X,t)$ is then obtained by inverting (21a) and taking the X-derivative. When the asymptotic, large time behaviour of $u'(X,t)$ is considered, the following results are obtained: in the region $V > 1$ there holds

$$u'(X,t) \underset{t\to\infty}{\cong} e^{\frac{V}{2}X} e^{-Vt} \left(\frac{V}{2} e^{-p(X)} - p'(X) \right) \times \left(A + \frac{V}{V^2 - V} \right) \tag{22a}$$

with

$$A = \int_{-\infty}^0 F(\xi)\, d\xi.$$

In the region $V < 1$ there obtains instead

$$u'(X,t) \underset{t\to\infty}{\cong} e^{\frac{V}{2}X} e^{-V^2 t} \frac{V}{(V^2 - V)} \times \left(p'(X) - \frac{V}{2} e^{-p(X)} \right) \tag{22b}$$

finally, at $V = 1$ it is

$$u'(X,t) \underset{t\to\infty}{\cong} t e^{\frac{1}{2}X} e^{-t} \left(\frac{e^{-p(X)}}{2} - p'(X) \right). \tag{22c}$$

We can then conclude that the small perturbation $u'(X,t)$ is asymptotically vanishing at $t \to \infty$.

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