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Soliton interactions in the vector NLS equation

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Abstract

Collisions of solitons for two coupled and N-coupled NLS equation are investigated from various viewpoints. By suitably employing Manakov's well-known formulae for the polarization shift of interacting vector solitons, it is shown that the multisoliton interaction process is pairwise and the net result of the interaction is independent of the order in which such collisions occur. Further, this is shown to be related to the fact that the map determining the interaction of two solitons with nontrivial internal degrees of freedom (e.g. vector solitons) satisfies the Yang–Baxter relation. The associated matrix factorization problem is discussed in detail. Soliton interactions are also described in terms of linear fractional transformations, and the problem of existence of a solution for a basic three-collision gate, which has recently been introduced, is analysed.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t = q_{xx} + 2|q|^2 q$$

is a well-known physically and mathematically significant nonlinear evolution equation. For example, it has been derived in such diverse fields as deep water waves, plasma physics and nonlinear fibre optics. In optics, NLS models wave propagation in Kerr media, where the nonlinearity is proportional to the intensity of the field. Mathematically speaking, NLS has been of great interest as a nonlinear evolution equation that is solvable via the inverse scattering transform (IST). Moreover, as is typical of such equations, the NLS has infinitely many conserved quantities, a Hamiltonian structure and soliton solutions.

The vector generalization

$$\mathrm{i}\mathbf{q}_t = \mathbf{q}_{xx} + 2\|\mathbf{q}\|^2\mathbf{q} \tag{1}$$

where $\mathbf{q} = (q_1, \ldots, q_N)$ is an N-component vector and $\|\cdot\|$ is the standard Euclidean norm, which we refer to as vector NLS (VNLS), arises, physically, under conditions similar to those described by the NLS when there are multiple wave-trains moving with nearly the same group velocity. Moreover, VNLS models physical systems in which the field has more than one component; for example, in optical fibres and waveguides, the propagating electric field has two components transverse to the direction of propagation. Manakov [1] first examined this equation, in the two-component case, as an asymptotic model for the propagation of the electric field in a waveguide. Subsequently, this system was derived as a key model for lightwave propagation in optical fibres [2–4]. Also, the matrix generalization of the Manakov system was derived as a model for spin systems [5]. Temporal VNLS solitons have recently been observed in optical fibres [26, 27]. In the spatial domain, VNLS solitons have been detected in AlGaAs planar waveguides [6].

It is well known that interacting scalar solitons affect each other only by a phase shift that depends only on the solitons' amplitudes and velocities, which are conserved quantities. Thus, when two soliton collisions occur sequentially, the outcome of the first collision does not affect the second collision as the phase shifts induced by the collisions are additive. On the other hand, the collision of solitons with internal degrees of freedom (e.g. vector or matrix solitons) is not so simple. Although such collisions are elastic, in the sense that the total energy of each soliton is conserved, there can be a significant redistribution of energy among the components. This redistribution of the soliton's energy follows directly from Manakov's formulae describing the effects of vector soliton collisions [1] (see also [7, 13]) and has been measured in experiments [8].

In this paper, we analyse various features of collisions of generic N-component VNLS solitons, including changes in energy distribution, phase and relative separation distance. Suitably employing Manakov's well-known formulae for the polarization shift of interacting vector solitons, we show that the multisoliton interaction process is pairwise and the net result of the interaction is independent of the order in which such collisions occur (see also [13]). We then show explicitly that this is related to the fact that the map determining the interaction of two solitons satisfies the Yang–Baxter relation.

The nontrivial interaction of vector solitons provides a mechanism for computation with solitons [9, 25]. Jakubowsky *et al* [9] expressed the energy redistributions as linear fractional transformations and described how vector-soliton logic gates could be developed. Later, Steiglitz [10] explicitly constructed such logic gates based on the shape changing collision properties and hence pointed out the possibility of designing an all optical computer equivalent to a Turing machine, at least in a mathematical sense. We consider the basic three-collision gate introduced in [10] and prove that there are parameter regimes for which a unique solution exists.

2. Multisoliton interactions

A pure one-soliton solution of (1),

$$\mathbf{q}(x,t) = -2\mathrm{i}\eta\,\mathrm{e}^{-2\mathrm{i}\xi x + 4\mathrm{i}(\xi^2-\eta^2)t}\,\mathrm{sech}(2\eta x - 8\xi\eta t - 2d)\mathbf{p} \tag{2}$$

where

$$\mathrm{e}^{2d} = \frac{\|\mathbf{C}\|}{2\eta} \equiv \frac{\left[\sum_{j=1}^{N}|\gamma^{(j)}|^2\right]^{1/2}}{2\eta} \qquad \mathbf{p} = \frac{\mathbf{C}^H}{\|\mathbf{C}\|} \tag{3}$$

is characterized by $N+1$ complex parameters, namely the discrete eigenvalue $k = \xi + \mathrm{i}\eta$ (where $\eta > 0$) and the corresponding norming constant $\mathbf{C} = (\gamma^{(1)}, \ldots, \gamma^{(N)})^T$. The norm 1 vector $\mathbf{p}$ is referred to as the polarization (vector) of the soliton, while the quantity

$$x_0 = \frac{d}{\eta} = \frac{1}{2\eta} \log \frac{\|\mathbf{C}\|}{2\eta} \tag{4}$$

denotes the position of the soliton envelope peak at $t = 0$. The inverse scattering theory for this equation in the two-component case was first studied by Manakov [1]. The generalization of the inverse scattering from the two-component case to the N-component case is straightforward [13].

From the inverse scattering transform, Manakov [1] also derived the general formula for a J-soliton interaction in the two-component VNLS. The generalization to the N-component case is written below (see also [13] for a detailed derivation). Assuming the discrete eigenvalues are such that $\mathrm{Re}\, k_1 < \mathrm{Re}\, k_2 < \cdots < \mathrm{Re}\, k_J$ (equivalently, the soliton velocities are ordered as $v_1 < v_2 < \cdots < v_J$) and the polarization vectors (normalized N-component row vectors) before/after the interaction are given by

$$\mathbf{p}_j^{\mp} = \frac{\left(\mathbf{s}_j^{\mp}\right)^H}{\|\mathbf{s}_j^{\mp}\|} \qquad j = 1, 2, \ldots, J \tag{5}$$

an asymptotic analysis yields

$$\mathbf{s}_J^{+} = \prod_{l=1}^{J-1} \frac{1}{a_l(k_J)} \prod_{\substack{l=1 \\ \text{right}}}^{J-1} (\mathbf{c}_l(k_J, \mathbf{s}_l^{-}))^{-1} \mathbf{s}_J^{-} \tag{6}$$

and, for $j = 1, \ldots, J-1$,

$$\mathbf{s}_j^{+} = \prod_{l=1}^{j-1} \frac{1}{a_l(k_j)} \prod_{l=j+1}^{J} a_l(k_j) \prod_{\substack{l=j+1 \\ \text{right}}}^{J} \mathbf{c}_l\big(k_j, \mathbf{s}_l^{+}\big) \prod_{\substack{l=1 \\ \text{right}}}^{j-1} (\mathbf{c}_l(k_j, \mathbf{s}_l^{-}))^{-1} \mathbf{s}_j^{-} \tag{7}$$

where the notation 'right' indicates that the matrix with index l is to the right of the matrix with index $l-1$ (i.e. the order is increasing from left to right) and the transmission coefficients are given by

$$a_j(k) = \frac{k - k_j}{k - k_j^*} \tag{8}$$

$$\mathbf{c}_j(k, \mathbf{s}_j) = \mathbf{I}_N - \frac{k_j - k_j^*}{k - k_j^*} \frac{1}{\|\mathbf{s}_j\|^2} \mathbf{s}_j^H \mathbf{s}_j \tag{9}$$

$$(\mathbf{c}_j(k, \mathbf{s}_j))^{-1} = \mathbf{I}_N + \frac{k_j - k_j^*}{k - k_j} \frac{1}{\|\mathbf{s}_j\|^2} \mathbf{s}_j^H \mathbf{s}_j. \tag{10}$$

Above, $\mathbf{I}_N$ is the $N \times N$ identity matrix and, in (7), we define $\prod_1^0 \equiv 1$ for $j = 1$.

The analysis of relations (6) and (7) solves the problem of a J-soliton collision. Given $\mathbf{s}_l^{-}$ for $l = 1, \ldots, J$, (6) allows one to find $\mathbf{s}_J^{+}$. Because expression (7) for $\mathbf{s}_j^{+}$ with $j < J$ depends, through the $\mathbf{c}$, on $\mathbf{s}_l^{+}$ for $l > j$, one can find $\mathbf{s}_j^{+}$ iteratively for any j by first obtaining $\mathbf{s}_{J-1}^{+}$, then $\mathbf{s}_{J-2}^{+}$ and so on. The polarization vectors $\mathbf{p}_j^{\pm}$ are then obtained from (5).

2.1. Two-soliton collision

Let us consider the interaction of two solitons in detail. Assuming that $v_1 < v_2$, i.e. $\mathrm{Re}\, k_1 < \mathrm{Re}\, k_2$, we obtain, from (6) and (7), the relations

$$\mathbf{s}_2^+ = \frac{1}{a_1(k_2)}(\mathbf{c}_1(k_2, \mathbf{s}_1^-))^{-1}\mathbf{s}_2^- \tag{11}$$

$$\mathbf{s}_1^+ = a_2(k_1)\mathbf{c}_2\big(k_1, \mathbf{s}_2^+\big)\mathbf{s}_1^-. \tag{12}$$

Note that formulae (11) and (12) are not symmetric with respect to the exchange of the subscripts 1 and 2. Taking into account the explicit expressions for a_j and $\mathbf{c}_j$, given by (8)–(10), one can solve (11) for $\mathbf{s}_2^+$, compute $\mathbf{c}_2\left(k_1, \mathbf{s}_2^+\right)$ from (9) and then substitute it into the right-hand side of (12) in order to get $\mathbf{s}_1^+$. It is convenient to define

$$\chi^2 = \frac{\left\|\mathbf{s}_2^+\right\|^2}{\|\mathbf{s}_2^-\|^2} \equiv \frac{1}{\left\|\mathbf{s}_2^+\right\|^2}(\mathbf{s}_2^-)^H\mathbf{s}_2^+ \tag{13}$$

which, according to (8), (10) and (11), is given by

$$\chi^2 = \left|\frac{k_1 - k_2^*}{k_1 - k_2}\right|^2 \left[1 + \frac{(k_1^* - k_1)(k_2 - k_2^*)}{|k_1 - k_2|^2}|\mathbf{p}_1^{-*} \cdot \mathbf{p}_2^-|^2\right]. \tag{14}$$

One can also check that

$$\frac{\left\|\mathbf{s}_1^+\right\|^2}{\|\mathbf{s}_1^-\|^2} = \frac{1}{\chi^2}. \tag{15}$$

From (11) and (12) it then follows that

$$\mathbf{p}_2^+ = \frac{1}{\chi}\frac{k_1 - k_2^*}{k_1^* - k_2^*}\left[\mathbf{p}_2^- + \frac{k_1^* - k_1}{k_2^* - k_1^*}(\mathbf{p}_1^{-*} \cdot \mathbf{p}_2^-)\mathbf{p}_1^-\right] \tag{16}$$

$$\mathbf{p}_1^+ = \frac{1}{\chi}\frac{k_1 - k_2^*}{k_1 - k_2}\left[\mathbf{p}_1^- + \frac{k_2^* - k_2}{k_2 - k_1}(\mathbf{p}_2^{-*} \cdot \mathbf{p}_1^-)\mathbf{p}_2^-\right] \tag{17}$$

with χ given by (14).

Due to the interaction, the components of the polarization vectors of each soliton change from $p_j^{-(l)}$ to $p_j^{+(l)}$, $j = 1, 2, l = 1, \ldots, N$. However, the total energy of each of the solitons is conserved: that is $\sum_{l=1}^N \left|p_1^{\pm(l)}\right|^2 = \sum_{l=1}^N \left|p_2^{\pm(l)}\right|^2 = 1$. (This is a consequence of the conservation of the L^2-norm for the solutions of the VNLS equation.)

Note that when either $\mathbf{p}_2^- = \mathrm{e}^{\mathrm{i}\theta}\mathbf{p}_1^-$ or $\mathbf{p}_1^{-*} \cdot \mathbf{p}_2^- = 0$, i.e. when the soliton polarizations are either parallel or orthogonal, the only change in the polarization vectors is an overall phase factor. For all other choices of the parameters, shape changing (intensity redistribution) collisions occur.

As in the collision of scalar solitons, the collision of vector solitons induces phase shifts, in addition to the shape changing that is the distinctive characteristic of vector-soliton collisions. The phase shifts depend both on k_1, k_2 and on the polarization vectors $\mathbf{p}_1^-, \mathbf{p}_2^-$. If we denote by $d_1^\pm, d_2^\pm$ the phases of the two solitons before/after the interaction, from (2) and (3) it follows that

$$\mathrm{e}^{2d_1^\pm} = \frac{\left\|\mathbf{s}_1^\pm\right\|}{2\eta_1} \qquad \mathrm{e}^{2d_2^\pm} = \frac{\left\|\mathbf{s}_2^\pm\right\|}{2\eta_2} \tag{18}$$

then, according to (14)–(17)

$$e^{2(d_2^+-d_2^-)} = \frac{\|\mathbf{s}_2^+\|}{\|\mathbf{s}_2^-\|} \equiv \chi \qquad e^{2(d_1^+-d_1^-)} = \frac{\|\mathbf{s}_1^+\|}{\|\mathbf{s}_1^-\|} \equiv \frac{1}{\chi} \tag{19}$$

i.e. the phase shifts which are opposite to one another. The absolute value of the phase shift is

$$\Phi = |\log\chi| = \left|\log\left[\left|\frac{k_1-k_2^*}{k_1-k_2}\right|\left[1+\frac{(k_1^*-k_1)(k_2-k_2^*)}{|k_1-k_2|^2}|\mathbf{p}_1^{-*}\cdot\mathbf{p}_2^-|^2\right]^{1/2}\right]\right|. \tag{20}$$

For parallel modes one has $\Phi = 2\left|\log\left|\frac{k_1-k_2^*}{k_1-k_2}\right|\right|$, for orthogonal modes $\Phi = \left|\log\left|\frac{k_1-k_2^*}{k_1-k_2}\right|\right|$, with intermediate values for other choices of the soliton polarizations.

Ultimately, the above phase shifts mean that the relative separation distance $x_{1,2}^{\pm}$ between the solitons (i.e. the 'centre' or the position of soliton 2 at $t \to \pm\infty$ minus the position of soliton 1 at $t \to \pm\infty$) also change due to collision. More precisely

$$x_{1,2}^- = \frac{d_2^-}{\eta_2} - \frac{d_1^-}{\eta_1} \equiv \frac{2d_2^-}{\mathrm{i}(k_2-k_2^*)} - \frac{2d_1^-}{\mathrm{i}(k_1-k_1^*)} \tag{21}$$

$$x_{1,2}^+ = \frac{d_2^+}{\eta_2} - \frac{d_1^+}{\eta_1} \equiv \frac{2d_2^+}{\mathrm{i}(k_2-k_2^*)} - \frac{2d_1^+}{\mathrm{i}(k_1-k_1^*)} \tag{22}$$

hence

$$\Delta x = x_{1,2}^+ - x_{1,2}^- = \frac{2\left(d_2^+-d_2^-\right)}{\mathrm{i}(k_2-k_2^*)} - \frac{2\left(d_1^+-d_1^-\right)}{\mathrm{i}(k_1-k_1^*)} = \frac{\eta_1+\eta_2}{2\eta_1\eta_2}\log\chi \tag{23}$$

with χ given by (14) and $k_j = \xi_j + \mathrm{i}\eta_j$ for $j = 1, 2$.

We remark that the intensity profiles, the phases and the relative separation distance of the two interacting vector solitons are all shifted by the soliton collision. Moreover, these shifts depend on relative polarization of the two solitons before the collision.

Note that equations (16) and (17) are not symmetric with the exchange of $1 \to 2$. However, such a notation expresses the invariance of the system under the substitution $t \to -t$, $\mathbf{q} \to \mathbf{q}^*$ (here a 'fast soliton' becomes a 'slow' soliton and vice versa) and this is exactly how the problem appears in physical situations, when one is interested in fixing the initial polarizations of the solitons and finding the polarizations of the solitons after their interaction.

It is convenient to denote the polarization of soliton j after interaction with soliton ℓ by

$$\mathbf{p}_{\{j,\ell\}} \equiv \frac{\mathbf{s}_{\{j,\ell\}}}{\|\mathbf{s}_{\{j,\ell\}}\|}. \tag{24}$$

According to this notation, equations (16) and (17) are written as

$$\mathbf{p}_{\{2,1\}} = \frac{1}{\chi}\frac{k_1-k_2^*}{k_1^*-k_2^*}\left[\mathbf{p}_2 + \frac{k_1^*-k_1}{k_2^*-k_1^*}(\mathbf{p}_1^*\cdot\mathbf{p}_2)\mathbf{p}_1\right] \tag{25}$$

$$\mathbf{p}_{\{1,2\}} = \frac{1}{\chi}\frac{k_1-k_2^*}{k_1-k_2}\left[\mathbf{p}_1 + \frac{k_2^*-k_2}{k_2-k_1}(\mathbf{p}_2^*\cdot\mathbf{p}_1)\mathbf{p}_2\right]. \tag{26}$$

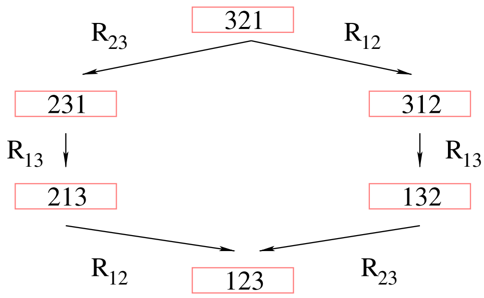


Figure 1. Three-soliton interaction.

2.2. *Multiple soliton collisions and order independence*

Take $J = 3$ solitons and assume $v_1 < v_2 < v_3$ so that the solitons are distributed along the x-axis according to $3-2-1$ as $t \to -\infty$ and $1-2-3$ as $t \to \infty$. According to equations (11) and (12) for a pairwise interaction between soliton j and soliton ℓ with $v_j > v_\ell$

$$\mathbf{s}_j^+ = \frac{1}{a_\ell(k_j)}(\mathbf{c}_\ell(k_j, \mathbf{s}_\ell^-))^{-1}\mathbf{s}_j^- \tag{27}$$

$$\mathbf{s}_\ell^+ = a_j(k_\ell)\mathbf{c}_j\big(k_\ell, \mathbf{s}_j^+\big)\mathbf{s}_\ell^-. \tag{28}$$

From (6) and (7) for $J = 3$ one has

$$\mathbf{s}_3^+ = \frac{1}{a_1(k_3)a_2(k_3)}(\mathbf{c}_1(k_3, \mathbf{s}_1^-))^{-1}(\mathbf{c}_2(k_3, \mathbf{s}_2^-))^{-1}\mathbf{s}_3^- \tag{29}$$

$$\mathbf{s}_2^+ = \frac{1}{a_1(k_2)}a_3(k_2)\mathbf{c}_3\big(k_2, \mathbf{s}_3^+\big)(\mathbf{c}_1(k_2, \mathbf{s}_1^-))^{-1}\mathbf{s}_2^- \tag{30}$$

$$\mathbf{s}_1^+ = a_2(k_1)a_3(k_1)\mathbf{c}_2\big(k_1, \mathbf{s}_2^+\big)\mathbf{c}_3\big(k_1, \mathbf{s}_3^+\big)\mathbf{s}_1^-. \tag{31}$$

At first it might appear that if we try to diagram these events in terms of the composition of pairwise interactions, we encounter a contradiction. It is true that, in general, the '$\mathbf{c}$' matrices do not commute with an arbitrary choice of arguments. However, using (9) and (25), (26), one can check that

$$\mathbf{c}_2(k_3, \mathbf{p}_2)\mathbf{c}_1(k_3, \mathbf{p}_1) = \mathbf{c}_1(k_3, \mathbf{p}_{\{1,2\}})\mathbf{c}_2(k_3, \mathbf{p}_{\{2,1\}}). \tag{32}$$

Thus, no contradiction arises from Manakov's formulae. We remark that equation (32) is exactly Veselov's matrix factorization relation [23], which we discuss in section 3.2.

Indeed, let us start by considering (29). It can be viewed in terms of a composition of pairwise interactions

$$\mathbf{s}_3^+ \equiv \mathbf{s}_{\{\{3,2\},1\}} = \frac{1}{a_1(k_3)}(\mathbf{c}_1(k_3, \mathbf{s}_1))^{-1}\mathbf{s}_{\{3,2\}} \tag{33}$$

corresponding to 3 interacting first with 2 ($3 \leftrightarrow 2$) and then with 1 ($3 \leftrightarrow 1$). The interaction $2 \leftrightarrow 1$ then follows, as schematically illustrated by the left branch in figure 1.

On the other hand, depending on the initial positions and relative velocities of the solitons, the sequence of soliton interactions might be different, i.e. corresponding to the right branch in figure 1. Note that we are not taking into account the case when more than two solitons interact simultaneously. Namely, one might have $2 \leftrightarrow 1$ requiring the consideration of the terms $\mathbf{s}_{\{2,1\}}$ and $\mathbf{s}_{\{1,2\}}$, then $3 \leftrightarrow 1$, resulting in $\mathbf{s}_{\{3,\{1,2\}\}}$ and $\mathbf{s}_{\{\{1,2\},3\}}$ and finally $3 \leftrightarrow 2$, the outgoing polarization of soliton 3 being

$$\mathbf{s}_{\{\{3,\{1,2\}\},\{2,1\}\}} = \frac{1}{a_1(k_3)a_2(k_3)}(\mathbf{c}_2(k_3, \mathbf{s}_{\{2,1\}}))^{-1}(\mathbf{c}_1(k_3, \mathbf{s}_{\{1,2\}}))^{-1}\mathbf{s}_3^-. \tag{34}$$

However, taking into account (32), from (33), (34) it follows that

$$\mathbf{s}_{\{\{3,2\},1\}} = \mathbf{s}_{\{\{3,\{1,2\}\},\{2,1\}\}} \tag{35}$$

which means that the polarization shift of soliton 3 is obtained via pairwise interactions with the remaining two and the result is independent of the order in which such interactions occur.

In the same way, one can check that the result also holds for solitons 2 and 1. For instance, from (30) one has

$$\mathbf{s}_2^+ \equiv a_3(k_2)\mathbf{c}_3\big(k_2, \mathbf{s}_3^+\big)\left[\frac{1}{a_1(k_2)}(\mathbf{c}_1(k_2, \mathbf{s}_1^-)^{-1}\mathbf{s}_2^-\right] \equiv a_3(k_2)\mathbf{c}_3\big(k_2, \mathbf{s}_3^+\big)\mathbf{s}_{\{2,1\}}. \tag{36}$$

That is, the total shift induced in the second soliton is the net effect of the composition of pairwise interactions of the form (27) and (28), corresponding to the order $1 \leftrightarrow 2$, then $1 \leftrightarrow 3$ and finally $2 \leftrightarrow 3$. The composition of pairwise interactions in a different order, i.e. $2 \leftrightarrow 3$, then $1 \leftrightarrow 3$ and finally $2 \leftrightarrow 1$, would give

$$\mathbf{s}_{\{2,\{1,\{3,2\}\}\}} = \frac{1}{a_1(k_2)}(\mathbf{c}_1(k_2, \mathbf{s}_{\{1,\{3,2\}\}})^{-1}\mathbf{s}_{\{2,3\}}. \tag{37}$$

Again, the two expressions (36) and (37) coincide provided

$$\mathbf{c}_1(k_2, \mathbf{s}_{\{1,\{3,2\}\}})\mathbf{c}_3\big(k_2, \mathbf{s}_3^+\big) = \mathbf{c}_3(k_2, \mathbf{s}_{\{3,2\}})\mathbf{c}_1(k_2, \mathbf{s}_1^-)$$

i.e.

$$\mathbf{c}_1(k_2, \mathbf{s}_{\{1,\{3,2\}\}})\mathbf{c}_3(k_2, \mathbf{s}_{\{\{3,\{1,2\}\},\{2,1\}\}}) = \mathbf{c}_3(k_2, \mathbf{s}_{\{3,2\}})\mathbf{c}_1(k_2, \mathbf{s}_1^-).$$

Taking into account (35), this last identity is reduced to

$$\mathbf{c}_1(k_2, \mathbf{s}_{\{1,\{3,2\}\}})\mathbf{c}_3(k_2, \mathbf{s}_{\{\{3,2\},1\}}) = \mathbf{c}_3(k_2, \mathbf{s}_{\{3,2\}})\mathbf{c}_1(k_2, \mathbf{s}_1^-)$$

which is exactly of the form (32).

In the same way one can show the result for soliton 1 starting from equation (31).

The fact that a J-soliton collision is equivalent to the composition of $J(J-1)/2$ pairwise interactions taking place in an arbitrary order (compatible with the choice for the soliton can be proved by an inductive argument that is anchored by the preceding analysis of the three-soliton interaction. To illustrate the induction, we first consider the four-soliton interaction in detail. Then, we generalize to the J-soliton case.

We denote the slowest soliton as 1, the next slowest soliton as 2, etc. In the case of four solitons, the fastest soliton is denoted as 4 and, in the limit $t \to -\infty$, the solitons are arranged in the order 4321 from left to right. As time evolves, faster solitons overtake slower solitons and shift their order through several intermediate steps (transpositions). In the limit $t \to +\infty$ the solitons are arranged in the order 1234. Prior to achieving this ultimate arrangement, the solitons can be in one of three possible arrangements, namely 1243, 2134 or 1324. By pairwise comparison of these three arrangements, we show that the phase shifts of the solitons in the limit $t \to +\infty$ (i.e the solitons in the arrangement 1234) are independent of the penultimate order. This is also illustrated in figure 2.

In both the arrangements 2134 and 1324 the fastest soliton is the rightmost soliton. Thus, in the evolution to the limit as $t \to +\infty$, this fastest soliton does not interact with the other three. We therefore can consider the arrangements 2134 and 1324 to be penultimate

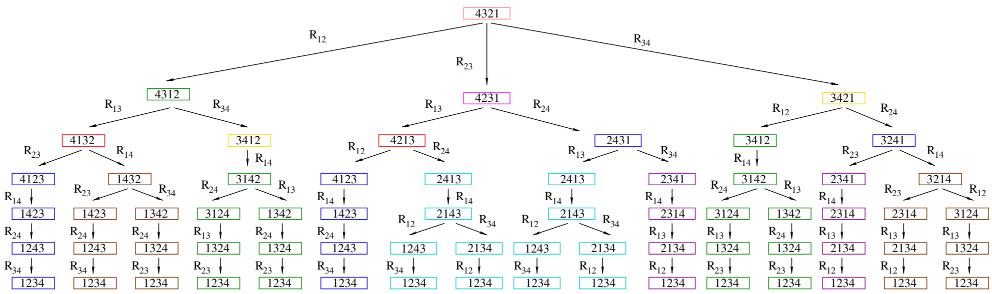


Figure 2. Four-soliton interaction.

intermediate steps in the evolution of the solitons from the arrangement 3214 to the arrangement 1234 in the long-time limit. The only way for the solitons to evolve to the arrangement 3214 from the arrangement 4321 (the order in the limit as $t \to -\infty$) is the following: $4321 \to 3421 \to 3241 \to 3214$. Hence, the phase shifts of the solitons in the arrangement 3214 are unique. In the evolution of the solitons from the order 3214 to 1234, soliton 4 (the fastest soliton) is always rightmost. In effect, the evolution from the arrangement 3214 to the long-time limit 1234 is a three-soliton interaction. In particular, this three-soliton interaction includes evolution of both the arrangement 2134 and the arrangement 1324 to the long-time limit 1234. As shown above, for any three-soliton interaction, the phase shifts in the limit $t \to \infty$ are independent of the order of the transpositions. Therefore, the phase shifts in the long-time limit are the same whether the solitons evolve through the penultimate arrangement 2134 or 1324.

The arrangements 1243 and 2134 are both obtained from the arrangement 2143 by a single transposition: in the evolution $2143 \to 1243$ the two leftmost solitons interact; in the evolution $2143 \to 2134$ the two rightmost solitons interact. Similarly both these arrangements evolve to the arrangement 1234 as a result of a single transposition. Thus, the arrangements 1243 and 2134 are both intermediate steps in the evolution of the solitons from the arrangement 2143 to the arrangement 1234. The key fact is that the evolution from 2143 to 1234 is the result of two isolated transpositions: (i) the interchange of the two leftmost solitons and (ii) the interchange of the two rightmost solitons. Due to this isolation, the temporal order of these two interchanges must be irrelevant to the phase shifts of the solitons in the long-time limit. To see that the shifts of the solitons are unique when they are in the order 2143, we repeat the preceding argument. The evolution of the solitons from the arrangement 4321 (in the limit as $t \to -\infty$) to the order 2143 proceeds as follows: $4321 \to 4231 \to 4213\ 2413 \to 2143$ or $4321 \to 4231 \to 2431\ 2413 \to 2143$. Again, the solitons evolve from the arrangement 4231 to the arrangement 2413 by a pair of isolated two-soliton interactions. Hence, in the arrangement 2413 and in the uniquely following arrangement 2143, the phase shifts of the individual solitons are uniquely determined. This implies that the evolution from the penultimate arrangements 1243 and 2134 to the long-time limit 1234 results in the same ultimate phase shifts. Having shown that evolution from the penultimate arrangements 2134 and 1324 results in the same phase shifts in the limit as $t \to \infty$, we conclude that all three penultimate arrangements 1243, 2134 or 1324 result in the same phase shifts in the evolution of the solitons to the arrangement 1234 in the limit as $t \to +\infty$.

To generalize to the J-soliton case, we observe that the previous examples in fact include all possible cases. Consider any two arrangements of J solitons, denoted as B_1 and B_2 respectively, such that, for each of these arrangements, a single transposition of solitons (i.e. a faster soliton overtakes a slower soliton) results in a common arrangement, denoted as C. There are two possibilities: the respective transpositions by which arrangements B_1 and B_2 evolve to arrangement C involve either (i) two distinct pairs of solitons or (ii) two pairs that include a common soliton. Case (i) corresponds to the second case above. In this case, the soliton interactions consist of two distinct transpositions that occur in an isolated manner and are therefore insensitive to the temporal order in which they occur. Case (ii) is essentially a three-soliton interaction in which B_1 and B_2 are the two possible penultimate arrangements in the evolution to the arrangement C. As shown above, the shifts in this three-soliton interaction are independent of the order of interaction. In particular, the shifts in arrangement C are the same whether the penultimate arrangement is B_1 or B_2. In general, for J solitons there can be up to $J-1$ arrangements of solitons that evolve, each via the transposition of a single pair of solitons, to a common arrangement, C. However, when compared pairwise, these reduce to the two cases given above. Hence, each pair of arrangements can be seen as the penultimate step in the evolution from a common preceding arrangement, denoted as A, to the arrangement C. Therefore, in particular, the phases of the soliton in arrangement C are independent of whether the immediately preceding arrangement was B_1 or B_2. Because this holds for all pairwise comparisons, we conclude that the phases of the solitons in arrangement C are independent of the immediately preceding arrangement, regardless of the number of such possible arrangements. To see that the phase shifts of the soliton induced by the evolution of the system from the arrangement in the limit as $t \to -\infty$ to the arrangement A are independent of the order of interactions, one uses the immediately preceding argument recursively, through a finite number of transpositions, to the arrangement in which the solitons are ordered from fastest to slowest (which is achieved as $t \to -\infty$). We conclude that, in the J-soliton case, the phase shifts between the limits as $t \to \pm\infty$, including the shifts in polarization, are independent of the order in which the solitons interact (see also [13]). We note that this result is obtained in [18] by a completely different argument and was also partially addressed in [15, 19, 20].

3. Soliton interactions as Yang–Baxter maps

Vector-soliton interactions can also be analysed in terms of Yang–Baxter maps. This alternate approach puts the symmetries that underlie the order-invariance of multiple vector-soliton collisions into a broader context.

The quantum Yang–Baxter equation is the following relation on a linear operator $R: V \otimes V \to V \otimes V$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where R_{ij} acts on the ith and jth components of the tensor product $V \otimes V \otimes V$ (see below). Following Drinfeld [21], we consider the set-theoretical version of the quantum Yang–Baxter equation. Let X be any set, $R : X \times X \to X \times X$ a map from the Cartesian product of X into itself. Let $R_{ij} : X^n \to X^n$, $X^n = X \times \cdots \times X$ be the maps which act as R on the ith and jth factors and identically on the others (i.e. leaves the other variables unchanged). More precisely, if $R(x, y) = (f(x, y), g(x, y))$, $x, y \in X$, then

$$\begin{aligned}
i < j \qquad & R_{ij}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, f(x_i, x_j), x_{i+1}, \cdots x_{j-1}, \\
& \quad g(x_i, x_j), x_{j+1}, \ldots, x_n) \\
i > j \qquad & R_{ij}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{j-1}, g(x_i, x_j), x_{j+1}, \cdots x_{i-1}, \\
& \quad f(x_i, x_j), x_{i+1}, \ldots, x_n).
\end{aligned}$$

In particular, for $n = 2$ the operators

$$R_{12} : X \times X \to X \times X \qquad R_{21} : X \times X \to X \times X$$

are such that

$$R_{12} \equiv R \qquad R_{21} = PRP$$

where $P : X \times X \to X \times X$ is the permutation of x and y, i.e. $P(x, y) = (y, x)$.

Following [22], we say R is a Yang–Baxter map if it satisfies the Yang–Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{38}$$

considered as the equality of the maps of $X \times X \times X$ into itself.

If, additionally, R satisfies the relation

$$R_{21}R = I \tag{39}$$

it is called a reversible Yang–Baxter map (this condition means that the map R is reversible with respect to the permutation P).

On the linear space $V = \mathbb{C}^X$ spanned by the set X, then any Yang–Baxter map R induces a linear operator in $V \otimes V$ which satisfies the quantum Yang–Baxter equation in the usual sense. Therefore, we have indeed a very special class of solutions to this equation. Here, the term Yang–Baxter maps is used as opposed to 'set-theoretical solutions to quantum Yang–Baxter equation'.

One can introduce a more general parameter-dependent Yang–Baxter equation as the relation

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2) \tag{40}$$

and the corresponding reversibility condition as

$$R_{21}(\mu, \lambda)R(\lambda, \mu) = I. \tag{41}$$

Veselov [22] also introduces the monodromy maps $T_i^{(n)}$, $i = 1, \ldots, n$ as the maps of X^n into itself defined by the following formulae:

$$T_i^{(n)} = R_{i,i+n-1}R_{i,i+n-2}\cdots R_{i,i+1} \tag{42}$$

where the indices are considered modulo n with the agreement to use n instead of 0. These matrices also play the role of the *transfer matrices* in the theory of quantum Yang–Baxter equation.

In [22] the following theorem is stated:

Theorem 1. *For any reversible Yang–Baxter map R, the monodromy maps $T_i^{(n)}$, $i = 1, \ldots, n$ commute with each other*

$$T_i^{(n)}T_j^{(n)} = T_j^{(n)}T_i^{(n)} \tag{43}$$

and satisfy the property

$$T_1^{(n)}T_2^{(n)}\cdots T_n^{(n)} = I. \tag{44}$$

Conversely, if the maps defined by formula (42) commute and satisfy relation (44) for any $n \geqslant 2$, then R is a reversible Yang–Baxter map.

Note that, for parameter-dependent Yang–Baxter maps, the dependence on the parameter in $R_{i,j}$ with $i > j$ in the definition of the monodromy maps is fixed according to (41).

3.1. Yang–Baxter maps in relation to the theory of solitons

In the case when the solitons have the internal degrees of freedom (see [16]) described by some manifold X (solitons with nontrivial internal parameters, e.g. matrix solitons), their pairwise interaction gives a map from $X \times X$ into itself which satisfies the Yang–Baxter relation. In principle, this indicates that the final result of multi-particle interaction is independent of the order of collisions. In [22–24] the matrix KdV equation is considered. Here we address the same problem for the VNLS equation.

In the case of VNLS, X is the complex vector space of dimension N and the change of polarizations of two solitons is described by the following map:

$$R(k_1, k_2) : (\mathbf{p}_1, \mathbf{p}_2) \to (\mathbf{p}_{\{1,2\}}, \mathbf{p}_{\{2,1\}})$$

defined by (25) and (26), i.e.

$$\mathbf{p}_{\{2,1\}} = \frac{1}{\chi} \frac{k_1 - k_2^*}{k_1^* - k_2^*} \left[\mathbf{p}_2 + \frac{k_1^* - k_1}{k_2^* - k_1^*} (\mathbf{p}_1^* \cdot \mathbf{p}_2) \mathbf{p}_1 \right] \tag{45}$$

$$\mathbf{p}_{\{1,2\}} = \frac{1}{\chi} \frac{k_1 - k_2^*}{k_1 - k_2} \left[\mathbf{p}_1 + \frac{k_2^* - k_2}{k_2 - k_1} (\mathbf{p}_2^* \cdot \mathbf{p}_1) \mathbf{p}_2 \right]. \tag{46}$$

One can show directly that equations (45), (46) define a reversible parameter-dependent Yang–Baxter map, but the calculations are quite long. A better alternative is to make use of matrix factorizations.

3.2. Matrix factorizations

Suppose we have a matrix $A(x, \lambda; \zeta)$ depending on the point $x \in X$, and two parameters $\lambda, \zeta \in \mathbb{C}$; the parameter $\zeta \in \mathbb{C}$ is called a spectral parameter. We assume that A depends on ζ polynomially or rationally.

Consider the product $L = A(y, \mu; \zeta)A(x, \lambda; \zeta)$, then change the order of the factors $L \to \tilde{L} = A(x, \lambda; \zeta)A(y, \mu; \zeta)$ and re-factorize it as $\tilde{L} = A(\tilde{y}, \mu; \zeta)A(\tilde{x}, \lambda; \zeta)$. Suppose that this re-factorization relation

$$A(x, \lambda; \zeta)A(y, \mu; \zeta) = A(\tilde{y}, \mu; \zeta)A(\tilde{x}, \lambda; \zeta) \tag{47}$$

uniquely determines $\tilde{x}$ and $\tilde{y}$, i.e. uniquely defines a map

$$R(\lambda, \mu)(x, y) = (\tilde{x}, \tilde{y}).$$

Then one can show that such a map determined by (47) satisfies the Yang–Baxter relation.

For the case of VNLS equation, given the map (45) and (46), in [13] it is shown (cf section 2.2) that the matrix in (10), i.e. the transmission coefficient, with the substitutions

$$\begin{array}{llll} \lambda \to k_1 & \mu \to k_2 & \zeta \to k & \\ x \to \mathbf{p}_1 & y \to \mathbf{p}_2 & \tilde{x} \to \mathbf{p}_{\{1,2\}} & \tilde{y} \to \mathbf{p}_{\{2,1\}} \\ A(x, \lambda, \zeta) \to \mathbf{c}_j(k, \mathbf{p}_j) & & & \end{array} \tag{48}$$

satisfies

$$\mathbf{c}_1(k, \mathbf{p}_1)\mathbf{c}_2(k, \mathbf{p}_2) = \mathbf{c}_2(k, \mathbf{p}_{\{2,1\}})\mathbf{c}_1(k, \mathbf{p}_{\{1,2\}}) \tag{49}$$

which is exactly the re-factorization relation in [22], where A is identified with the transmission coefficient as prescribed by (48). Relation (49) then proves that the map is indeed a Yang–Baxter map and, consequently, that the soliton interactions are pairwise and independent of the order in which the collisions take place. In other words, this is equivalent to saying the diagram in figure 1 is commutative.

In order to extend this result to the general case of J solitons, $J \geqslant 4$, we first note the following relation satisfied by any Yang–Baxter map:

$$R_{ij} R_{kl} = R_{kl} R_{ij} \tag{50}$$

for any i, j, k, l with $i \neq k, l$, $j \neq k, l$. In terms of soliton collisions, as we specified in section 2.2 (cf also figure 2), this essentially means that it is irrelevant which of the two collisions takes place first whenever they involve separate pairs of solitons. Then one can either use the inductive argument illustrated at the end of section 2.2 (note that this argument, although involved, does not require the reversibility of the Yang–Baxter map, even though one can check directly that such a reversibility condition indeed holds for (25) and (26)), or one can use theorem 1, which, for $n = J$, $i = 1, \ldots, J$, together with (50), in principle, provides the required identities.

We remark that the matrix KdV case considered in [22–24] has

$$A(\xi, \eta, \lambda; \zeta) = I + \frac{2\lambda}{\zeta - \lambda} \frac{\xi \eta^T}{(\xi \cdot \eta)}$$

while for VNLS from (9) with the prescriptions (48) one has

$$A(x, k_j; k) = I - \frac{k_j - k_j^*}{k - k_j^*} \frac{x^H x}{\|x\|^2}$$

where x is an N-component row vector. One can check directly that the re-factorization relation for these matrices leads to the map for the polarization. In the next section we show that this can be done in a more general setting, in analogy with [23].

3.3. More general Yang–Baxter maps

Let V be an N-dimensional complex vector space, $P : V \to V$ be a projector of rank m ($m = 1, \ldots, N-1$): $P^2 = P$. Any such projector is uniquely determined by its kernel $K = \ker P$ and image $L = \mathrm{im} P$, which are two subspaces of V of dimensions m and $N - m$, respectively, complementary to each other: $K \oplus L = V$. The space of all projectors X of rank m is an open set in the product of two Grassmannians $G(m, n) \times G(n - m, n)$, where, as usual, the Grassmannian $G(k, n)$ is the set of k-dimensional subspaces in an n-dimensional vector space. Consider the following matrix:

$$A(P_j, k_j; k) = I - \frac{k_j - k_j^*}{k - k_j^*} P_j$$

and the related re-factorization relation

$$\left(I - \frac{k_1 - k_1^*}{k - k_1^*} P_1\right)\left(I - \frac{k_2 - k_2^*}{k - k_2^*} P_2\right) = \left(I - \frac{k_2 - k_2^*}{k - k_2^*} \tilde{P}_2\right)\left(I - \frac{k_1 - k_1^*}{k - k_1^*} \tilde{P}_1\right) \tag{51}$$

which we can rewrite in polynomial form as

$$\begin{aligned} &((k - k_1^*)I + (k_1^* - k_1)P_1)((k - k_2^*)I + (k_2^* - k_2)P_2) \\ &\quad = ((k - k_2^*)I + (k_2^* - k_2)\tilde{P}_2)((k - k_1^*)I + (k_1^* - k_1)\tilde{P}_1). \end{aligned} \tag{52}$$

The claim is that if $k_1 \neq k_2$ it has a unique solution for $\tilde{P}_1$ and $\tilde{P}_2$. Indeed, this follows from the general theory of matrix polynomials but we can see it directly.

Indeed, let us compare the kernels of both sides of relation (52) when the spectral parameter is $k = k_1^*$. On the right-hand side we have $\tilde{K}_1$ while the left-hand side gives

$$((k_1^* - k_2^*)I + (k_2^* - k_2)P_2)^{-1}K_1 = \left(I + \frac{k_2^* - k_2}{k_1^* - k_2^*}P_2\right)^{-1} K_1$$

so that

$$\tilde{K}_1 = \left(I + \frac{k_2 - k_2^*}{k_1^* - k_2}P_2\right) K_1.$$

Similarly, taking the image of both sides of (52) at $k = k_2^*$ we will have

$$\tilde{L}_2 = \left(I + \frac{k_1^* - k_1}{k_2^* - k_1^*}P_1\right) L_2$$

and one can find $\tilde{K}_2$ and $\tilde{L}_1$ by inverting (51) and then repeating the procedure, first by evaluating it at $k = k_2$ and then at $k = k_1$, getting, respectively

$$\tilde{K}_2 = \left(I + \frac{k_1 - k_1^*}{k_1^* - k_2}P_1\right) K_2 \qquad \tilde{L}_1 = \left(I + \frac{k_2 - k_2^*}{k_1 - k_2}P_1\right) L_1.$$

The formulae obtained in this way determine a parameter-dependent Yang–Baxter map on the set of projectors. One can check that for $m = 1$ one has the formulae for two vector soliton interaction.

4. Generalized linear fractional transformations

In this section we investigate soliton interactions using linear fractional transformations (LFT).

Equations (16) and (17) for the 'intensity redistribution' among the modes of the two solitons can be viewed as a generalized linear fractional transformation [15]. For any $j = 1, \ldots, N$, the jth component of the polarization vectors $\mathbf{p}_1^+, \mathbf{p}_2^+$ can be written as

$$p_2^{+(j)} = \frac{1}{\chi}\frac{k_1 - k_2^*}{k_1^* - k_2^*}\left[p_2^{-(j)} + \frac{k_1^* - k_1}{k_2^* - k_1^*}\sum_{l=1}^{N}\left(p_1^{-(l)}\right)^* p_2^{-(l)} p_1^{-(j)}\right]$$

$$p_1^{+(j)} = \frac{1}{\chi}\frac{k_1 - k_2^*}{k_1 - k_2}\left[p_1^{-(j)} + \frac{k_2^* - k_2}{k_2 - k_1}\sum_{l=1}^{N}\left(p_2^{-(l)}\right)^* p_1^{-(l)} p_2^{-(j)}\right]$$

i.e.,

$$p_2^{+(j)} = \frac{1}{\chi}\frac{k_1 - k_2^*}{k_1^* - k_2^*}\sum_{l=1}^{N} C_{j,l}^{(1)} p_2^{-(l)} \qquad C_{j,l}^{(1)} = \delta_{jl} + \frac{k_1^* - k_1}{k_2^* - k_1^*}\left(p_1^{-(l)}\right)^* p_1^{-(j)} \tag{53}$$

$$p_1^{+(j)} = \frac{1}{\chi}\frac{k_1 - k_2^*}{k_1 - k_2}\sum_{l=1}^{N} C_{j,l}^{(2)} p_1^{-(l)} \qquad C_{j,l}^{(2)} = \delta_{jl} + \frac{k_2^* - k_2}{k_2 - k_1}\left(p_2^{-(l)}\right)^* p_2^{-(j)} \tag{54}$$

where the coefficients $C_{i,j}^{(1)}$ are independent of the polarization of the second soliton and $C_{i,j}^{(2)}$ are independent of the polarization of the first. Then, for $j = 1, \ldots, N-1$, we define

$$\rho_{j,N}^{2\pm} = \frac{p_2^{\pm(j)}}{p_2^{\pm(N)}} \qquad \rho_{j,N}^{1\pm} = \frac{p_1^{\pm(j)}}{p_1^{\pm(N)}}$$

where $\rho^{1,\pm}_{N,N} = \rho^{2,\pm}_{N,N} = 1$. The formulae for the transformation of the polarization yield the generalized LFTs:

$$\rho^{1+}_{j,N} = \frac{\sum_{l=1}^{N} C^{(2)}_{j,l}\rho^{1,-}_{l,N}}{\sum_{l=1}^{N} C^{(2)}_{N,l}\rho^{1,-}_{l,N}} \qquad \rho^{2+}_{j,N} = \frac{\sum_{l=1}^{N} C^{(1)}_{j,l}\rho^{2,-}_{l,N}}{\sum_{l=1}^{N} C^{(1)}_{N,l}\rho^{2,-}_{l,N}} \tag{55}$$

where

$$C^{(1)}_{j,l} = \delta_{j,l} + \frac{k_1^* - k_1}{k_2^* - k_1^*}\left(\sum_{i=1}^{N}\left|\rho^{1,-}_{i,N}\right|^2\right)^{-1}\rho^{1,-}_{j,N}\left(\rho^{1,-}_{l,N}\right)^* \tag{56}$$

$$C^{(2)}_{j,l} = \delta_{j,l} + \frac{k_2^* - k_2}{k_2 - k_1}\left(\sum_{i=1}^{N}\left|\rho^{2,-}_{i,N}\right|^2\right)^{-1}\rho^{2,-}_{j,N}\left(\rho^{2,-}_{l,N}\right)^*. \tag{57}$$

Such transformations were suggested in [15] but the coefficients were not given explicitly for general N. Moreover, the expressions in [15] for $N = 2$ are very complicated and do not reveal the physical collision properties in a clear way. In contrast, with the use of Manakov's formulae the extension to any number of components is straightforward.

We note that, as was observed in [25], two-soliton collisions in the higher component systems can be reduced to two-soliton collisions in the two-component system by using the invariance of solutions of VNLS under unitary transformations. The previous formulae (55), however, are explicit and can be used directly (i.e., without rotating the coordinate system).

We can think of a soliton in a state ρ_1^- as an operator $L_{\rho_1^-}$ that transforms the state of any other soliton by colliding with it. In particular, one can show that: (i) every such operator has a simply determined inverse; (ii) the only fixed points of such an operator are ρ_1^- and its inverse; (iii) no such operator effects a pure rotation of the complex state for all operands; (iv) by concatenating such operators a pure rotation can be achieved [9]. The application of these ideas to implementing all-optical digital computation without employing physically discrete components is also discussed in [9]. Such a computational machine would be based on the propagation and collisions of solitons and would use conservative logic operations, since the soliton collisions preserve the total energy and number of solitons.

Three complex numbers $k, \gamma^{(1)}, \gamma^{(2)}$ fully characterize the bright soliton in the Manakov ($N = 2$) system. Since k (the 'eigenvalue') is unchanged by collisions, two degrees of freedom can be removed immediately from the state characterization. Manakov removed an additional degree of freedom by normalizing the polarization vector $\boldsymbol{\gamma}$. Therefore, single complex-valued polarization state $\rho = \gamma^{(1)}/\gamma^{(2)}$, with only two degrees of freedom, suffices to characterize two-soliton collisions when the constants k of both solitons are given. We will use the pair (ρ, k) to refer to a soliton with variable state $\rho = \frac{q_1(x,t)}{q_2(x,t)} = \gamma^{(1)}/\gamma^{(2)}$, where q_1 and q_2 are the components of the vector $\mathbf{q}$, and (constant) parameter k.

For concreteness, we give the linear fractional transformations that describe soliton interactions in the two-component case. Let k_1 and k_2 represent the constant soliton parameters, ρ_1^- and ρ_2^- the respective soliton states before the impact and suppose the collision transforms ρ_1^- into ρ_1^+ and ρ_2^- into ρ_2^+, i.e.

$$L_{\rho_1^-,k_1}(\rho_2^-, k_2) = \rho_2^+, \qquad R_{\rho_2^-,k_2}(\rho_1^-, k_1) = \rho_1^+. \tag{58}$$

Then, it follows from (55)–(57) that the state transformation for soliton 2 is given by

$$\rho_2^+ = \frac{a_1\rho_2^- + b_1}{c_1\rho_2^- + d_1} \tag{59}$$

where the coefficients are given by

$$a_1 = 1 + \frac{k_2^* - k_1}{k_2^* - k_1^*}|\rho_1^-|^2 \tag{60}$$

$$b_1 = \frac{k_1^* - k_1}{k_2^* - k_1^*}\rho_1^- \tag{61}$$

$$c_1 = \frac{k_1^* - k_1}{k_2^* - k_1^*}(\rho_1^-)^* \tag{62}$$

$$d_1 = \frac{k_2^* - k_1}{k_2^* - k_1^*} + |\rho_1^-|^2 \tag{63}$$

and depend only on the particle in the state ρ_1^-, while the state transformation for soliton 1 is given by

$$\rho_1^+ = \frac{a_2\rho_1^- + b_2}{c_2\rho_1^- + d_2} \tag{64}$$

where the coefficients are functions of the particle in state ρ_2^-

$$a_2 = 1 + \frac{k_2^* - k_1}{k_2 - k_1}|\rho_2^-|^2 \tag{65}$$

$$b_2 = \frac{k_2^* - k_2}{k_2 - k_1}\rho_2^- \tag{66}$$

$$c_2 = \frac{k_2^* - k_2}{k_2 - k_1}(\rho_2^-)^* \tag{67}$$

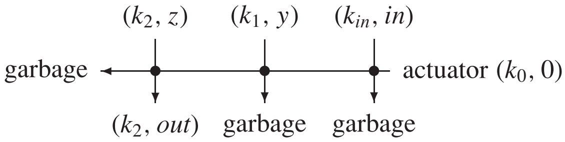
$$d_2 = \frac{k_2^* - k_1}{k_2 - k_1} + |\rho_2^-|^2. \tag{68}$$

We remark that a straightforward calculation shows that for every operator, L_ρ, there is a unique inverse, L_σ, where $\sigma = -1/\rho^*$. We refer to a particle ρ followed by its inverse $-1/\rho^*$ as an inverse pair. Collision with an inverse pair leaves any sequence of particles unchanged. This property is especially useful in designing logic operators because data encoded as inverse particle pairs leave operators unchanged. Hence the unaltered logic operators can be used for subsequent logic operations on new data. Moreover, every operator L_ρ has exactly two distinct fixed points, ρ and $-1/\rho^*$. It follows that a particle is transparent to itself and to the particle corresponding to its inverse operator, and to no others.

In [9] it is shown that there is no single collision operator that is a pure rotation or a multiplication by a scalar. However, a simple nontrivial operator which is pure rotation by $\pi/2$ or multiplication by i is achieved by composing $R_{0,1-i}(\rho, 1+i)$ and $R_{\infty,5-i}(\rho, 1+i)$. The composition of two i operators results in the -1 operator, which, with appropriate encoding of information, can be used as a logical NOT processor. More generally, in [11] NOT, COPY, FANOUT and NAND gates are obtained with sequences of basic three-soliton collision gates, with the proviso that it is possible to time-gate the beam input into the medium. Therefore, time-gated Manakov spatial solitons are computationally universal.

4.1. Three-collision gate

Let us consider the three-collision gate



The actuator is a left-moving soliton in the state 0, hence it is characterized by $\rho_1^- = 0$, $k_0 = \xi_0 + \mathrm{i}\eta_0$ with $\xi_0 < 0$. All the other solitons are right-moving and soliton parameters for in, y, z will be denoted, respectively, by k_{in}, k_1, k_2, which are chosen so that they do not interact with each other, but only with the 'actuator' (for instance, $\mathrm{Re}\,k_{in} \geqslant \mathrm{Re}\,k_1 \geqslant \mathrm{Re}\,k_2 > 0$, but also other arrangements are possible).

Let us introduce the following notation:

$$\rho_2^+ = R(\rho_1^-, \rho_2^-) \qquad \rho_1^+ = L(\rho_1^-, \rho_2^-) \tag{69}$$

where R and L are defined by means of (59) and (64). The successive left-moving products are $L(0, in)$ and $L[L(0, in), y]$, namely one has first the actuator interacting with the soliton labelled in, which gives for the actuator the outgoing state $L(0, in)$; subsequently, the actuator in the state $L(0, in)$ interacts with the soliton in the state y, which will leave the actuator in the state $L[L(0, in), y]$. Finally, the actuator interacts with z, so that the output state out, which is the state of z after this last interaction, is given by $R\{L[L(0, in), y], z\}$.

A COPY gate, for instance, is such that $out = in$ and this corresponds to requiring that 0 maps to 0 and 1 maps to 1. Thus, a COPY gate is obtained by the solution of the two simultaneous complex equations in two complex unknowns:

$$R\{L[L(0, j), y], z\} = j \qquad j = 0, 1. \tag{70}$$

Taking into account the explicit expression of the coefficients of the linear fractional transformations (60)–(63) and (65)–(68), one gets

$$L(0,0) = \frac{b_2(0)}{d_2(0)} = 0 \qquad L(0,1) = \frac{b_2(1)}{d_2(1)} = \frac{k_{in}^* - k_{in}}{k_{in}^* - 2k_0 + k_{in}}, \tag{71}$$

and analogously

$$L[L(0,1), y] = \frac{(k_1 - k_0)(k_{in}^* - k_{in}) + (k_1^* - k_0)(k_{in}^* - k_{in})|y|^2 + (k_{in}^* - 2k_0 + k_{in})(k_1^* - k_1)y}{(k_1^* - k_1)(k_{in}^* - k_{in})y^* + (k_{in}^* - 2k_0 + k_{in})(k_1^* - k_0) + (k_1 - k_0)(k_{in}^* - 2k_0 + k_{in})|y|^2} \tag{72}$$

$$L[L(0,0), y] = \frac{(k_1^* - k_1)y}{(k_1^* - k_0) + (k_1 - k_0)|y|^2} \tag{73}$$

and the conditions (70) become

$$\frac{A_j z + B_j}{C_j z + D_j} = j \qquad j = 0, 1 \tag{74}$$

where

$$A_j = 1 + \frac{k_2^* - k_0}{k_2^* - k_0^*}|L[L(0,j), y]|^2 \qquad B_j = \frac{k_0^* - k_0}{k_2^* - k_0^*}L[L(0,j), y] \tag{75}$$

$$C_j = \frac{k_0^* - k_0}{k_2^* - k_0^*}(L[L(0,j), y])^* \qquad D_j = \frac{k_2^* - k_0}{k_2^* - k_0^*} + |L\,[L(0,j), y]\,|^2. \tag{76}$$

Equations (74) yield

$$(C_1 - A_1)B_0 = (D_1 - B_1)A_0 \tag{77}$$

which is a complex equation for y and

$$z = -\frac{B_0}{A_0} \tag{78}$$

which determines z in terms of y.

If we denote by $L_0(y) = L[L(0, 0), y]$ and $L_1(y) = L[L(0, 1), y]$ in (72) and (73), (77) can be written as

$$\begin{aligned}(k_0^* - k_0)L_0(y)[k_2^* - k_0^* + (k_2^* - k_0)L_1(y)L_1(y)^* + (k_0 - k_0^*)L_1(y)^*] \\ = [k_0 - k_2^* + (k_0^* - k_0)L_1(y) + (k_0^* - k_2^*)L_1(y)L_1^*(y)] \\ \times [k_2^* - k_0^* + (k_2^* - k_0)L_0(y)L_0(y)^*].\end{aligned} \tag{79}$$

It is not known whether, in general (i.e., for any choice of soliton parameters), equation (79) has a unique solution or even if a solution exists. Below, we show that a unique solution exists in certain parameter regimes.

Let us write $k_j = \xi_j + \mathrm{i}\eta_j$ for $j = 0, 1, 2$ and $k_{in} = \xi_{in} + \mathrm{i}\eta_{in}$ and choose, according to the sequences of soliton interactions in a three-collision gate,

$$\xi_0 < 0 < \xi_2 < \xi_1 < \xi_{in}$$

and $\eta_j, \eta_{in} > 0$ for all $j = 0, 1, 2$. The issue of finding solutions for the algebraic equation can be investigated asymptotically. Choosing, for instance, $\eta_0 = \eta_1 = \eta_{in}$, one has

$$L(0, 1) = \frac{\eta_0}{\eta_0 + \mathrm{i}(\xi_{in} - \xi_0)}$$

and

$$L_0(y) = \frac{y}{1 + \mathrm{i}(2\eta_0)^{-1}(1 + |y|^2)(\xi_1 - \xi_0)} \tag{80}$$

$$\begin{aligned}L_1(y) = \{&y(1 + y^*) + \mathrm{i}(2\eta_0)^{-1}[(\xi_1 - \xi_0)(1 + |y|^2) + 2(\xi_{in} - \xi_0)y]\}/\{(1 + y^*) \\ &+ \mathrm{i}(2\eta_0)^{-1}[(\xi_1 - \xi_0)(1 + |y|^2) + 2(\xi_{in} - \xi_0)] \\ &- 2(2\eta_0)^{-2}(\xi_{in} - \xi_0)(\xi_1 - \xi_0)(1 + |y|^2)\}.\end{aligned} \tag{81}$$

If we also take $\eta_2 = \eta_0$, the algebraic equation (79) for y becomes

$$\begin{aligned}L_0(y)(1 - L_1(y))(L_0(y) - L_1(y))^* \\ + \frac{\mathrm{i}}{2\eta_0}(\xi_2 - \xi_0)[L_0(y)(1 + |L_1(y)|^2)(1 + L_0(y))^* + (1 - L_1(y))(1 + |L_0(y)|^2)] \\ - \frac{1}{(2\eta_0)^2}(\xi_2 - \xi_0)^2(1 + |L_0(y)|^2)(1 + |L_1(y)|^2) = 0.\end{aligned} \tag{82}$$

Since from (80) and (81)

$$L_0(y) = \frac{y}{1 + \frac{\mathrm{i}}{2\eta_0} Y_0}, \qquad L_1(y) = \frac{yY_1 + \frac{\mathrm{i}}{2\eta_0}(Y_0 + yX_1)}{Y_1 + \frac{\mathrm{i}}{2\eta_0}(Y_0 + X_1) - \frac{1}{(2\eta_0)^2} Y_0 X_1} \tag{83}$$

where

$$Y_0 = (\xi_1 - \xi_0)(1 + |y|^2) \qquad Y_1 = 1 + y^* \qquad X_1 = 2(\xi_{in} - \xi_0) \tag{84}$$

we can write

$$L_0(y) = y + L^{(1)}_{0,\mathrm{rem}}(y) \qquad L^{(1)}_{0,\mathrm{rem}}(y) = -\frac{\mathrm{i}}{2\eta_0} \frac{yY_0}{1 + \frac{\mathrm{i}}{2\eta_0} Y_0} \tag{85}$$

$$L^{(1)}_{0,\mathrm{rem}}(y) = -\frac{\mathrm{i}}{2\eta_0} yY_0 + L^{(2)}_{0,\mathrm{rem}}(y) \qquad L^{(2)}_{0,\mathrm{rem}}(y) = -\frac{1}{(2\eta_0)^2} \frac{yY_0^2}{1 + \frac{\mathrm{i}}{2\eta_0} Y_0} \tag{86}$$

and analogously

$$L_1(y) = y + L^{(1)}_{1,\mathrm{rem}}(y) \qquad L^{(1)}_{1,\mathrm{rem}}(y) = \frac{\mathrm{i}}{2\eta_0} Y_0 \frac{1 - y - \frac{\mathrm{i}}{2\eta_0} yX_1}{Y_1 + \frac{\mathrm{i}}{2\eta_0}(Y_0 + X_1) - \frac{1}{(2\eta_0)^2} Y_0 X_1} \tag{87}$$

$$L^{(1)}_{1,\mathrm{rem}}(y) = \frac{\mathrm{i}}{2\eta_0} \frac{(1 - y)Y_0}{Y_1} + L^{(2)}_{1,\mathrm{rem}}(y) \tag{88}$$

$$L^{(2)}_{1,\mathrm{rem}}(y) = \frac{1}{(2\eta_0)^2} \frac{(1 - y)Y_0}{Y_1} \frac{(Y_0 + X_1) + \frac{y}{1-y} Y_1 X_1 - \frac{\mathrm{i}}{2\eta_0} Y_0 X_1}{Y_1 - \frac{\mathrm{i}}{2\eta_0}(Y_0 + X_1) - \frac{1}{(2\eta_0)^2} Y_0 X_1}. \tag{89}$$

Note that $L^{(1)}_{j,\mathrm{rem}}(y)$ for both $j = 1, 2$ is of order $O\big(\eta_0^{-1}\big)$ and $L^{(2)}_{j,\mathrm{rem}}(y)$ is of order $O\big(\eta_0^{-2}\big)$ for large η_0. This suggests one could look for order 1 asymptotic solutions in y of this equation corresponding to $\eta_0 \gg 1$ and finite ξ. Substituting into (82), the leading order terms in η_0 yield

$$\mathrm{Red}\, E(y) \equiv -\frac{\mathrm{i}}{2\eta_0} \frac{(1 + |y|^2)^2}{1 + y} [(\xi_1 - \xi_0)y(1 - y) + (\xi_2 - \xi_0)(1 + y)] = 0$$

and this equation admits exactly two solutions, say $y^{(1)}$ and $y^{(2)}$. Moreover, these solutions are distinct and real if, as we assumed, $(\xi_1 - \xi_0)(\xi_2 - \xi_0) > 0$. Precisely, one can write equation (82) in the form

$$\mathrm{Red}\, E(y) = -E_{\mathrm{rem}}(y) \tag{90}$$

where

$$\begin{aligned}
E_{\mathrm{rem}}(y) = {} & \frac{\mathrm{i}}{2\eta_0}(\xi_1 - \xi_0) \frac{(1 + |y|^2)^2}{1 + y} \big[(1 - y)L^{(1)}_{0,\mathrm{rem}}(y) - yL^{(1)}_{1,\mathrm{rem}}(y) - L^{(1)}_{0,\mathrm{rem}}(y)L^{(1)}_{1,\mathrm{rem}}(y)\big] \\
& + L_0(y)(1 - L_1(y))\big(L^{(2)}_{1,\mathrm{rem}}(y) - L^{(2)}_{0,\mathrm{rem}}(y)\big)^* \\
& + \frac{\mathrm{i}}{2\eta_0}(\xi_2 - \xi_0)\big[L_0(y)(1 + |L_1(y)|^2)\big(L^{(1)}_{0,\mathrm{rem}}(y)\big)^* - L^{(1)}_{1,\mathrm{rem}}(y)(1 + |L_0(y)|^2) \\
& + L_0(y)(1 + L_0(y))^*\big(\big|L^{(1)}_{1,\mathrm{rem}}(y)\big|^2 + 2\,\mathrm{Re}\big(y^* L^{(1)}_{1,\mathrm{rem}}(y)\big)\big) \\
& + L^{(1)}_{0,\mathrm{rem}}(y)(1 + |L_1(y)|^2)(1 + L_0(y))^* \\
& + (1 - L_1(y))\big(\big|L^{(1)}_{0,\mathrm{rem}}(y)\big|^2 + 2\,\mathrm{Re}\big(y^* L^{(1)}_{0,\mathrm{rem}}(y)\big)\big)\big] \\
& - \frac{(\xi_2 - \xi_0)^2}{(2\eta_0)^2}(1 + |L_0(y)|^2)(1 + |L_1(y)|^2)
\end{aligned} \tag{91}$$

and all the terms in the 'remainder' $E_{\mathrm{rem}}(y)$ are at least proportional to η_0^{-2}.

In a neighbourhood of one of the roots of the 'reduced' equation Red $E(y) = 0$, say $y = y^{(1)}$, equation (90) can be written as

$$y - y^{(1)} = -\frac{2\mathrm{i}\eta_0}{\xi_1 - \xi_0} \frac{1 + y}{(1 + |y|^2)^2 (y - y^{(2)})} E_{\mathrm{rem}}(y). \tag{92}$$

The last one can be written as a fixed point system of equations (note that in (92) y is complex)

$$x_1 = F(x_1, x_2) \qquad x_2 = G(x_1, x_2) \tag{93}$$

and if $\{\tilde{x}_1, \tilde{x}_2\}$ is a solution of the equation and $\left\{x_1^{(0)}, x_2^{(0)}\right\}$ is an approximation to $\{\tilde{x}_1, \tilde{x}_2\}$, the successive approximations can be generated by

$$x_1^{(j+1)} = F\left(x_1^{(j)}, x_2^{(j)}\right) \qquad x_2^{(j+1)} = G\left(x_1^{(j)}, x_2^{(j)}\right).$$

This iteration will converge if

(i) F, G and their partial derivatives are continuous in a neighbourhood I of the root $\{\tilde{x}_1, \tilde{x}_2\}$, where I consists of all $\{x_1, x_2\}$ with $|x_1 - \tilde{x}_1| < \epsilon$, $|x_2 - \tilde{x}_2| < \epsilon$, for some positive ϵ;

(ii) the following inequalities are satisfied

$$\left|\partial_{x_1} F\right| + \left|\partial_{x_2} F\right| \leqslant K \qquad \left|\partial_{x_1} G\right| + \left|\partial_{x_2} G\right| \leqslant K$$

for all points $\{x_1, x_2\}$ in I and some $K < 1$;

(iii) the initial approximation $\left\{x_1^{(0)}, x_2^{(0)}\right\}$ is chosen in I.

In the case of equation (92), since $F = \frac{1}{2\eta_0}\tilde{F}$, $G = \frac{1}{2\eta_0}\tilde{G}$ where $\tilde{F}$, $\tilde{G}$ are rational functions of x_1, x_2, it follows that it is always possible to choose η_0 large enough so that conditions (i)–(iii) are satisfied and, consequently, the fixed point iteration procedure converges to a solution.

In [10] a similar nonlinear algebraic system was obtained and analysed numerically. However, to our knowledge, no proof for the existence of a solution has been given. Here we have shown that, in fact, there are parameter regimes for which a unique solution for this basic three-collision gate is guaranted to exist.

Using the same basic three-soliton collisions, one can also implement the NOT gate, i.e. a gate such that *out* = 0 when *in* = 1 and vice versa. The NOT gate is obtained by the solution of the following two simultaneous complex equations in two complex unknowns:

$$R\{L[L(0, j), y], z\} = 1 - j \qquad j = 0, 1. \tag{94}$$

Other logical operations are implemented similarly.

4.2. Order invariance of three-soliton collisions via LFT

It is known that the composition of two linear fractional transformations

$$f(z) = f_1(f_2(z))$$

where

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

is still a linear fractional transformation with coefficients

$$a_3 = a_1 a_2 + b_1 c_2, \qquad b_3 = a_1 b_2 + b_1 d_2$$
$$c_3 = c_1 a_2 + d_1 c_2, \qquad d_3 = c_1 b_2 + d_1 d_2.$$

Hence, if we introduce for $j = 1, 2$

$$a_{j3} = 1 + \frac{k_3^* - k_j}{k_3^* - k_j^*}|\rho_j^-|^2 \qquad b_{j3} = \frac{k_j^* - k_j}{k_3^* - k_j^*}\rho_j^- \tag{95}$$

$$c_{j3} = \frac{k_j^* - k_j}{k_3^* - k_j^*}(\rho_j^-)^* \qquad d_{j3} = \frac{k_3^* - k_j}{k_3^* - k_j^*} + |\rho_j^-|^2 \tag{96}$$

the state of soliton 3 after the subsequent interactions with solitons 2 and 1 is characterized by

$$\rho_3^+ = \frac{A\rho_3^- + B}{C\rho_3^- + D} \tag{97}$$

where

$$A = a_{13}a_{23} + b_{13}c_{23} \qquad B = a_{13}b_{23} + b_{13}d_{23} \tag{98}$$

$$C = c_{13}a_{23} + d_{13}c_{23} \qquad D = c_{13}b_{23} + d_{23}d_{13}. \tag{99}$$

On the the other hand, if the order of the soliton interaction is $2 \leftrightarrow 1$, $3 \leftrightarrow 1$ and finally $3 \leftrightarrow 2$, the composition of the corresponding linear fractional transformations yields for soliton 3

$$\tilde{\rho}_3^+ = \frac{\tilde{A}\rho_3^- + \tilde{B}}{\tilde{C}\rho_3^- + \tilde{D}} \tag{100}$$

where

$$\tilde{A} = \tilde{a}_{13}\tilde{a}_{23} + \tilde{b}_{23}\tilde{c}_{13} \qquad \tilde{B} = \tilde{a}_{23}\tilde{b}_{13} + \tilde{b}_{23}\tilde{d}_{13} \tag{101}$$

$$\tilde{C} = \tilde{c}_{23}\tilde{a}_{13} + \tilde{d}_{23}\tilde{c}_{13} \qquad \tilde{D} = \tilde{c}_{23}\tilde{b}_{13} + \tilde{d}_{13}\tilde{d}_{23}. \tag{102}$$

with

$$\tilde{a}_{j3} = 1 + \frac{k_3^* - k_j}{k_3^* - k_j^*}\left|\rho_j^+\right|^2 \qquad \tilde{b}_{j3} = \frac{k_j^* - k_j}{k_3^* - k_j^*}\rho_j^+ \tag{103}$$

$$\tilde{c}_{j3} = \frac{k_j^* - k_j}{k_3^* - k_j^*}\left(\rho_j^+\right)^* \qquad \tilde{d}_{j3} = \frac{k_3^* - k_j}{k_3^* - k_j^*} + \left|\rho_j^+\right|^2 \tag{104}$$

and ρ_j^+ for $j = 1, 2$ are given by (59)–(68).

Choosing, without loss of generality, $\rho_1^- = 0$ and arbitrary ρ_2^-, from (59)–(68) it follows that

$$\rho_1^+ = \frac{(k_2^* - k_2)\rho_2^-}{(k_2 - k_1)|\rho_2^-|^2 + k_2^* - k_1} \qquad \rho_2^+ = \frac{k_2^* - k_1^*}{k_2^* - k_1}\rho_2^-$$

and substituting into (101) and (102), one can check that

$$\tilde{A} = cA \qquad \tilde{B} = cB \qquad \tilde{C} = cC \qquad \tilde{D} = cD$$

where the common multiplier is given by

$$c = \frac{|k_2 - k_1|^2|\rho_2^-|^4 + 2|k_2^* - k_1|^2|\rho_2^-|^2 + |k_2^* - k_1|^4||k_2 - k_1|^{-2}}{|k_2^* - k_1|^2[|\rho_2^-|^2 + (k_2^* - k_1)(k_2 - k_1)^{-1}][|\rho_2^-|^2 + (k_2 - k_1^*)(k_2^* - k_1^*)^{-1}]}.$$

This is yet another proof that the net shift in a soliton polarization state is independent of the order of soliton interaction in a two-component, three-soliton collision. The inductive argument given previously allows one to extend this result to the collision of an arbitrary number of solitons.

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