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Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with nonvanishing boundary conditions

Mark J Ablowitz[1], **Gino Biondini**[2] **and Barbara Prinari**[3]

[1] Department of Applied Mathematics, University of Colorado, Boulder, CO, USA
[2] Department of Mathematics, State University of New York at Buffalo, Buffalo, NY, USA
[3] Dipartimento di Fisica and Sezione INFN, Università del Salento, Lecce, Italy

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Abstract
The inverse scattering transform for an integrable discretization of the defocusing nonlinear Schrodinger equation with nonvanishing boundary values at infinity is constructed. This problem had been previously studied, and many key results had been established. Here, a suitable transformation of the scattering problem is introduced in order to address the open issue of analyticity of eigenfunctions and scattering data. Moreover, the inverse problem is formulated as a Riemann–Hilbert problem on the unit circle, and a modification of the standard procedure is required in order to deal with the dependence of asymptotics of the eigenfunctions on the potentials. The discrete analog of Gel'fand–Levitan–Marchenko equations is also derived. Finally, soliton solutions and solutions in the small-amplitude limit are obtained and the continuum limit is discussed.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t = q_{xx} - 2\sigma|q|^2 q \tag{1.1}$$

is a universal model for weakly nonlinear dispersive waves, and as such it appears in many different physical contexts. It is well known that the initial-value problem for equation (1.1) on the infinite line ($-\infty < x < \infty$) can be solved via the inverse scattering transform (IST) [1], and the properties of IST for (1.1) have been extensively investigated in the literature, both in the focusing ($\sigma = -1$) and in the defocusing ($\sigma = 1$) cases. In particular, the defocusing case with nonvanishing boundary conditions was first studied in 1973 [2]; the problem was subsequently clarified and generalized in various works [3–9]. A detailed study can be found in the monograph [10]. In particular, it is well known that equation (1.1) with $\sigma = 1$ admits soliton solutions with nontrivial boundary conditions, the so-called dark/gray solitons, which have the form

$$q(x,t) = q_0\, \mathrm{e}^{2\mathrm{i}q_0^2 t}[\cos\alpha + \mathrm{i}\sin\alpha \tanh[\sin\alpha\, q_0(x - 2q_0\cos\alpha t - x_0)]] \tag{1.2}$$

with q_0, α and x_0 being arbitrary real parameters. Such solutions satisfy the boundary conditions

$$q(x,t) \to q_\pm(t) = q_0\, \mathrm{e}^{2\mathrm{i}q_0^2 t \pm \mathrm{i}\alpha} \qquad \text{as} \quad x \to \pm\infty$$

and appear as localized dips of intensity $q_0^2 \sin^2\alpha$ on the background field q_0. The properties of dark solitons have been extensively discussed in the review article [11]. It is interesting to note, however, that while the IST for the scalar NLS equation was developed many years ago, the formulation of IST for the vector nonlinear Schrödinger (VNLS) equation has been only recently completed [12]. The IST for certain matrix NLS systems has been studied in [13].

In this paper, we study the IST for a semi-discrete (discrete in space, continuous in time) version of the NLS equation (1.1). In general, a discretization of an integrable partial differential equation (PDE) is likely to be non-integrable. That is, even though the integrable PDE is the compatibility condition of a linear operator pair, one is not guaranteed to have a pair of linear equations corresponding to a generic discretization of the PDE. On the other hand, for the differential-difference equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} q_n = \frac{1}{h^2}(q_{n+1} - 2q_n + q_{n-1}) - \sigma |q_n|^2 (q_{n+1} + q_{n-1}), \tag{1.3}$$

which is referred to here as the integrable discrete NLS (IDNLS) equation, and which is a $O(h^2)$ finite-difference approximation of (1.1), there is such an associated operator pair (cf section 2). The corresponding scattering problem is usually referred to as the Ablowitz–Ladik scattering problem (cf [14, 15] and the monograph [16]). Besides being used as a basis for numerical schemes for its continuous counterpart, the IDNLS equation has also numerous physical applications, related to the dynamics of anharmonic lattices [17], self-trapping on a dimer [18], Heisenberg spin chains [19, 20] etc.

The purpose of this work is to develop the IST under nonvanishing boundary conditions for the following system of differential-difference equations on the doubly infinite lattice:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau} Q_n = Q_{n+1} - 2Q_n + Q_{n-1} - Q_n R_n (Q_{n+1} + Q_{n-1}), \tag{1.4a}$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau} R_n = R_{n+1} - 2R_n + R_{n-1} - Q_n R_n (R_{n+1} + R_{n-1}), \tag{1.4b}$$

with $n \in \mathbb{Z}$. Equations (1.4) include the IDNLS equations (1.3) via the reductions[4] $R_n = \sigma Q_n^*$, with $Q_n = hq_n$ and $\tau = t/h^2$. The IST for equations (1.4) with vanishing boundary conditions was studied in [15]. The case of interest here, namely equations (1.4) with $R_n = Q_n^*$ and nonvanishing boundaries, was also studied in [21], and we often refer to some key results already established there. In some important respects, however, we part from the approach in [21], and we solve the problem differently, most notably by relaxing the implicit requirement in [21] that the eigenfunctions be entire functions of the scattering parameter (which otherwise precludes the possibility of studying non-soliton solutions). We establish the analyticity properties of eigenfunctions and scattering data from the direct scattering problem for potentials in a suitable function class; we formulate the inverse problem as a Riemann–Hilbert problem which also takes into account the asymptotic dependence (at large

[4] Throughout this work, the conjugate of a complex number will be denoted by an asterisk *. Accordingly, overbars, which (following standard notations) will be used extensively, do *not* mean complex conjugate.

and small values of the scattering parameter) of the eigenfunctions on the potentials, and we discuss the small-amplitude and continuum limits of the problem.

The paper is organized as follows. In section 2, we discuss the direct scattering problem. As in the continuous case, the spectral parameter of the associated scattering problem for the IDNLS equation is an element of a two-sheeted Riemann surface. Unlike the continuous system, however, for the discrete problem the Riemann surface has four branch points, located on the unit circle. The Riemann surface in the discrete case is genus 1, that is, topologically equivalent to a torus. In spite of this, due to the symmetries in the location of the branch points, the elliptic Riemann surface admits an involutive automorphism and can therefore be uniformized in algebraic form. Following [21], in section 2.2 we therefore introduce an algebraic parametrization for the uniformization coordinate. Sections 2.3 and 2.4 are devoted to the study of the analyticity of the scattering eigenfunctions. In [21], the eigenfunctions are assumed to be entire, and the equations of the inverse problem derived accordingly, which imposes strong restrictions on the class of admissible potentials. We show that in general the eigenfunctions are analytic inside or outside the unit circle of the uniform variable when $\sum_{j=\mp\infty}^{n} |Q_j - Q_\mp| < \infty$ for any finite n, where $Q_\pm = \lim_{n\pm\infty} Q_n$. In section 2.5, we study the asymptotic behavior of the eigenfunctions for relevant values of the scattering parameter and show this behavior explicitly depends on the potentials. In section 2.6, we study the properties of the scattering coefficients and their symmetries and we discuss the discrete spectrum. The inverse problem is formulated in section 3 as a Riemann–Hilbert (RH) problem associated with analytic eigenfunctions. The formulation must be modified with respect to the standard case in order to take into account that the asymptotic behavior of the eigenfunctions for relevant values of the scattering parameter explicitly depends on the potentials, which are unknown in the inverse problem. The RH problem is then transformed into a closed linear system of algebraic-integral equations. In section 4, the Gel'fand–Levitan–Marchenko (GLM) equations are derived, which are usually key for studying issues of existence and uniqueness of solutions of the inverse problem. The time evolution of the scattering data is discussed in section 5, where an infinite set of conserved quantities is also obtained. Explicit solutions are given in section 6, where the one-soliton solution is derived. In section 7, the linearized solution of the IDNLS equation is obtained and found to be consistent with that of the small-amplitude limit obtained from the RH formulation. Also, in section 8 the continuum limit is explicitly carried out. Finally, the proof of various statements in the text is given in the appendix.

2. Direct scattering problem

The Lax pair for system (1.4) is given by [14, 15]

$$v_{n+1} = \begin{pmatrix} z & Q_n \\ R_n & 1/z \end{pmatrix} v_n, \tag{2.1a}$$

$$\frac{\partial v_n}{\partial \tau} = \begin{pmatrix} \mathrm{i}Q_n R_{n-1} - \frac{\mathrm{i}}{2}(z-1/z)^2 & -\mathrm{i}zQ_n + \mathrm{i}Q_{n-1}/z \\ \mathrm{i}R_n/z - \mathrm{i}zR_{n-1} & -\mathrm{i}Q_{n-1}R_n + \frac{\mathrm{i}}{2}(z-1/z)^2 \end{pmatrix} v_n, \tag{2.1b}$$

where v_n is a two-component vector, $z \in \mathbb{C}$ is the scattering parameter and $Q_n(\tau)$, $R_n(\tau)$ are the potentials. That is, the compatibility condition between (2.1a) and (2.1b) (namely, $\partial v_{n+1}/\partial\tau = (\partial v_m/\partial\tau)_{m=n+1}$) is equivalent to the evolution equations (1.4) for $Q_n(\tau)$ and $R_n(\tau)$. It is also convenient to write (2.1) as

$$v_{n+1} = \mathbf{L}_n v_n, \tag{2.2a}$$

$$\frac{\partial v_n}{\partial \tau} = \mathbf{M}_n v_n, \tag{2.2b}$$

where $\mathbf{M}_n v_n$ denotes the right-hand side of (2.1*b*), and where

$$\mathbf{L}_n = \mathbf{Z} + \mathbf{Q}_n, \qquad \mathbf{Z} = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \qquad \mathbf{Q}_n = \begin{pmatrix} 0 & Q_n \\ R_n & 0 \end{pmatrix}. \tag{2.3}$$

Throughout this work, we will use boldface fonts to denote 2×2 matrices.

2.1. Eigenfunctions

The direct scattering problem requires characterizing the spectrum of (2.2*a*) and the corresponding eigenfunctions in terms of the potentials Q_n and R_n. As customary, when discussing the direct and inverse problems we will omit the dependence on τ of the potentials and eigenfunctions, since τ only plays the role of a constant parameter in these contexts.

We seek solutions of the scattering problem (2.2*a*) with $R_n = Q_n^*$ and with the potentials satisfying the following boundary conditions:

$$\lim_{n\to\pm\infty} Q_n = Q_\pm \equiv Q_o\, \mathrm{e}^{\mathrm{i}\theta_\pm}, \qquad \lim_{n\to\pm\infty} R_n = R_\pm \equiv Q_o\, \mathrm{e}^{-\mathrm{i}\theta_\pm}, \tag{2.4}$$

where Q_o is a real and positive constant. The solutions of (2.2*a*) then satisfy asymptotically as $n \to \pm\infty$,

$$v_{n+1} \sim \begin{pmatrix} z & Q_o\, \mathrm{e}^{\mathrm{i}\theta_\pm} \\ Q_o\, \mathrm{e}^{-\mathrm{i}\theta_\pm} & 1/z \end{pmatrix} v_n. \tag{2.5}$$

Denoting by $v_n^{(j)}$ (for $j = 1, 2$) the jth component of the vector v_n, (2.5) yields

$$v_{n+2}^{(j)} = (z + 1/z) v_{n+1}^{(j)} + \left(Q_o^2 - 1\right) v_n^{(j)} \qquad j = 1, 2. \tag{2.6}$$

Looking for solutions of (2.6) in the form $v_n^{(j)} \sim \alpha^n$ we get $\alpha + r^2/\alpha = z + 1/z$, with

$$r = \sqrt{1 - Q_o^2}. \tag{2.7}$$

Throughout this work, we assume $0 < Q_o < 1$. As a consequence, $0 < r < 1$. Introducing $\lambda = \alpha/r$, a solution of (2.6) is given by $v_n^{(j)} \sim \lambda^n r^n$, with λ such that

$$r(\lambda + 1/\lambda) = z + 1/z. \tag{2.8}$$

Similarly, looking for solutions in the form $v_n^{(j)} \sim 1/\beta^n$, we get $1/\beta + r^2\beta = z + 1/z$. Introducing $\lambda = \beta r$, it follows that λ satisfies the same equation (2.8). We conclude that two independent solutions of (2.6) are given by

$$v_n^{(j)} \sim \lambda^n r^n, \qquad j = 1, 2, \qquad n \to \pm\infty,$$
$$v_n^{(j)} \sim r^n/\lambda^n, \qquad j = 1, 2, \qquad n \to \pm\infty,$$

where λ as a function of z is defined by (2.8). Therefore one can uniquely define solutions of the scattering problem (2.1*a*) by their asymptotic behavior at large n. We therefore introduce the eigenfunctions $\phi_n(z)$, $\bar{\phi}_n(z)$, $\psi_n(z)$ and $\bar{\psi}_n(z)$ which are the analogs of the Jost eigenfunctions in the case of vanishing boundary conditions, and are defined by

$$\bar{\psi}_n(z) \sim \lambda^n r^n \begin{pmatrix} Q_+ \\ \lambda r - z \end{pmatrix} \qquad n \to +\infty \tag{2.9a}$$

$$\phi_n(z) \sim \lambda^n r^n \begin{pmatrix} Q_- \\ \lambda r - z \end{pmatrix} \qquad n \to -\infty \tag{2.9b}$$

$$\psi_n(z) \sim \lambda^{-n} r^n \begin{pmatrix} \lambda r - z \\ -R_+ \end{pmatrix} \qquad n \to +\infty \tag{2.9c}$$

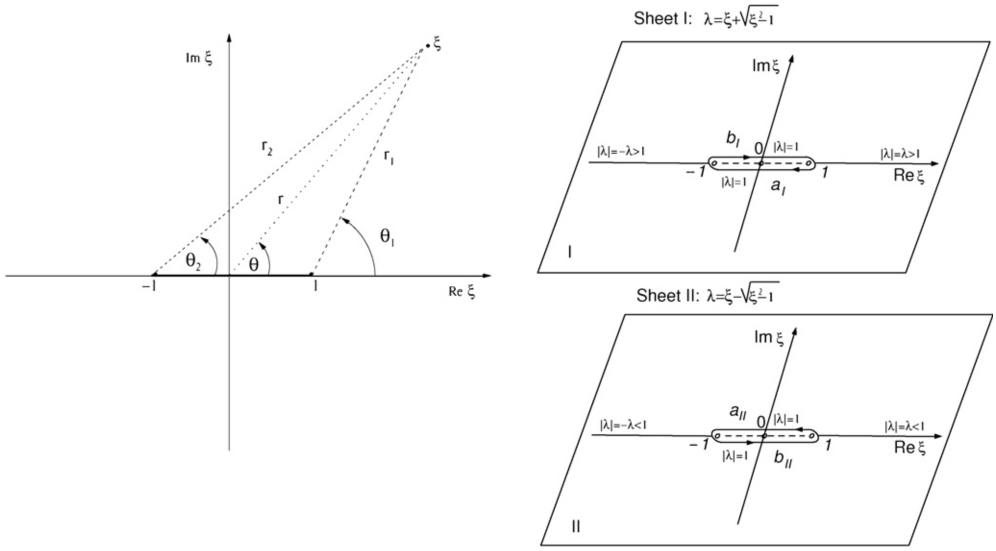

Figure 1. Left: the choice of branch cut in the complex ξ-plane. Right: the two-sheeted covering of the complex ξ-plane defined by $\lambda = \xi + (\xi^2 - 1)^{1/2}$.

$$\bar{\phi}_n(z) \sim \lambda^{-n} r^n \begin{pmatrix} \lambda r - z \\ -R_- \end{pmatrix} \qquad n \to -\infty. \tag{2.9d}$$

It is convenient to introduce the variable $\xi(z) = (z + 1/z)/2r$, such that (2.8) yields

$$\lambda(z) = \xi \pm \sqrt{\xi^2 - 1}, \qquad 1/\lambda(z) = \xi \mp \sqrt{\xi^2 - 1}. \tag{2.10}$$

From definition (2.10) it then follows that for each $z \in \mathbb{C}$ there are two possible values of λ, and the branch points of $\lambda(z)$ are located at $\xi^2 = 1$. Namely, from (2.10), the branch points are given by the solutions of $(z + 1/z)^2 = 4r^2$, that is $z^2 \mp 2rz + 1 = 0$. Therefore, in terms of the variable z the eigenfunctions have four branch points located on the unit circle, which we denote by $\pm z_0$ and $\pm z_0^*$, with

$$z_0 = r + \mathrm{i}Q_o. \tag{2.11}$$

It is therefore natural to define the Riemann surface of equation $(\lambda - \xi)^2 = \xi^2 - 1$ obtained by gluing together two copies of the extended complex ξ-plane, which we will call I and II, cut along the segment $(-1, 1)$. One can introduce the local polar coordinates

$$\xi - 1 = r_1\,\mathrm{e}^{\mathrm{i}\theta_1}, \qquad \xi + 1 = r_2\,\mathrm{e}^{\mathrm{i}\theta_2}, \qquad \xi = r\,\mathrm{e}^{\mathrm{i}\theta}, \qquad 0 \leqslant \theta_1, \theta_2, \theta < 2\pi, \tag{2.12}$$

with the magnitudes r_1, r_2 and r uniquely fixed by the location of the point ξ : $r_1 = |\xi - 1|$, $r_2 = |\xi + 1|$ and $r = |\xi|$ (cf figure 1(a)). Then on the sheet I one can define

$$\lambda(\xi) = r\,\mathrm{e}^{\mathrm{i}\theta} + (r_1 r_2)^{1/2}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}. \tag{2.13}$$

If we let $\Theta = (\theta_1 + \theta_2)/2$, then Θ is discontinuous along the segment $(-1, 1)$; indeed, one has $\lambda = \pm r + \mathrm{i}\sqrt{r_1 r_2}$ on a_{I} and $\lambda = \pm r - \mathrm{i}\sqrt{r_1 r_2}$ on b_{I} (cf figure 1(b)). Conversely, on sheet II one defines

$$\lambda(\xi) = r\,\mathrm{e}^{\mathrm{i}\theta} - (r_1 r_2)^{1/2}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}, \tag{2.14}$$

again with a cut along the segment $(-1, 1)$. The lower branch of the cut on sheet I, denoted by a_{I}, is then glued with the upper branch on sheet II, called a_{II}, while b_{I} is glued with b_{II}

(cf figure 1(b)). Points on this Riemann surface are uniquely identified by a pair (z, λ). When there is no ambiguity as to the location of the point on a sheet, however, we will omit the explicit dependence on $\lambda = \lambda(z)$.

In the following, we will show that the eigenfunctions $\phi_n(z)$ and $\psi_n(z)$ are analytic on sheet I, while $\bar{\phi}_n(z)$ and $\bar{\psi}_n(z)$ are analytic on sheet II. This is related to the fact that one can prove that $|\lambda| > 1$ on sheet I and $|\lambda| < 1$ on sheet II. Along the cuts, λ is clearly discontinuous, but $|\lambda| = 1$ on each side (see the appendix for details). (Note that, alternatively, one could consider a two-sheeted Riemann surface with a cut along the semi-infinite lines $(-\infty, -1) \cup (1, \infty)$. With this choice, however, $|\lambda| - 1$ does not have definite sign on either sheet, therefore in the following we will take the cut to be on the segment.)

The Wronskian of two solutions of the scattering problem (2.1a) is defined as usual as

$$\mathrm{Wr}(v_n, w_n) = \det(v_n, w_n),$$

and it satisfies the recursion relation

$$\mathrm{Wr}(v_{n+1}, w_{n+1}) = (1 - Q_n R_n)\,\mathrm{Wr}(v_n, w_n).$$

Then, since $\mathrm{Wr}(r^{-n}\bar{\phi}_n, r^{-n}\phi_n) \sim (\lambda r - z)^2 + Q_o^2$ as $n \to -\infty$ and $\mathrm{Wr}(r^{-n}\psi_n, r^{-n}\bar{\psi}_n) \sim (\lambda r - z)^2 + Q_o^2$ as $n \to +\infty$ by virtue of (2.9), one has

$$\mathrm{Wr}(\phi_n(z), \bar{\phi}_n(z)) = -\left[(\lambda r - z)^2 + Q_o^2\right] r^{2n} \prod_{k=-\infty}^{n-1} \frac{1 - Q_k R_k}{1 - Q_o^2} \tag{2.15a}$$

$$\mathrm{Wr}(\bar{\psi}_n(z), \psi_n(z)) = -\left[(\lambda r - z)^2 + Q_o^2\right] r^{2n} \prod_{k=n}^{+\infty} \frac{1 - Q_o^2}{1 - Q_k R_k}. \tag{2.15b}$$

That is, $\phi_n(z)$ and $\bar{\phi}_n(z)$ are linearly independent (see remark 1), and so are $\psi_n(z)$ and $\bar{\psi}_n(z)$. Therefore, introducing the 2×2 matrices

$$\mathbf{\Phi}_n(z) = (\phi_n(z)\ \bar{\phi}_n(z)), \qquad \mathbf{\Psi}_n(z) = (\bar{\psi}_n(z)\ \psi_n(z)), \tag{2.16}$$

we conclude that both $\mathbf{\Phi}_n(z)$ and $\mathbf{\Psi}_n(z)$ are fundamental solutions of (2.1a).

Remark 1. The multiplicative factors depending on the potentials in (2.15) are also present in the decaying case, and one must require that they never vanish. (This is obvious in the focusing case, $R_n = -Q_n^*$, but must be imposed as a 'small norm' condition in the defocusing regime, that is, when $R_n = Q_n^*$.) Moreover, we show in the appendix that $(\lambda r - z)^2 + Q_o^2 = 0$ iff $z = \pm z_0$ or $z = \pm z_0^*$, with z_0 defined in (2.11). We conclude that the eigenfunctions $\phi_n(z), \bar{\phi}_n(z)$ are linearly independent for all z except at the branch points of $\lambda(z)$, and so are $\psi_n(z), \bar{\psi}_n(z)$. In this respect, the situation is analogous to what happens in the continuum limit, where the Wronskian of the eigenfunctions is proportional to $\lambda = \sqrt{k^2 - q_0^2}$, and hence different from zero except, possibly, at the branch points $k = \pm q_0$.

Since the $\mathbf{\Phi}_n(z)$ and $\mathbf{\Psi}_n(z)$ are both fundamental solutions of (2.1a), they are related to an n-independent invertible transformation. Namely, we can introduce the scattering matrix $\mathbf{T}(z)$ such that

$$\mathbf{\Phi}_n(z) = \mathbf{\Psi}_n(z)\mathbf{T}(z). \tag{2.17}$$

Explicitly,

$$\phi_n(z) = b(z)\psi_n(z) + a(z)\bar{\psi}_n(z), \qquad \bar{\phi}_n(z) = \bar{a}(z)\psi_n(z) + \bar{b}(z)\bar{\psi}_n(z),$$

where

$$\mathbf{T}(z) = \begin{pmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{pmatrix}. \tag{2.18}$$

Then (2.15) implies

$$\det \mathbf{T}(z) = \frac{\det \mathbf{\Phi}_n(z)}{\det \mathbf{\Psi}_n(z)} = \frac{\mathrm{Wr}(\phi_n(z), \bar{\phi}_n(z))}{\mathrm{Wr}(\bar{\psi}_n(z), \psi_n(z))} = \prod_{k=-\infty}^{\infty} \frac{1 - Q_k R_k}{1 - Q_o^2} \equiv c_\infty. \tag{2.19}$$

The scattering coefficients can also be written as

$$a(z) = \frac{\mathrm{Wr}(\phi_n(z), \psi_n(z))}{\mathrm{Wr}(\bar{\psi}_n(z), \psi_n(z))}, \qquad \bar{a}(z) = -\frac{\mathrm{Wr}(\bar{\phi}_n(z), \bar{\psi}_n(z))}{\mathrm{Wr}(\bar{\psi}_n(z), \psi_n(z))} \tag{2.20a}$$

$$b(z) = -\frac{\mathrm{Wr}(\phi_n(z), \bar{\psi}_n(z))}{\mathrm{Wr}(\bar{\psi}_n(z), \psi_n(z))}, \qquad \bar{b}(z) = \frac{\mathrm{Wr}(\bar{\phi}_n(z), \psi_n(z))}{\mathrm{Wr}(\bar{\psi}_n(z), \psi_n(z))}. \tag{2.20b}$$

Note that if z_k is a zero of $a(z)$, then from (2.20) it follows that $\mathrm{Wr}(\phi_n(z_k), \psi_n(z_k)) = 0$, that is,

$$\phi_n(z_k) = b_k \psi_n(z_k) \tag{2.21a}$$

for some complex constant b_k. Similarly, at a zero $\bar{z}_k$ of $\bar{a}(z)$ there exists a complex constant $\bar{b}_k$ such that

$$\bar{\phi}_n(\bar{z}_k) = \bar{b}_k \bar{\psi}_n(\bar{z}_k). \tag{2.21b}$$

Because the asymptotic behavior of the eigenfunctions depends on r^n and $0 < r < 1$, we define the continuous spectrum of the scattering problem (2.1a) as any value of z for which modified eigenfunctions $r^{-n} v_n(z)$ are bounded for all $n \in \mathbb{Z}$. This corresponds to values z such that $|\lambda(z)| = 1$, with λ given by (2.8). Such continuous spectrum is then found to be the union of two arcs of the unit circle $|z| = 1$ given by $|\mathrm{Re}\, z| < r$ (see the appendix for details). Namely,

$$\mathcal{C} = \{z \in \mathbb{C} : |z| = 1 \text{ and } |\mathrm{Re}\, z| < r\}. \tag{2.22a}$$

On the other hand, the discrete spectrum (if there is any) consists of values z_k such that the problem possess modified eigenfunctions $r^{-n} v_n(z_k)$ which vanish as $n \to \pm\infty$. Following [21], it can be shown that, if $Q_o < 1$, the discrete eigenvalues lie on the unit circle and, for the sake of simplicity, we exclude the possibility of discrete eigenvalues embedded in the continuous spectrum. We will therefore take the discrete eigenvalues (if any) to be located in the complement of the unit circle with respect to continuous spectrum. Namely,

$$\mathcal{D} = \{z_1, \ldots, z_J \in \mathbb{C} : |z_k| = 1 \text{ and } |\mathrm{Re}\, z_k| > r \ \forall k = 1, \ldots, J\}. \tag{2.22b}$$

The proof that for $Q_o < 1$, necessarily $|z_k| = 1$ mirrors the one in [21] and it is given, for completeness, in the appendix.

2.2. *Uniformization*

The eigenfunctions and the scattering data are not single-valued functions of z. Indeed, as follows from (2.8), the function $\lambda(z)$ is double-valued, and has four branch points located at $\pm z_0$ and $\pm z_0^*$. Hence, the scattering problem is defined on the two-sheeted Riemann surface for $\lambda(z)$. In principle, this problem can be eliminated by introducing a uniformization coordinate. A two-sheeted Riemann surface with four branch points is equivalent to a torus (genus 1), and in general is uniformized by means of elliptic functions. Due to the symmetries of the scattering problem, however, it is possible to find an algebraic parametrization. In [21], such a uniformization variable is introduced by means of the conformal mapping

$$\zeta(z) = \lambda(z)/z, \tag{2.23}$$

which implies the following relations:

$$z^2 = \frac{\zeta - r}{\zeta(r\zeta - 1)}, \qquad \lambda^2 = \zeta\frac{\zeta - r}{r\zeta - 1}, \qquad z\lambda = \frac{\zeta - r}{r\zeta - 1}. \tag{2.24}$$

Hence all these quantities, as well as all quantities which are even functions of λ and z, are meromorphic functions of ζ.

The function $\zeta(z)$ maps the arcs of the unit circle $(z_0, -z_0^*)$ and $(z_0^*, -z_0)$ onto the unit circle in the ζ-plane. The edges of the continuous spectrum, $\pm z_0$ and $\pm z_0^*$, are such that $\lambda(\pm z_0) \equiv \lambda(\pm z_0^*) = \pm 1$. Therefore, since $|z_0| = 1$, from (2.23) it follows that they are transformed into the points ζ_0 and ζ_0^*, where $\zeta(\pm z_0) = \zeta_0^* = z_0^*$ and $\zeta(\pm z_0^*) = \zeta_0 = z_0$. Note also that as $\zeta \to r$ one has $(z, \lambda) \to 0$ and $\zeta \to 1/r$ corresponds to $(z, \lambda) \to \infty$. Therefore, the points r and $1/r$ in the complex ζ-plane play the same role as the points 0 and ∞.

Remark 2. In the appendix we show that

$$|\lambda| \leqslant 1 \quad \text{iff} \quad |\zeta| \leqslant 1, \tag{2.25a}$$

$$|z| \geqslant 1 \quad \text{iff} \quad (|\zeta|^2 - 1)[|\zeta - 1/r|^2 - Q_o/r] \leqslant 0. \tag{2.25b}$$

The values of ζ corresponding to the distinguished points of the unit circle in the z-plane are tabulated in the appendix. Equations (2.25) and the correspondences in table 1 yield the mapping $z \to \zeta$ shown in figure 2. Note that the mapping is not 1-to-1, but rather 2-to-1. That is, two different values of z (both in the same sheet) correspond to any single value of ζ. This however does pose obstacles in the analysis.

Because the asymptotic behavior of the eigenfunctions as $|n| \to \infty$ contains the terms $r^n\lambda^{\pm n}$, in order to more effectively study analyticity properties we now introduce the 2×2 matrix

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}. \tag{2.26}$$

Using $\mathbf{\Lambda}$, we can define modified eigenfunctions whose asymptotic behavior as $n \to \pm\infty$ is independent of λ, as in [21]. Indeed, note that if $\mathbf{\Phi}_n(z)$ and $\mathbf{\Psi}_n(z)$ are defined by (2.16), then from (2.9) one has

$$r^{-n}\mathbf{\Phi}_n(z)\mathbf{\Lambda}^{-n} \sim \begin{pmatrix} Q_- & \lambda r - z \\ \lambda r - z & -R_- \end{pmatrix} \qquad n \to -\infty, \tag{2.27a}$$

$$r^{-n}\mathbf{\Psi}_n(z)\mathbf{\Lambda}^{-n} \sim \begin{pmatrix} Q_+ & \lambda r - z \\ \lambda r - z & -R_+ \end{pmatrix} \qquad n \to +\infty. \tag{2.27b}$$

The asymptotics of the matrices $r^{-n}\mathbf{\Phi}_n(z)\mathbf{\Lambda}^{-n}$ and $r^{-n}\mathbf{\Psi}_n(z)\mathbf{\Lambda}^{-n}$ show that the diagonal parts are meromorphic functions of ζ, while the off-diagonal parts contain terms like $r\lambda - z$, which are not even functions of λ or z. The scattering matrix $\mathbf{T}(z)$ shares the same property. (This follows from (2.27) in the asymptotics at large n, but in fact holds true for all integers n, and it is proven from the scattering problem [21].) We can obtain meromorphic functions of ζ, however, by performing a further transformation on the eigenfunctions. That is, we introduce

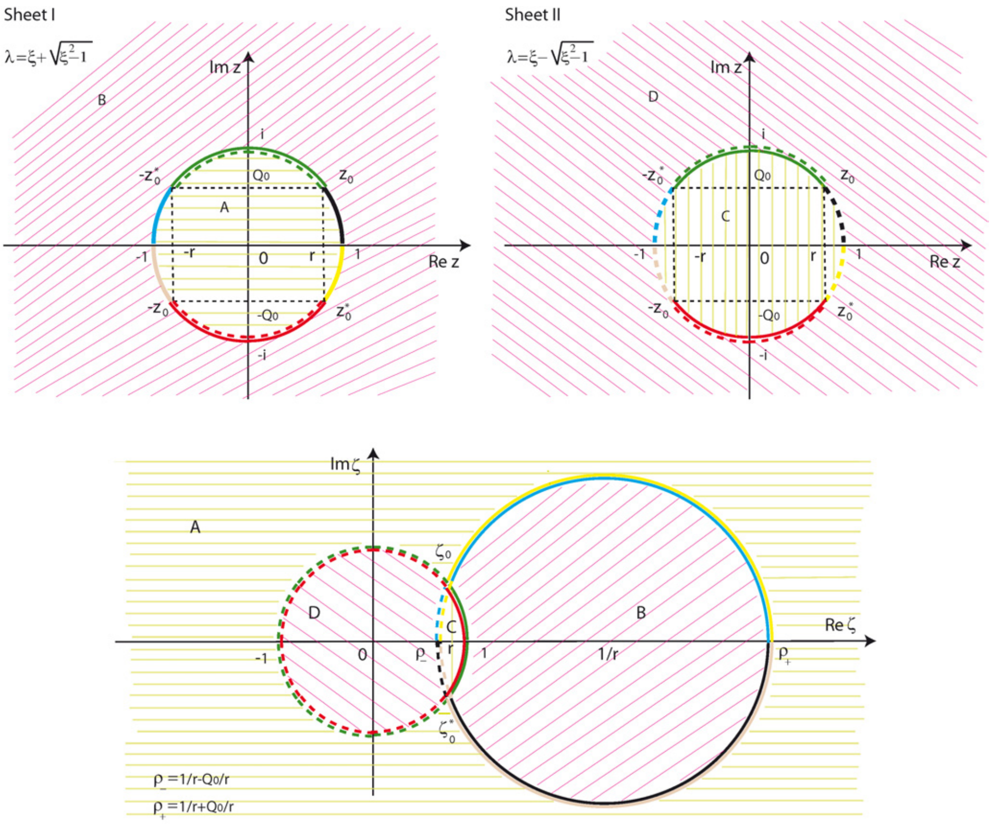

Figure 2. The mapping $z \to \zeta$. The exterior of the unit disk in sheets I and II is mapped respectively onto the interior of the disks $|\zeta| < 1$ and $|\zeta - 1/r| < Q_o/r$ minus their intersection (regions B and D). The interior of the unit disk in sheet I is mapped onto the region outside both disks in the ζ-plane (region A). Finally, the interior of the unit disk in sheet II is mapped onto the intersections of the two disks in the ζ-plane (region C).

(This figure is in colour only in the electronic version)

the modified eigenfunctions

$$(M_n(\zeta)\bar{M}_n(\zeta)) = r^{-n}\mathbf{A}(\lambda)\mathbf{\Phi}_n(z)\mathbf{A}^{-1}(\lambda)\mathbf{\Lambda}^{-n}, \qquad (2.28a)$$

$$(\bar{N}_n(\zeta)N_n(\zeta)) = r^{-n}\mathbf{A}(\lambda)\mathbf{\Psi}_n(z)\mathbf{A}^{-1}(\lambda)\mathbf{\Lambda}^{-n}, \qquad (2.28b)$$

where

$$\mathbf{A}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}. \qquad (2.29)$$

Note that the choice of $\mathbf{A}$ is obviously not unique. This choice however will also be convenient when studying the analyticity properties of the eigenfunctions (cf section 2.4). Componentwise, (2.28) is

$$M_n(\zeta) = \lambda^{-n}r^{-n}\begin{pmatrix} \phi_n^{(1)}(\zeta) \\ \lambda(\zeta)\phi_n^{(2)}(\zeta) \end{pmatrix}, \qquad \bar{M}_n(\zeta) = \lambda^{n}r^{-n}\begin{pmatrix} \bar{\phi}_n^{(1)}(\zeta)/\lambda(\zeta) \\ \bar{\phi}_n^{(2)}(\zeta) \end{pmatrix}, \qquad (2.30a)$$

$$N_n(\zeta) = \lambda^{n}r^{-n}\begin{pmatrix} \psi_n^{(1)}(\zeta)/\lambda(\zeta) \\ \psi_n^{(2)}(\zeta) \end{pmatrix}, \qquad \bar{N}_n(\zeta) = \lambda^{-n}r^{-n}\begin{pmatrix} \bar{\psi}_n^{(1)}(\zeta) \\ \lambda(\zeta)\bar{\psi}_n^{(2)}(\zeta) \end{pmatrix}. \qquad (2.30b)$$

These modified eigenfunctions satisfy the difference equations (modified scattering problems)

$$r M_{n+1}(\zeta) = \begin{pmatrix} z/\lambda & Q_n/\lambda^2 \\ R_n & 1/(\lambda z) \end{pmatrix} M_n(\zeta), \qquad r \bar{N}_{n+1}(\zeta) = \begin{pmatrix} z/\lambda & Q_n/\lambda^2 \\ R_n & 1/(\lambda z) \end{pmatrix} \bar{N}_n(\zeta), \tag{2.31a}$$

$$r \bar{M}_{n+1}(\zeta) = \begin{pmatrix} \lambda z & Q_n \\ \lambda^2 R_n & \lambda/z \end{pmatrix} \bar{M}_n(\zeta), \qquad r N_{n+1}(\zeta) = \begin{pmatrix} \lambda z & Q_n \\ \lambda^2 R_n & \lambda/z \end{pmatrix} N_n(\zeta). \tag{2.31b}$$

Note that the combinations of scattering parameters in (2.31) can all be expressed as uniquely terms of ζ via (2.23) and (2.24).

The asymptotic behavior of these modified eigenfunctions at large n, which follows from (2.28) and (2.9), is independent of n, as desired. Indeed, since $(\lambda r - z)\lambda = \lambda^2 r - \lambda z = \zeta - r$, and $(\lambda r - z)/\lambda = r - 1/\zeta$, one has

$$M_n(\zeta) \sim \begin{pmatrix} Q_- \\ \zeta - r \end{pmatrix}, \qquad \bar{M}_n(\zeta) \sim \begin{pmatrix} r - 1/\zeta \\ -R_- \end{pmatrix} \qquad n \to -\infty, \tag{2.32a}$$

$$\bar{N}_n(\zeta) \sim \begin{pmatrix} Q_+ \\ \zeta - r \end{pmatrix}, \qquad N_n(\zeta) \sim \begin{pmatrix} r - 1/\zeta \\ -R_+ \end{pmatrix} \qquad n \to +\infty. \tag{2.32b}$$

After transformation by means of the matrix $\mathbf{A}(\lambda)$, relation (2.17) becomes

$$(M_n(\zeta)\bar{M}_n(\zeta)) = (\bar{N}_n(\zeta)N_n(\zeta))\mathbf{\Lambda}^n\mathbf{S}(\zeta)\mathbf{\Lambda}^{-n},$$

where the modified scattering matrix $\mathbf{S}(\zeta)$ is given by

$$\mathbf{S}(\zeta) = \mathbf{A}(\lambda)\mathbf{T}(z)\mathbf{A}^{-1}(\lambda) \equiv \begin{pmatrix} a(\zeta) & \bar{b}(\zeta)/\lambda(\zeta) \\ \lambda(\zeta)b(\zeta) & \bar{a}(\zeta) \end{pmatrix}. \tag{2.33}$$

Note that both $\mathbf{S}(\zeta)$ and the product $\mathbf{\Lambda}^n\mathbf{S}(\zeta)\mathbf{\Lambda}^{-n}$ for integer n are meromorphic functions of ζ. Explicitly, we have

$$M_n(\zeta) = \bar{N}_n(\zeta)a(\zeta) + \lambda(\zeta)^{-2n}N_n(\zeta)\beta(\zeta), \tag{2.34a}$$

$$\bar{M}_n(\zeta) = N_n(\zeta)\bar{a}(\zeta) + \lambda(\zeta)^{2n}\bar{N}_n(\zeta)\bar{\beta}(\zeta), \tag{2.34b}$$

where $\beta(\zeta) = \lambda(\zeta)b(\zeta)$, $\bar{\beta}(\zeta) = \bar{b}(\zeta)/\lambda(\zeta)$. We note that equations (2.34) are key to the formulation of the inverse problem.

Taking into account (2.20) and (2.30), we then have the following Wronskian representations for the scattering coefficients:

$$a(\zeta) = \frac{\mathrm{Wr}(M_n(\zeta), N_n(\zeta))}{\mathrm{Wr}(\bar{N}_n(\zeta), N_n(\zeta))}, \qquad \bar{a}(\zeta) = -\frac{\mathrm{Wr}(\bar{M}_n(\zeta), \bar{N}_n(\zeta))}{\mathrm{Wr}(\bar{N}_n(\zeta), N_n(\zeta))}, \tag{2.35}$$

$$\beta(\zeta) = -\lambda^{2n}(\zeta)\frac{\mathrm{Wr}(M_n(\zeta), \bar{N}_n(\zeta))}{\mathrm{Wr}(\bar{N}_n(\zeta), N_n(\zeta))}, \qquad \bar{\beta}(\zeta) = \lambda^{-2n}(\zeta)\frac{\mathrm{Wr}(\bar{M}_n(\zeta), N_n(\zeta))}{\mathrm{Wr}(\bar{N}_n(\zeta), N_n(\zeta))}. \tag{2.36}$$

Note that $\mathrm{Wr}(\bar{N}_n, N_n) = r^{-2n}\,\mathrm{Wr}(\bar{\psi}_n, \psi_n)$ and $(\lambda r - z)^2 = (\zeta - r)(r - 1/\zeta)$ so that (cf (2.15) and (A.2))

$$\mathrm{Wr}(N_n(\zeta), \bar{N}_n(\zeta)) = r(\zeta + 1/\zeta - 2r)/\Delta_n, \tag{2.37}$$

where Δ_n is defined by

$$\Delta_n = \prod_{k=n}^{\infty} \frac{1 - Q_k R_k}{1 - Q_o^2}. \tag{2.38}$$

Therefore,

$$a(\zeta) = -\Delta_n \frac{\mathrm{Wr}(M_n(\zeta), N_n(\zeta))}{r(\zeta + 1/\zeta - 2r)}, \qquad \bar{a}(\zeta) = \Delta_n \frac{\mathrm{Wr}(\bar{M}_n(\zeta), \bar{N}_n(\zeta))}{r(\zeta + 1/\zeta - 2r)}, \tag{2.39a}$$

$$\begin{aligned} \beta(\zeta) &= \Delta_n \left[\zeta \frac{\zeta - r}{r\zeta - 1}\right]^n \frac{\mathrm{Wr}(M_n(\zeta), \bar{N}_n(\zeta))}{r(\zeta + 1/\zeta - 2r)}, \\ \bar{\beta}(\zeta) &= -\Delta_n \left[\zeta \frac{\zeta - r}{r\zeta - 1}\right]^{-n} \frac{\mathrm{Wr}(\bar{M}_n(\zeta), N_n(\zeta))}{r(\zeta + 1/\zeta - 2r)}. \end{aligned} \tag{2.39b}$$

Note also that from (2.21) and (2.30) it follows that at zeros ζ_k of $a(\zeta)$ the following relation holds:

$$M_n(\zeta_k) = b_k(\lambda(\zeta_k))^{1-2n} N_n(\zeta_k). \tag{2.40a}$$

Similarly, at zeros $\bar{\zeta}_k$ of $\bar{a}(\zeta)$,

$$\bar{M}_n(\bar{\zeta}_k) = \bar{b}_k(\lambda(\bar{\zeta}_k))^{2n-1} \bar{N}_n(\bar{\zeta}_k). \tag{2.40b}$$

2.3. Green's functions

The analyticity properties of the modified eigenfunctions satisfying the difference equations (2.31) can be studied effectively using Green's functions. Both $M_n(\zeta)$ and $\bar{N}_n(\zeta)$ satisfy the same equation, which can be written as

$$v_{n+1}(\zeta) = \frac{1}{r}\begin{pmatrix} z/\lambda & Q_\mp/\lambda^2 \\ R_\mp & 1/(z\lambda) \end{pmatrix} v_n(\zeta) + \frac{1}{r}\begin{pmatrix} 1/\lambda^2 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{Q}_n - \mathbf{Q}_\mp) v_n(\zeta), \tag{2.41}$$

where $\mathbf{Q}_n$ is defined by (2.3) and

$$\mathbf{Q}_\pm = \lim_{n\to\pm\infty} \mathbf{Q}_n \equiv \begin{pmatrix} 0 & Q_\pm \\ R_\pm & 0 \end{pmatrix}.$$

Solutions of (2.41) can be written in the form of a 'summation' equation (discrete version of an integral equation)

$$v_n(\zeta) = w_\mp + \sum_{k=-\infty}^{+\infty} \mathbf{G}^\mp_{n-k}(\zeta)(\mathbf{Q}_k - \mathbf{Q}_\mp) v_k(\zeta), \tag{2.42}$$

where the inhomogeneous term $w_\pm$ is such that

$$\left[\mathbf{I} - \frac{1}{r}\begin{pmatrix} z/\lambda & Q_\mp/\lambda^2 \\ R_\mp & 1/(z\lambda) \end{pmatrix}\right] w_\mp = 0. \tag{2.43}$$

Here $\mathbf{I}$ is the 2×2 identity matrix, and Green's functions $\mathbf{G}^\pm_n(\zeta)$ satisfy the difference equations

$$\mathbf{G}^\pm_{n+1}(\zeta) - \frac{1}{r}\begin{pmatrix} z/\lambda & Q_\pm/\lambda^2 \\ R_\pm & 1/(z\lambda) \end{pmatrix} \mathbf{G}^\pm_n(\zeta) = \frac{1}{r}\begin{pmatrix} 1/\lambda^2 & 0 \\ 0 & 1 \end{pmatrix} \delta_{n,0}, \tag{2.44}$$

with $\delta_{n,0}$ being the Kronecker delta. Note that all the combinations of z and λ that appear above can be written in terms of ζ according to (2.24), but for brevity it is convenient to leave them and only make the substitution at the end. Note that Green's function is not unique, and indeed it is the choice of Green's function and the choice of the inhomogeneous term that together determine the eigenfunctions and their analyticity properties. First of all, note that both of the vectors $w_\pm = (Q_\pm, \zeta - r)^T$ satisfy (2.43). (The superscript T denotes matrix transpose.) Then, using Fourier transforms, we obtain for Green's functions the following expressions (see the appendix for details):

$$\mathbf{G}_n^{\mathrm{out}}(\zeta) = \theta(n-1)\frac{1/\lambda^2}{r^2(1-1/\lambda^2)}\left\{\begin{pmatrix} \frac{Q_o^2}{\zeta-r} & Q_- \\ R_- & \zeta-r \end{pmatrix} - \lambda^{-2(n-1)}\begin{pmatrix} -\frac{r\zeta-1}{\zeta} & Q_- \\ R_- & -Q_o^2\frac{\zeta}{r\zeta-1} \end{pmatrix}\right\}, \tag{2.45a}$$

$$\mathbf{G}_n^{\mathrm{in}}(\zeta) = -\theta(-n)\frac{1/\lambda^2}{r^2(1-1/\lambda^2)}\left\{\begin{pmatrix} \frac{Q_o^2}{\zeta-r} & Q_+ \\ R_+ & \zeta-r \end{pmatrix} - \lambda^{-2(n-1)}\begin{pmatrix} -\frac{r\zeta-1}{\zeta} & Q_+ \\ R_+ & -Q_o^2\frac{\zeta}{r\zeta-1} \end{pmatrix}\right\}, \tag{2.45b}$$

where θ_n is the Heaviside unit step function. The term $1-1/\lambda^2$ gives singularities for $\lambda = \pm 1$, i.e. at the branch points ζ_0 and ζ_0^*. One can take a 'formal' limit of (2.45) as $\zeta \to \zeta_0 = r + \mathrm{i}Q_o$ or $\zeta \to \zeta_0^* = r - \mathrm{i}Q_o$, however. For example, as $\zeta \to r \pm \mathrm{i}Q_o$, one obtains

$$\mathbf{G}_n^{\mathrm{out}}(\zeta) \sim \theta(n-1)\frac{n-1}{r^2}\begin{pmatrix} \mp\mathrm{i}Q_o & Q_- \\ R_- & \pm\mathrm{i}Q_o \end{pmatrix}, \tag{2.46}$$

which shows that this Green's function is well defined at the branch points, though linearly growing in n. Therefore, the associated eigenfunctions are also well defined at the branch points, provided that the potentials $Q_n - Q_\pm$ decay fast enough as $n \to \pm\infty$ to take care of the growth of Green's functions. Due to the θ-functions, $\mathbf{G}_n^{\mathrm{out}}(\zeta)$ admits an analytic extension for $|\lambda| > 1$, hence for $|\zeta| > 1$, while $\mathbf{G}_n^{\mathrm{in}}(\zeta)$ is analytic for $|\lambda| < 1$, hence for $|\zeta| < 1$. On the other hand, taking into account the difference equations (2.42), we see that $\mathbf{G}_n^{\mathrm{out}}(\zeta)$ (with the $-$ signs on the right-hand side (RHS) for $R_\pm$, $Q_\pm$) is Green's function associated with the eigenfunction $M_n(\zeta)$, which then is also expected to admit analytic extension outside the unit circle $|\zeta| = 1$, while $\mathbf{G}_n^{\mathrm{in}}(\zeta)$ (with the + signs on the RHS $R_\pm$, $Q_\pm$) is associated with $\bar{N}_n$, which therefore can be analytically continued inside the unit circle.

To be precise, the Green's functions are meromorphic either inside or outside the unit circle, since they exhibit poles at points $\zeta = 0, \infty, r$ and $1/r$. However, it turns out that in the integral equations the terms combine in such a way that the eigenfunctions do not have singularities at these points. This problem is addressed in section 2.4, using an approach analogous to the one introduced in [4, 5].

In a similar way, one can show that solutions of the difference equations (2.31*b*) can be written as

$$v_n(\zeta) = \tilde{w}_\mp + \sum_{k=-\infty}^{+\infty} \tilde{\mathbf{G}}_{n-k}^{\mp}(\zeta)(\mathbf{Q}_k - \mathbf{Q}_\mp)v_k(\zeta), \tag{2.47}$$

with inhomogeneous terms $\tilde{w}_\pm = (r - 1/\zeta, -R_\pm)^T$ and Green's functions

$$\tilde{\mathbf{G}}_n^{\mathrm{out}}(\zeta) = \theta(n-1)\frac{\lambda^2}{r^2(1-\lambda^2)}\left\{\begin{pmatrix} -\frac{r\zeta-1}{\zeta} & Q_\pm \\ R_\pm & -Q_o^2\frac{\zeta}{r\zeta-1} \end{pmatrix} - \lambda^{2(n-1)}\begin{pmatrix} -\frac{Q_o^2}{\zeta-r} & Q_\pm \\ R_\pm & \zeta-r \end{pmatrix}\right\},$$

$$\tilde{\mathbf{G}}_n^{\mathrm{in}}(\zeta) = -\theta(-n)\frac{\lambda^2}{r^2(1-\lambda^2)}\left\{\begin{pmatrix} -\frac{r\zeta-1}{\zeta} & Q_\pm \\ R_\pm & -Q_o^2\frac{\zeta}{r\zeta-1} \end{pmatrix} - \lambda^{2(n-1)}\begin{pmatrix} -\frac{Q_o^2}{\zeta-r} & Q_\pm \\ R_\pm & \zeta-r \end{pmatrix}\right\}.$$

Clearly, $\tilde{\mathbf{G}}_n^{\mathrm{out}}(\zeta)$ is meromorphic inside the circle $|\zeta| = 1$, and $\tilde{\mathbf{G}}_n^{\mathrm{in}}(\zeta)$ is meromorphic outside the circle $|\zeta| = 1$. By looking at the 'summation' equations (2.47), and taking into account the θ-functions, $\tilde{\mathbf{G}}_n^{\mathrm{out}}(\zeta) \to \tilde{\mathbf{G}}_n^{-}(\zeta)$ and $\tilde{\mathbf{G}}_n^{\mathrm{in}}(\zeta) \to \tilde{\mathbf{G}}_n^{+}(\zeta)$. This indicates that $\bar{M}_n(\zeta)$ is analytic inside the unit circle $|\zeta| = 1$, and $N_n(\zeta)$ is analytic outside, as we show next.

2.4. Analyticity

To study analyticity of the eigenfunctions, it is convenient to use the approach of [4] and introduce a modified scattering problem, related to (2.1*a*), but *with potentials that decay as* $|n| \to \infty$. Recall that the discrete spectral problem (2.1*a*) can be written as $v_{n+1} = \mathbf{L}_n v_n$, where $\mathbf{L}_n$ was defined in (2.3). As $n \to \pm\infty$, one has $\mathbf{L}_n \sim \mathbf{L}_\pm$, where

$$\mathbf{L}_\pm \equiv \begin{pmatrix} z & Q_\pm \\ R_\pm & 1/z \end{pmatrix},$$

and where $Q_\pm = Q_o\,\mathrm{e}^{\mathrm{i}\theta_\pm}$, $R_\pm = Q_o\,\mathrm{e}^{-\mathrm{i}\theta_\pm}$ (cf (2.4)). The eigenvalues and eigenvectors of the matrix $\mathbf{L}_\pm$ are given by $\mathbf{L}_\pm\mathbf{U}_\pm = \mathbf{U}_\pm\mathbf{D}$ with $\mathbf{D} = \mathrm{diag}(\lambda r, r/\lambda)$ and

$$\mathbf{U}_\pm = \begin{pmatrix} Q_\pm & \lambda r - z \\ \lambda r - z & -R_\pm \end{pmatrix}.$$

As in section 2.2, if $\mathbf{v}_n$ is a 2×2 matrix solution of (2.2*a*), we introduce

$$\hat{\mathbf{v}}_n = \mathbf{A}\mathbf{v}_n\mathbf{A}^{-1}, \tag{2.48}$$

with $\mathbf{A}(\lambda)$ defined by (2.29). The 2×2 matrix $\hat{\mathbf{v}}_n$ then solves the scattering problem

$$\hat{\mathbf{v}}_{n+1} = \hat{\mathbf{L}}_n\hat{\mathbf{v}}_n, \tag{2.49}$$

where

$$\hat{\mathbf{L}}_n = \mathbf{A}\mathbf{L}_n\mathbf{A}^{-1} \equiv \begin{pmatrix} z & Q_n/\lambda \\ \lambda R_n & 1/z \end{pmatrix}.$$

Note that

$$\lim_{n\to\pm\infty} \hat{\mathbf{L}}_n = \hat{\mathbf{L}}_\pm = \begin{pmatrix} z & Q_\pm/\lambda \\ \lambda R_\pm & 1/z \end{pmatrix}.$$

Since $\hat{\mathbf{L}}_\pm$ is similar to $\mathbf{L}_\pm$, it has the same eigenvalues, namely $\{\lambda r, r/\lambda\}$. The eigenvectors of $\mathbf{L}_\pm$ are given by the matrix $\mathbf{U}_\pm$, and therefore a matrix of eigenvectors of $\hat{\mathbf{L}}_\pm$ is given by $\mathbf{A}\mathbf{U}_\pm$. Since eigenvectors are defined up to a multiplicative factor, however, it is convenient to multiply the second column of $\mathbf{A}\mathbf{U}_\pm$ so that $\hat{\mathbf{U}}_\pm$ is an even function of λ, z. For instance, we can take the matrix of eigenvectors of $\hat{\mathbf{L}}_\pm$ to be

$$\hat{\mathbf{U}}_\pm = \begin{pmatrix} Q_\pm & (\lambda r - z)/\lambda \\ \lambda(\lambda r - z) & -R_\pm \end{pmatrix} = \mathbf{A}\mathbf{U}_\pm\mathbf{A}^{-1}.$$

Now, in analogy with [4], we introduce

$$\hat{\mathbf{U}}_n = \begin{pmatrix} \tilde{Q}_n & (\lambda r - z)/\lambda \\ \lambda(\lambda r - z) & -\tilde{R}_n \end{pmatrix}, \tag{2.50}$$

where the modified potentials $\tilde{Q}_n$ and $\tilde{R}_n$ are such that

$$\tilde{Q}_n \to Q_\pm, \qquad \tilde{R}_n \to R_\pm \qquad \text{as} \quad n \to \pm\infty, \tag{2.51a}$$

and they satisfy the constraint

$$\tilde{Q}_n\tilde{R}_n = Q_o^2 \qquad \forall n \in \mathbb{Z}. \tag{2.51b}$$

We then define

$$\tilde{\mathbf{v}}_n = \hat{\mathbf{U}}_n^{-1}\hat{\mathbf{v}}_n, \tag{2.52}$$

so that $\tilde{\mathbf{v}}_n \to \mathbf{I}$ as $n \to \pm\infty$ and observe that $\tilde{\mathbf{v}}_{n+1} = \big(\hat{\mathbf{U}}_{n+1}^{-1} - \hat{\mathbf{U}}_n^{-1}\big)\hat{\mathbf{v}}_{n+1} + \hat{\mathbf{U}}_n^{-1}\hat{\mathbf{v}}_{n+1}$. Thus, $\tilde{\mathbf{v}}_n$ satisfies the modified scattering problem

$$\tilde{\mathbf{v}}_{n+1} = \tilde{\mathbf{L}}_n\tilde{\mathbf{v}}_n, \tag{2.53a}$$

where

$$\tilde{\mathbf{L}}_n = \left(\hat{\mathbf{U}}_{n+1}^{-1} - \hat{\mathbf{U}}_n^{-1}\right)\hat{\mathbf{L}}_n\hat{\mathbf{U}}_n + \hat{\mathbf{U}}_n^{-1}\hat{\mathbf{L}}_n\hat{\mathbf{U}}_n. \tag{2.53b}$$

Note that from (2.50) it follows that

$$\hat{\mathbf{U}}_n^{-1} = \frac{1}{Q_o^2 + (\lambda r - z)^2}\begin{pmatrix} \tilde{R}_n & (\lambda r - z)/\lambda \\ \lambda(\lambda r - z) & -\tilde{Q}_n \end{pmatrix}. \tag{2.54}$$

Note also that the first term in $\tilde{\mathbf{L}}_n$ in (2.53*b*) decays asymptotically as $n \to \pm\infty$ since $\hat{\mathbf{U}}_n^{-1}$ and $\hat{\mathbf{U}}_{n+1}^{-1}$ have the same limit (cf (2.51*a*)). On the other hand, the second term in (2.53*b*) can be decomposed into the sum of two pieces: one which is diagonal and contains the eigenvalues (this is because asymptotically $\hat{\mathbf{U}}_n$ diagonalizes $\hat{\mathbf{L}}_n$) and a second term which again decays due to (2.51). In fact, one has

$$\hat{\mathbf{U}}_n^{-1}\hat{\mathbf{L}}_n\hat{\mathbf{U}}_n = \begin{pmatrix} \lambda r & 0 \\ 0 & r/\lambda \end{pmatrix} + \frac{1}{Q_o^2 + (\lambda r - z)^2}$$
$$\begin{pmatrix} (\lambda r - z)\left(Q_n\tilde{R}_n + R_n\tilde{Q}_n - 2Q_o^2\right) & \left\{\tilde{R}_n\left(Q_o^2 - Q_n\tilde{R}_n\right) + (\lambda r - z)^2(R_n - \tilde{R}_n)\right\}/\lambda \\ \lambda\left\{\tilde{Q}_n\left(Q_o^2 - \tilde{Q}_n R_n\right) + (\lambda r - z)^2(Q_n - \tilde{Q}_n)\right\} & -(\lambda r - z)\left(Q_n\tilde{R}_n + R_n\tilde{Q}_n - 2Q_o^2\right) \end{pmatrix}$$

and

$$\left(\hat{\mathbf{U}}_{n+1}^{-1} - \hat{\mathbf{U}}_n^{-1}\right)\hat{\mathbf{L}}_n\hat{\mathbf{U}}_n = \frac{1}{Q_o^2 + (\lambda r - z)^2}$$
$$\begin{pmatrix} (\tilde{R}_{n+1} - \tilde{R}_n)[\lambda r Q_n + z(\tilde{Q}_n - Q_n)] & (\tilde{R}_{n+1} - \tilde{R}_n)[z(\lambda - rz) - Q_n\tilde{R}_n]/\lambda \\ -\lambda(\tilde{Q}_{n+1} - \tilde{Q}_n)[(\lambda - rz)/z + \tilde{Q}_n R_n] & (\tilde{Q}_{n+1} - \tilde{Q}_n)[rR_n/\lambda + (\tilde{R}_n - R_n)/z] \end{pmatrix},$$

and therefore it follows that

$$\tilde{\mathbf{L}}_n = \begin{pmatrix} \lambda r & 0 \\ 0 & r/\lambda \end{pmatrix} + \tilde{\mathbf{V}}_n(\zeta), \tag{2.55}$$

where $\tilde{\mathbf{V}}_n(\zeta)$ decays as $n \to \pm\infty$. What is relevant here is that all terms depending on potentials are decaying at both space infinities, and that all entries (including the eigenvalues) are odd functions of λ and z. Therefore, if we define modified eigenfunctions to eliminate the factors $\lambda^n r^n$ or $\lambda^{-n} r^n$, we obtain a scattering problem which contains rational functions of the uniformization variable ζ only. This is important since it considerably simplifies the study of analyticity through a Neumann series approach.

For instance, let us introduce a modified eigenfunction

$$\tilde{M}_n(z, \lambda) = \lambda^{-n} r^{-n} \tilde{v}_n(z, \lambda), \tag{2.56}$$

where the column vector $\tilde{v}_n(z, \lambda)$ is a solution of (2.53*a*). Note that $\tilde{M}_n(z, \lambda)$ has asymptotic behavior

$$\tilde{M}_n(z, \lambda) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n \to -\infty.$$

Moreover, $\tilde{M}_n$ satisfies the difference equation

$$\tilde{M}_{n+1}(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda^2(\zeta) \end{pmatrix}\tilde{M}_n(\zeta) + \mathbf{W}_n(\zeta)\tilde{M}_n(\zeta), \tag{2.57}$$

where $\mathbf{W}_n(\zeta)$ is an energy-dependent matrix potential of the form

$$\mathbf{W}_n(\zeta) = \frac{1}{r(\zeta + 1/\zeta - 2r)}\tilde{\mathbf{W}}_n(\zeta), \tag{2.58}$$

and the matrix $\tilde{\mathbf{W}}_n$ has entries

$$(\tilde{\mathbf{W}}_n)_{11} = \{(r\zeta - 1)[f_n + f_n^*] + h_n^*[Q_n - g_n]\}/(r\zeta), \tag{2.59a}$$

$$(\tilde{\mathbf{W}}_n)_{12} = \{[(\lambda r - z)^2 g_n^* - \tilde{R}_n f_n^*] + h_n^*[z(\lambda r - z) - Q_n \tilde{R}_n]\}/(\lambda^2 r), \tag{2.59b}$$

$$(\tilde{\mathbf{W}}_n)_{21} = \{-f_n \tilde{Q}_n + g_n(\lambda r - z)^2 - h_n[r\zeta - 1 + \tilde{Q}_n R_n]\}/r, \tag{2.59c}$$

$$(\tilde{\mathbf{W}}_n)_{22} = -(r\zeta - 1)(f_n + f_n^*)/(r\zeta) + h_n[R_n/\lambda^2 - g_n^*/(r\lambda z)], \tag{2.59d}$$

where we introduced the short-hand notations

$$f_n = \tilde{Q}_n R_n - Q_o^2, \qquad g_n = Q_n - \tilde{Q}_n, \qquad h_n = \tilde{Q}_{n+1} - \tilde{Q}_n. \tag{2.60}$$

The potentials f_n, g_n and h_n are all decaying as $n \to \pm\infty$. Therefore $\mathbf{W}_n(\zeta) \to 0$ as $n \to \pm\infty$. Also, note that all the coefficients in (2.59) can be explicitly expressed in terms of the uniformization variable ζ. Using (2.24), one can show that all entries of $\mathbf{W}_n(\zeta)$ are bounded functions of ζ, except possibly at the points where the overall factor $1/(\zeta + 1/\zeta - 2r)$ in (2.58) diverges, i.e. at the branch points. Therefore, we can introduce a ζ-independent matrix $\mathbf{W}_n$ such that for all $\zeta \neq \zeta_0, \zeta_0^*$, one has $\|\mathbf{W}_n(\zeta)\| \leqslant \|\mathbf{W}_n\|$, where $\|\cdot\|$ is any matrix norm, which is used to prove convergence of the Neumann series (see the appendix).

The relation between the eigenfunction $M_n(\zeta)$ and $\tilde{M}_n(\zeta)$ is given by the transformation (2.52):

$$\tilde{M}_n(\zeta) = \hat{\mathbf{U}}_n^{-1}(\zeta) M_n(\zeta), \qquad M_n(\zeta) = \hat{\mathbf{U}}_n(\zeta)\tilde{M}_n(\zeta),$$

or, componentwise,

$$M_n^{(1)}(\zeta) = \tilde{Q}_n \tilde{M}_n^{(1)}(\zeta) + (r - 1/\zeta)\tilde{M}_n^{(2)}(\zeta), \tag{2.61a}$$

$$M_n^{(2)}(\zeta) = (\zeta - r)\tilde{M}_n^{(1)}(\zeta) - \tilde{R}_n \tilde{M}_n^{(2)}(\zeta). \tag{2.61b}$$

Therefore, if, as we prove below, $\tilde{M}_n(\zeta)$ is analytic outside the unit circle, then $M_n(\zeta)$ is, with at most a pole at $\zeta = \infty$ if $\tilde{M}_n(\zeta) = O(1)$ as $\zeta \to \infty$.

We now write a summation equation for $\tilde{M}_n(\zeta)$ from the difference equation (2.57):

$$\tilde{M}_n(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=-\infty}^{+\infty} \tilde{\mathbf{G}}_{n-j}(\zeta)\mathbf{W}_j(\zeta)\tilde{M}_j(\zeta),$$

where Green's function $\tilde{\mathbf{G}}_n$ satisfies the difference equation

$$\tilde{\mathbf{G}}_{n+1} - \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda^2(\zeta) \end{pmatrix} \tilde{\mathbf{G}}_n = \delta_{n,0} I.$$

A solution to this equation can be found in terms of a discrete Fourier transform

$$\tilde{\mathbf{G}}_n(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{|p|=1} p^{n-1} \begin{pmatrix} 1/(p-1) & 0 \\ 0 & 1/(p - 1/\lambda^2(\zeta)) \end{pmatrix} \mathrm{d}p.$$

Perturbing the contour of integration away from the unit circle so that the points $p = 0, 1, 1/\lambda^2$ are all inside the contour yields obtain for Green's function the simple expression

$$\tilde{\mathbf{G}}_n(\zeta) = \theta(n-1) \begin{pmatrix} 1 & 0 \\ 0 & (\lambda^2(\zeta))^{1-n} \end{pmatrix}$$

and therefore the discrete integral equation for $\tilde{M}_n(\zeta)$ becomes

$$\tilde{M}_n(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=-\infty}^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & (\lambda^2(\zeta))^{j+1-n} \end{pmatrix} \mathbf{W}_j(\zeta)\tilde{M}_j(\zeta). \tag{2.62}$$

In the appendix, we use a Neumann series solution to (2.62) to prove that $\tilde{M}_n(\zeta)$ is an analytic function of ζ outside the unit circle, if the potentials f_n, g_n, h_n in $\mathbf{W}_n(\zeta)$ are summable, i.e. $\sum_{n=-\infty}^{+\infty} |f_n| < \infty$ etc. Correspondingly, from (2.60) this requires $Q_n, R_n \in \ell_{1,0}$, where we define

$$\ell_{1,0} = \left\{ f_n : \sum_{j=\mp\infty}^{n} |f_j - \lim_{k\to\mp\infty} f_k| < \infty \; \forall n \in \mathbb{Z} \right\}.$$

Since $\tilde{M}_n(\zeta)$ is analytic for $|\zeta| > 1$, then (2.61) yield analyticity for $M_n(\zeta)$ in the same region, apart from a pole at ∞. In a similar way one can prove analyticity for the other eigenfunctions. In conclusion, for potentials $Q_n, R_n \in \ell_{1,0}$, $M_n(\zeta)$ and $N_n(\zeta)$ are analytic functions of ζ outside the unit circle $|\zeta| = 1$, and $\bar{M}_n(\zeta)$ and $\bar{N}_n(\zeta)$ inside and continuous on the circle, excluding, possibly, the branch points ζ_0 and ζ_0^*. In this respect, note that $\mathbf{W}_n(\zeta)$ has an overall factor $1/(\zeta + 1/\zeta - 2r)$ which vanishes precisely at the points ζ_0 and ζ_0^* that correspond to the four branch points $\pm z_0$ and $\pm z_0^*$ of the function $\lambda(z)$. However, the behavior of the eigenfunctions $M_n(\zeta)$, $\bar{M}_n(\zeta)$, $N_n(\zeta)$ and $\bar{N}_n(\zeta)$ at the branch points can be studied directly in terms of the original scattering problem (cf (2.46)), to conclude that all eigenfunctions are also well defined at the branch points.

Note that in [21] the eigenfunctions are implicitly assumed to be entire, and the equations of the inverse problem are derived accordingly. We stress that, in order for the eigenfunctions to be entire functions of ζ, strong assumptions on the decay of $Q_n - Q_\pm$ as $n \to \pm\infty$ are required. Here we find that under more general summability conditions on the potentials, $M_n(\zeta)$, $N_n(\zeta)$ and $a(\zeta)$ are analytic in the ζ-plane outside the circle $|\zeta| = 1$ (apart from a pole at ∞ for $M_n(\zeta)$), while $\bar{M}_n(\zeta)$, $\bar{N}_n(\zeta)$ and $\bar{a}(\zeta)$ are analytic in the ζ-plane for $|\zeta| < 1$ (apart from a pole at $\zeta = 0$ for $\bar{M}_n(\zeta)$). The other scattering coefficients $b(\zeta)$ and $\bar{b}(\zeta)$ are in general only defined on the circle $|\zeta| = 1$.

2.5. Asymptotics of eigenfunctions and scattering coefficients in the uniformization variable

First of all, note that from (2.23)–(2.24) we have the following:

$$z^2 \sim \begin{cases} r/\zeta & \zeta \to 0 \\ 1/(r\zeta) & \zeta \to \infty \\ -(\zeta - r)/\left(rQ_o^2\right) & \zeta \to r \\ Q_o^2/(r\zeta - 1) & \zeta \to 1/r, \end{cases} \qquad \lambda^2 \sim \begin{cases} r\zeta & \zeta \to 0 \\ \zeta/r & \zeta \to \infty \\ -(\zeta - r)r/Q_o^2 & \zeta \to r \\ Q_o^2 r^{-2}/(r\zeta - 1) & \zeta \to 1/r \end{cases}$$

$$\lambda z \sim \begin{cases} r & \zeta \to 0 \\ 1/r & \zeta \to \infty \\ -(\zeta - r)/Q_o^2 & \zeta \to r \\ Q_o^2 r^{-2}/(r\zeta - 1) & \zeta \to 1/r. \end{cases}$$

Then, performing a WKB expansion on the modified scattering problem (2.31), we obtain (see the appendix for details)

$$M_n(\zeta) \sim \begin{pmatrix} Q_{n-1} \\ \zeta \end{pmatrix}, \qquad N_n(\zeta) \sim \frac{1}{\Delta_n} \begin{pmatrix} r \\ -R_n \end{pmatrix} \qquad \zeta \to \infty, \tag{2.63a}$$

$$\bar{N}_n(\zeta) \sim \frac{1}{\Delta_n} \begin{pmatrix} Q_n \\ -r \end{pmatrix}, \qquad \bar{M}_n(\zeta) \sim \begin{pmatrix} -1/\zeta \\ -R_{n-1} \end{pmatrix} \qquad \zeta \to 0, \tag{2.63b}$$

where Δ_n is defined by (2.38). Now observe also that $\zeta \to r$ corresponds to $\lambda \to zr$ and from (2.8), since $r^2 \neq 1$ (i.e. $Q_o \neq 0$), this means $z \to 0$ and $\lambda \to 0$. Similarly, $\zeta = 1/r$

corresponds to $z, \lambda \to \infty$. This suggests that the values $\zeta = r, 1/r$ play a special role and should be treated on the same footing as $\zeta = 0, \infty$, respectively. A WKB expansion about these points yields

$$\bar{M}_n(\zeta) \sim -R_- \begin{pmatrix} Q_{n-1}/r \\ 1 \end{pmatrix}, \qquad \bar{N}_n(\zeta) \sim \frac{Q_+}{\Delta_n} \begin{pmatrix} 1 \\ (\zeta - r)R_n/Q_o^2 \end{pmatrix} \qquad \zeta \to r, \tag{2.64a}$$

and

$$M_n(\zeta) \sim Q_- \begin{pmatrix} 1 \\ R_{n-1}/r \end{pmatrix}, \qquad N_n(\zeta) \sim -\frac{R_+}{\Delta_n} \begin{pmatrix} -Q_n r^2(\zeta - 1/r)/Q_o^2 \\ 1 \end{pmatrix} \qquad \zeta \to 1/r. \tag{2.64b}$$

Finally, recalling (2.39*a*), from the above expansions we obtain

$$a(\zeta) = -\Delta_n \frac{W(M_n(\zeta), N_n(\zeta))}{r(\zeta + 1/\zeta - 2r)} \sim 1 \qquad \zeta \to \infty, \tag{2.65a}$$

$$\bar{a}(\zeta) = \Delta_n \frac{W(\bar{M}_n(\zeta), \bar{N}_n(\zeta))}{r(\zeta + 1/\zeta - 2r)} \sim 1 \qquad \zeta \to 0, \tag{2.65b}$$

and

$$\bar{a}(\zeta) \sim \frac{R_- Q_+}{1 - r^2} \equiv \mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)} \qquad \zeta \to r, \tag{2.66a}$$

$$a(\zeta) \sim \frac{Q_- R_+}{1 - r^2} \equiv \mathrm{e}^{-\mathrm{i}(\theta_+ - \theta_-)} \qquad \zeta \to 1/r. \tag{2.66b}$$

The asymptotic behavior of the eigenfunctions and scattering coefficients will be key to properly formulate the inverse problem.

2.6. Symmetries and properties of the scattering data

From the scattering problem (2.1*a*), one can see that the matrix $\mathbf{L}_n = \mathbf{Z} + \mathbf{Q}_n$ obeys the involution

$$\mathbf{L}_n(z) = \boldsymbol{\sigma}_1 \mathbf{L}_n^*(1/z^*)\boldsymbol{\sigma}_1,$$

where, as usual, $\boldsymbol{\sigma}_1$ is the Pauli matrix

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note also that the involution $z \to 1/z^*$ corresponds to $\zeta \to 1/\zeta^*$. Comparing the boundary conditions (2.9) and recalling that $\lambda \to 1/\lambda^*$, and $r/\lambda - 1/z = z - \lambda r$, one can check that

$$\psi_n^*(1/z^*, 1/\lambda^*) = -\boldsymbol{\sigma}_1 \bar{\psi}_n(z, \lambda) \qquad \phi_n^*(1/z^*, 1/\lambda^*) = -\boldsymbol{\sigma}_1 \bar{\phi}_n(z, \lambda). \tag{2.67}$$

Or, in a matrix form (cf (2.17)),

$$\boldsymbol{\Psi}_n^*(1/z^*, 1/\lambda^*) = -\boldsymbol{\sigma}_1 \boldsymbol{\Psi}_n(z, \lambda)\boldsymbol{\sigma}_1 \qquad \boldsymbol{\Phi}_n^*(1/z^*, 1/\lambda^*) = -\boldsymbol{\sigma}_1 \boldsymbol{\Phi}_n(z, \lambda)\boldsymbol{\sigma}_1. \tag{2.68}$$

From (2.21) it then follows that the complex constants b_k and $\bar{b}_k$ relating the values of the eigenfunctions at discrete eigenvalues satisfy the symmetry relation

$$\bar{b}_k = b_k^* \qquad k = 1, \ldots, J. \tag{2.69}$$

Substituting (2.68) into (2.17) yields

$$\mathbf{T}^*(1/z^*, 1/\lambda^*) = \boldsymbol{\sigma}_1 \mathbf{T}(z, \lambda)\boldsymbol{\sigma}_1. \tag{2.70}$$

There is a second involution that relates the values of the eigenfunctions on two different sheets in the z-plane and in particular across the cuts (cf figure 1). In fact, the scattering problem (2.1*a*) is independent of λ, and comparing the asymptotic values of the eigenfunctions as $n \to \pm\infty$ yields

$$\phi_n(z, 1/\lambda) = \frac{z - r/\lambda}{R_-}\bar{\phi}_n(z, \lambda), \qquad \psi_n(z, 1/\lambda) = \frac{r/\lambda - z}{Q_+}\bar{\psi}_n(z, \lambda). \quad (2.71)$$

Combining (2.67) and (2.71) one obtains

$$\phi_n(z, 1/\lambda) = \frac{r/\lambda - z}{R_-}\boldsymbol{\sigma}_1\phi_n^*(1/z^*, 1/\lambda^*), \qquad \bar{\phi}_n(z, 1/\lambda) = \frac{z - r/\lambda}{Q_-}\boldsymbol{\sigma}_1\bar{\phi}_n^*(1/z^*, 1/\lambda^*) \quad (2.72a)$$

$$\psi_n(z, 1/\lambda) = \frac{z - r/\lambda}{Q_+}\boldsymbol{\sigma}_1\psi_n^*(1/z^*, 1/\lambda^*), \qquad \bar{\psi}_n(z, 1/\lambda) = \frac{r/\lambda - z}{R_+}\boldsymbol{\sigma}_1\bar{\psi}_n^*(1/z^*, 1/\lambda^*). \quad (2.72b)$$

Note, in particular, that for the points that correspond to the continuous spectrum one has $1/z^* = z$ and $1/\lambda^* = \lambda$. Therefore, from the definitions (2.17) it follows

$$b^*(z, \lambda) = -\frac{R_-}{Q_+}b(z, 1/\lambda), \qquad a^*(z, \lambda) = \frac{R_-}{R_+}a(z, 1/\lambda) \quad (2.73a)$$

$$\bar{b}^*(z, \lambda) = -\frac{Q_-}{R_+}\bar{b}(z, 1/\lambda), \qquad \bar{a}^*(z, \lambda) = \frac{Q_-}{Q_+}\bar{a}(z, 1/\lambda). \quad (2.73b)$$

Hence, the reflection coefficients $\rho(z, \lambda) = \lambda(z)b(z, \lambda)/a(z, \lambda)$ and $\bar{\rho}(z, \lambda) = \bar{b}(z, \lambda)/(\lambda(z)\bar{a}(z, \lambda))$ satisfy the symmetry relations

$$\rho^*(z, \lambda) = -\frac{R_+}{Q_+}\rho(z, 1/\lambda), \qquad \bar{\rho}^*(z, \lambda) = -\frac{Q_+}{R_+}\bar{\rho}(z, 1/\lambda). \quad (2.73c)$$

Since $z \to 1/z^*$ corresponds to $\zeta \to 1/\zeta^*$, in terms of the variable ζ the scattering matrix (2.33) also satisfies the same involution

$$\mathbf{S}^*(1/\zeta^*) = \boldsymbol{\sigma}_1\mathbf{S}(\zeta)\boldsymbol{\sigma}_1, \quad (2.74)$$

as follows by noting that (2.29) yields $\mathbf{A}^*(1/\lambda^*) = \mathbf{A}^{-1}(\lambda)$ and $\mathbf{A}(\lambda(\zeta))\boldsymbol{\sigma}_1\mathbf{A}(\lambda(\zeta)) = \lambda\boldsymbol{\sigma}_1$. Explicitly, we can write the following symmetry relations among the scattering coefficients:

$$\bar{a}(\zeta) = a^*(1/\zeta^*), \qquad \bar{b}(\zeta) = b^*(1/\zeta^*). \quad (2.75)$$

In particular, this implies that ζ_k is a zero of $a(\zeta)$ outside the unit circle iff $\bar{\zeta}_k = 1/\zeta_k^*$ is a zero of $\bar{a}(\zeta)$ inside the unit circle. As a consequence, the discrete eigenvalues, i.e., the zeros of $a(\zeta)$ outside the unit circle and of $\bar{a}(\zeta)$ inside, are paired.

Remark 3. In the appendix we show that

(a) The continuous spectrum (2.22*a*) in the z-plane is mapped onto the unit circle of the ζ-plane (that is, $|\zeta| = 1$) excluding the points ζ_0 and ζ_0^*.
(b) The discrete spectrum (2.22*b*) in the z-plane is mapped onto a set of points $\{\zeta_1, \ldots, \zeta_J\}$ which lie on the circle of center $1/r$ and radius Q_o/r in the ζ-plane; that is, on the circle $|\zeta - 1/r|^2 = Q_o^2/r^2$.

We conclude that, in terms of ζ, the discrete spectrum is a set of zeros $\bar{\zeta}_k$ of $\bar{a}(\zeta)$:

$$\bar{a}(\bar{\zeta}_k) = 0, \qquad |\bar{\zeta}_k| < 1, \qquad k = 1, 2, \ldots, J,$$

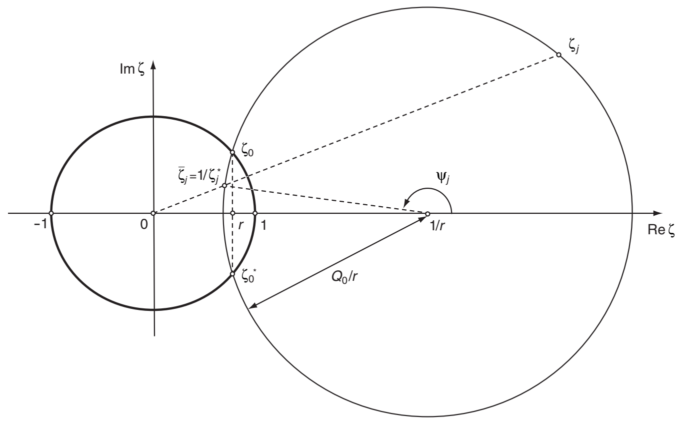

Figure 3. The uniformization variable ζ; continuous spectrum $|\zeta| = 1$, discrete spectrum $|\zeta - 1/r| = Q_o/r$.

located on the arc of circle centered at $\zeta = 1/r$ and radius Q_o/r, lying inside the unit circle (cf [21], and see also figure 3 and remark 3), and the corresponding zeros $\zeta_k = 1/\bar{\zeta}_k^*$ of $a(z)$ (cf (2.75)) outside the unit circle. Given their location, the discrete eigenvalues lying inside the circle $|\zeta| = 1$ can be naturally parametrized by means of the angles ψ_k, cf figure 3, defined by

$$\bar{\zeta}_k = (1 + Q_o\,\mathrm{e}^{\mathrm{i}\psi_k})/r, \qquad |\pi - \psi_k| < \arctan(r/Q_o). \tag{2.76}$$

Note that from (2.75) one can also obtain a symmetry relation for the derivatives of the scattering coefficients $\bar{a}(\zeta)$ and $a(\zeta)$. In fact, one has

$$\bar{a}'(\zeta) = -(a'(1/\zeta^*))^*/\zeta^2. \tag{2.77}$$

In particular, at a discrete eigenvalue (2.77) becomes

$$(a'(\zeta_k))^* = -\bar{\zeta}_k^2\bar{a}'(\bar{\zeta}_k). \tag{2.78}$$

Noting that $\det \mathbf{S} = \det \mathbf{T}$, from (2.19) and (2.75) we get the discrete analog of unitarity for points on the continuous spectrum ($|\zeta| = 1$, $\zeta \neq \zeta_0, \zeta_0^*$):

$$|\bar{a}(\zeta)|^2 - |\bar{b}(\zeta)|^2 = \prod_{n=-\infty}^{\infty}(1 - Q_nR_n)/r^2.$$

Using (2.17) and taking into account that the diagonal elements of the scattering matrix are not modified by the transformation (2.28), equations (2.20) can be written as

$$\bar{a}(\zeta) = \left[\boldsymbol{\Psi}_n^{-1}(\zeta)\boldsymbol{\Phi}_n(\zeta)\right]_{22}, \qquad a(\zeta) = \left[\boldsymbol{\Psi}_n^{-1}(\zeta)\boldsymbol{\Phi}_n(\zeta)\right]_{11}, \tag{2.79}$$

where the lower indices denote the corresponding entries in the 2×2 matrices. Differentiating the above relations with respect to the scattering parameter yields

$$\bar{a}'(\zeta) = \left[-\boldsymbol{\Psi}_n^{-1}(\zeta)\frac{\mathrm{d}\boldsymbol{\Psi}_n(\zeta)}{\mathrm{d}\zeta}\mathbf{T}(\zeta) + \mathbf{T}(\zeta)\boldsymbol{\Phi}_n^{-1}(\zeta)\frac{\mathrm{d}\boldsymbol{\Phi}_n(\zeta)}{\mathrm{d}\zeta}\right]_{22}, \tag{2.80a}$$

$$a'(\zeta) = \left[-\boldsymbol{\Psi}_n^{-1}(\zeta)\frac{\mathrm{d}\boldsymbol{\Psi}_n(\zeta)}{\mathrm{d}\zeta}\mathbf{T}(\zeta) + \mathbf{T}(\zeta)\boldsymbol{\Phi}_n^{-1}(\zeta)\frac{\mathrm{d}\boldsymbol{\Phi}_n(\zeta)}{\mathrm{d}\zeta}\right]_{11}. \tag{2.80b}$$

From the scattering problem (2.1*a*) one can check that $\mathrm{d}\mathbf{\Psi}_n/\mathrm{d}\zeta$ and $\mathrm{d}\mathbf{\Phi}_n/\mathrm{d}\zeta$ both satisfy the same difference equation

$$\mathbf{v}_{n+1}^{-1}\frac{\mathrm{d}\mathbf{v}_{n+1}}{\mathrm{d}\zeta}-\mathbf{v}_n^{-1}\frac{\mathrm{d}\mathbf{v}_n}{\mathrm{d}\zeta}=\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\mathbf{v}_{n+1}^{-1}\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{v}_n.$$

Therefore, one can formally write

$$\mathbf{\Phi}_n^{-1}(\zeta)\frac{\mathrm{d}\mathbf{\Phi}_n(\zeta)}{\mathrm{d}\zeta}=\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\sum_{j=-\infty}^{n-1}\mathbf{\Phi}_{j+1}^{-1}(\zeta)\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{\Phi}_j(\zeta),$$

$$\mathbf{\Psi}_n^{-1}(\zeta)\frac{\mathrm{d}\mathbf{\Psi}_n(\zeta)}{\mathrm{d}\zeta}=-\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\sum_{j=n}^{\infty}\mathbf{\Psi}_{j+1}^{-1}(\zeta)\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{\Psi}_j(\zeta),$$

using the decay of the corresponding eigenfunctions $\mathbf{\Phi}_n$ and $\mathbf{\Psi}_n$ as $n\to\mp\infty$ when evaluated at a discrete eigenvalue in the proper region of analyticity. Substituting into (2.80) yields

$$\bar{a}'(\bar{\zeta}_k)=\left[\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\sum_{j=-\infty}^{\infty}\mathbf{\Psi}_{j+1}^{-1}\mathbf{Z}\sigma_3\mathbf{\Psi}_j\mathbf{T}\right]_{22}(\bar{\zeta}_k),\tag{2.81a}$$

$$a'(\zeta_k)=\left[\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\sum_{j=-\infty}^{\infty}\mathbf{\Psi}_{j+1}^{-1}\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{\Psi}_j\mathbf{T}\right]_{11}(\zeta_k).\tag{2.81b}$$

The previous relations can be further simplified into (see the appendix):

$$\frac{\bar{a}'(\bar{\zeta}_k)}{\bar{b}(\bar{\zeta}_k)}=-\frac{z(\bar{\zeta}_k)}{(\bar{\zeta}_k-r)}\frac{1}{R_+}\,\mathrm{Re}\left\{z(\bar{\zeta}_k)\sum_{j=-\infty}^{\infty}r^{-2j}\Delta_j\bar{\psi}_{j-1}^{(1)}(\bar{\zeta}_k)\big(\bar{\psi}_j^{(1)}(\bar{\zeta}_k)\big)^*\right\},\tag{2.82a}$$

$$\frac{a'(\zeta_k)}{b(\zeta_k)}=-\frac{z(\zeta_k)}{(\zeta_k-r)}\frac{1}{Q_+}\,\mathrm{Re}\left\{z(\zeta_k)\sum_{j=-\infty}^{\infty}r^{-2j}\Delta_j\big(\psi_j^{(1)}(\zeta_k)\big)^*\psi_{j-1}^{(1)}(\zeta_k)\right\}.\tag{2.82b}$$

Equations (2.82) will be used in the following to obtain symmetry relations for the norming constants. Moreover, the above relations also imply that the zeros of $a(\zeta)$ and $\bar{a}(\zeta)$ are simple (see [21] for details).

A trace formula for the scattering coefficients $a(\zeta)$ and $\bar{a}(\zeta)$ can be derived from the knowledge of their analyticity properties and asymptotic behaviors. Namely, if $\bar{a}(\zeta)$ has J simple zeros at points $\bar{\zeta}_k$, for any ζ such that $|\zeta|<1$ one obtains [21]

$$\bar{a}(\zeta)=\prod_{k=1}^{J}\bar{\zeta}_k^*\frac{\zeta-\bar{\zeta}_k}{\zeta\bar{\zeta}_k^*-1}\exp\left[\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{\log|\bar{a}(w)|^2}{w-\zeta}\,\mathrm{d}w\right].\tag{2.83}$$

Condition (2.66*a*) then gives the analog of the θ-condition introduced by Faddeev and Takhtajan for the continuous NLS equation, i.e.

$$\mathrm{e}^{\mathrm{i}(\theta_+-\theta_-)}=\prod_{k=1}^{J}\bar{\zeta}_k^*\frac{r-\bar{\zeta}_k}{r\bar{\zeta}_k^*-1}\exp\left[\frac{1}{2\pi\mathrm{i}}\oint_{|w|=1}\frac{\log|\bar{a}(w)|^2}{w-r}\,\mathrm{d}w\right].\tag{2.84}$$

As observed in [21, 23], the peculiarity of this problem, compared to the case of vanishing boundaries, is that eigenfunctions and scattering data possess singularities at the edges of the continuous spectrum. The existence of the above mentioned singularities is formally a result of nontrivial dependence on the spectral parameter of the limiting values of the eigenfunctions, which in turn are determined by the behavior at infinity of the scattering potential.

Finally, let us briefly discuss the behavior of the scattering coefficients at the branch points. In section 2.3, it was shown that the eigenfunctions are well defined at the branch points ζ_0 and ζ_0^*. From (2.37) it then follows that

$$\mathrm{Wr}(\bar{N}_n(\zeta_0), N_n(\zeta_0)) = \mathrm{Wr}(\bar{N}_n(\zeta_0^*), N_n(\zeta_0^*)) = 0.$$

Therefore, at these values of ζ the eigenfunctions are proportional to each other; namely, $\bar{N}_n(\zeta_0) = \gamma N_n(\zeta_0)$ and $\bar{N}_n(\zeta_0^*) = \bar{\gamma} N_n(\zeta_0^*)$. Comparing the behavior at large n of both eigenfunctions, one gets $\gamma = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_+}$ and $\bar{\gamma} = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_+}$. Then from (2.39) it follows that the scattering coefficients have poles at points ζ_0 and ζ_0^*. Namely,

$$a(\zeta \sim \zeta_0) = \frac{\alpha_+}{\zeta - \zeta_0} + O(1), \qquad b(\zeta \sim \zeta_0) = \frac{\beta_+}{\zeta - \zeta_0} + O(1)$$

as $\zeta \to \zeta_0$, while

$$a(\zeta \sim \zeta_0^*) = \frac{\alpha_-}{\zeta - \zeta_0^*} + O(1), \qquad a(\zeta \sim \zeta_0^*) = \frac{\beta_-}{\zeta - \zeta_0^*} + O(1)$$

as $\zeta \to \zeta_0^*$. Finally, note that $\lambda^2(\zeta_0) = \lambda^2(\zeta_0^*) = 1$. Therefore from (2.39) it also follows that $\beta_\pm = \mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\theta_+}\alpha_\pm$. The special case when either α_+ or α_- vanish gives rise to what in scattering theory is referred to as a *virtual* level (cf [10]).

3. Inverse problem

The inverse problem consists in reconstructing the potentials from the scattering data. We accomplish this by defining and solving a suitable Riemann–Hilbert (RH) problem. To this end, we write the 'jump' conditions (2.34) in terms of the uniform variable ζ as

$$\frac{M_n(\zeta)}{a(\zeta)} = \bar{N}_n(\zeta) + \lambda(\zeta)^{-2n} N_n(\zeta)\rho(\zeta), \qquad (3.1a)$$

$$\frac{\bar{M}_n(\zeta)}{\bar{a}(\zeta)} = N_n(\zeta) + \lambda(\zeta)^{2n} \bar{N}_n(\zeta)\bar{\rho}(\zeta), \qquad (3.1b)$$

where we have introduced the reflection coefficients

$$\rho(\zeta) = \beta(\zeta)/a(\zeta), \qquad \bar{\rho}(\zeta) = \bar{\beta}(\zeta)/\bar{a}(\zeta). \qquad (3.2)$$

The functions $M_n(\zeta)/a(\zeta)$ and $N_n(\zeta)$ are analytic outside the unit circle $|\zeta| > 1$, while $\bar{M}_n(\zeta)/\bar{a}(\zeta)$ and $\bar{N}_n(\zeta)$ are analytic for $|\zeta| < 1$, with singularities (poles) at $\zeta = 0$, at $\zeta = \infty$ and at the zeros of $a(\zeta)$ and $\bar{a}(\zeta)$ respectively. To obtain an appropriate RH problem from (3.1), one must take into account the asymptotic behavior of the eigenfunctions as well as the poles, which we do next.

The results in section 2.5 imply that the asymptotic behavior of the quantities in (3.1) is given by

- As $\zeta \to \infty$:

$$\mu_n(\zeta) \equiv \frac{M_n(\zeta)}{a(\zeta)} \sim \begin{pmatrix} Q_{n-1} + o(1) \\ \zeta + O(1) \end{pmatrix}, \qquad N_n(\zeta) \sim \frac{1}{\Delta_n}\begin{pmatrix} r \\ -R_n \end{pmatrix} + O(1/\zeta). \qquad (3.3a)$$

- As $\zeta \to 0$:

$$\bar{\mu}_n(\zeta) \equiv \frac{\bar{M}_n(\zeta)}{\bar{a}(\zeta)} \sim -\begin{pmatrix} 1/\zeta + O(1) \\ R_{n-1} + o(1) \end{pmatrix}, \qquad \bar{N}_n(\zeta) \sim \frac{1}{\Delta_n}\begin{pmatrix} Q_n \\ -r \end{pmatrix} + O(\zeta). \qquad (3.3b)$$

- As $\zeta \to r$:

$$\bar{M}_n(\zeta) \sim -R_- \begin{pmatrix} Q_{n-1}/r \\ 1 \end{pmatrix} + O(\zeta - r), \qquad \bar{N}_n(\zeta) \sim \begin{pmatrix} Q_+/\Delta_n \\ 0 \end{pmatrix} + O(\zeta - r), \tag{3.3c}$$

$$\bar{a}(\zeta) \sim \mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)} + O(\zeta - r). \tag{3.3d}$$

- As $\zeta \to 1/r$:

$$M_n(\zeta) \sim Q_- \begin{pmatrix} 1 \\ R_{n-1}/r \end{pmatrix} + O(\zeta - 1/r), \qquad N_n(\zeta) \sim \begin{pmatrix} 0 \\ -R_+/\Delta_n \end{pmatrix} + O(\zeta - 1/r), \tag{3.3e}$$

$$a(\zeta) \sim \mathrm{e}^{-\mathrm{i}(\theta_+ - \theta_-)} + O(\zeta - 1/r). \tag{3.3f}$$

Note further that the points $z = 0$ and $z = \infty$ on sheet I are mapped onto the points $\zeta = \infty$ and $\zeta = 1/r$ respectively. The points $z = 0$ and $z = \infty$ on sheet II, instead, are mapped onto the points $\zeta = r$ and $\zeta = 0$. Therefore, in terms of the uniformization variable ζ, the points $\zeta = r$ and $\zeta = 1/r$ play the same role as the points 0 and ∞.

As for the behavior at the zeros of $a(\zeta)$ and $\bar{a}(\zeta)$, recall that, from the symmetry (2.75) it follows that $a(\zeta)$ has simple zeros at points $\zeta = \zeta_k$ ($|\zeta_k| > 1$) if and only if $\bar{a}(\zeta)$ has zeros at the corresponding points $\zeta = \bar{\zeta}_k \equiv 1/\zeta_k^*$. Also, from (2.40), one has

$$\mathrm{Res}\left(\frac{\bar{M}_n(\zeta)}{\bar{a}(\zeta)}; \zeta = \bar{\zeta}_k\right) = \bar{C}_k[\lambda(\bar{\zeta}_k)]^{2n} \bar{N}_n(\bar{\zeta}_k), \qquad \bar{C}_k = \frac{\bar{b}_k}{\lambda(\bar{\zeta}_k)\bar{a}'(\bar{\zeta}_k)}, \tag{3.4a}$$

$$\mathrm{Res}\left(\frac{M_n(\zeta)}{a(\zeta)}; \zeta = \zeta_k\right) = C_k[\lambda(\zeta_k)]^{-2n} N_n(\zeta_k), \qquad C_k = \frac{b_k \lambda(\zeta_k)}{a'(\zeta_k)}. \tag{3.4b}$$

If the potentials $Q_n - Q_\pm$ decay rapidly enough as $n \to \pm\infty$, such that $\beta(\zeta), \bar{\beta}(\zeta)$ can be extended off the unit circle in correspondence of the discrete eigenvalues, then the norming constants can be written as $\bar{C}_k = \bar{\beta}(\bar{\zeta}_k)/\bar{a}'(\bar{\zeta}_k)$ and $C_k = \beta(\zeta_k)/a'(\zeta_k)$. Note also that recalling (2.39*b*) and (2.78), and noting that $\lambda^*(z) = 1/\lambda(1/z^*)$, from (3.4) it follows

$$\bar{C}_k = -\bar{\zeta}_k^2 C_k^*. \tag{3.5}$$

Moreover, since $\bar{C}_k = \bar{\beta}(\bar{\zeta}_k)/\bar{a}'(\bar{\zeta}_k) \equiv \bar{b}(\bar{\zeta}_k)/(\bar{a}'(\bar{\zeta}_k)\lambda(\bar{\zeta}_k))$

$$\bar{C}_k = R_+(r\bar{\zeta}_k - 1)\frac{1}{\mathrm{Re}\left\{z(\bar{\zeta}_k)\sum_{n=-\infty}^{\infty} r^{-2n}\Delta_n \bar{\psi}_{n-1}^{(1)}(\bar{\zeta}_k)\big(\bar{\psi}_j^{(1)}(\bar{\zeta}_k)\big)^*\right\}}. \tag{3.6}$$

Recalling that the discrete eigenvalues are parametrized by the angles ψ_k according to (2.76), from (3.6) it then follows that

$$\bar{C}_k = \pm|\bar{C}_k|\,\mathrm{e}^{-\mathrm{i}\theta_+ + \mathrm{i}\psi_k}. \tag{3.7}$$

Equation (3.5) then fixes the argument of C_k accordingly.

Now consider (3.1*a*). Since $M_n(\zeta)/a(\zeta)$ grows linearly as $\zeta \to \infty$, we divide it by $\zeta - r$, obtaining

$$\frac{M_n(\zeta)}{(\zeta - r)a(\zeta)} = \frac{\bar{N}_n(\zeta)}{\zeta - r} + \frac{\lambda(\zeta)^{-2n}}{\zeta - r} N_n(\zeta)\rho(\zeta). \tag{3.8}$$

The function on the left-hand side (LHS) is still analytic outside the unit circle (since $r < 1$). Now, however, it goes to a constant as $\zeta \to \infty$ (cf (3.3*a*)). On the other hand, the function

$\bar{N}_n(\zeta)/(\zeta - r)$ on the right-hand side (RHS) now has a pole at $\zeta = r$ inside the unit circle, whose residue however is known. We thus subtract from both sides of (3.8) the behavior of the left-hand side at ∞, and the pole contributions at the zeros of $a(\zeta)$:

$$\frac{M_n(\zeta)}{(\zeta - r)a(\zeta)} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^{J} \frac{\mathrm{Res}(M_n/a;\, \zeta_k)}{(\zeta_k - r)(\zeta - \zeta_k)}$$
$$= \frac{\bar{N}_n(\zeta)}{\zeta - r} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^{J} \frac{\mathrm{Res}(M_n/a;\, \zeta_k)}{(\zeta_k - r)(\zeta - \zeta_k)} + \frac{\lambda(\zeta)^{-2n}}{\zeta - r} N_n(\zeta)\rho(\zeta). \tag{3.9}$$

We now introduce the 'inside' and 'outside' projectors

$$\begin{aligned} \bar{P}[f](\zeta) &= \frac{1}{2\pi \mathrm{i}} \lim_{\substack{\zeta' \to \zeta \\ |\zeta'|<1}} \int_{|w|=1} \frac{f(w)}{w - \zeta'}\, \mathrm{d}w \qquad |\zeta| < 1 \\ P[f](\zeta) &= \frac{1}{2\pi \mathrm{i}} \lim_{\substack{\zeta' \to \zeta \\ |\zeta'|>1}} \int_{|w|=1} \frac{f(w)}{w - \zeta'}\, \mathrm{d}w \qquad |\zeta| > 1, \end{aligned} \tag{3.10}$$

which are the projection operators for functions analytic inside and outside the unit circle, respectively. We then apply the 'inside' projector $\bar{P}[\cdot]$ to both sides of (3.9). Since the LHS is analytic outside the circle, and decaying as $\zeta \to \infty$, its $\bar{P}$ projection will be identically zero. Hence for any ζ inside the unit circle ($|\zeta| < 1$) the RHS yields

$$\frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \left[\frac{\bar{N}_n(w)}{w - r} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^{J} \frac{\mathrm{Res}(M_n/a;\, \zeta_k)}{(\zeta_k - r)(w - \zeta_k)} + \frac{\lambda(w)^{-2n}}{w - r} N_n(w)\rho(w) \right] \frac{\mathrm{d}w}{w - \zeta} = 0.$$

That is (since ζ is inside the unit circle, and so is r)

$$\bar{N}_n(\zeta) = \bar{N}_n(r) + \begin{pmatrix} 0 \\ \zeta - r \end{pmatrix} + \sum_{k=1}^{J} \frac{(\zeta - r)\, \mathrm{Res}(M_n/a;\, \zeta_k)}{(\zeta_k - r)(\zeta - \zeta_k)}$$
$$- \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\zeta - r)}{w - \zeta} \frac{\lambda(w)^{-2n}}{w - r} N_n(w)\rho(w)\, \mathrm{d}w.$$

Then, taking into account that $\bar{N}_n(r)$ is given by (3.3*c*) and that

$$\mathrm{Res}(M_n/a;\, \zeta_k) = C_k \lambda(\zeta_k)^{-2n} N_n(\zeta_k),$$

we obtain, according to (3.4*a*), for any ζ with $|\zeta| < 1$

$$\bar{N}_n(\zeta) = \begin{pmatrix} Q_+/\Delta_n \\ \zeta - r \end{pmatrix} + \sum_{k=1}^{J} \frac{(\zeta - r) C_k \lambda(\zeta_k)^{-2n}}{(\zeta_k - r)(\zeta - \zeta_k)} N_n(\zeta_k)$$
$$- \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\zeta - r)}{w - \zeta} \frac{\lambda(w)^{-2n}}{w - r} N_n(w)\rho(w)\, \mathrm{d}w. \tag{3.11a}$$

Equation (3.11*a*) is the first part of the equations that will provide the solution of the inverse problem. Similarly, we consider (3.1*b*), and we divide it by $\zeta - 1/r$. After subtracting the pole at $\zeta = 0$ due to $\bar{M}_n(\zeta)$ and the pole contributions at the zeros of $\bar{a}(\zeta)$, we apply the 'outside' projector $P[\cdot]$ introduced in (3.10) for any ζ outside the unit circle ($|\zeta| > 1$). This yields

$$\frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \left[\frac{N_n(w)}{w - 1/r} - \frac{r}{w} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{k=1}^{J} \frac{\mathrm{Res}(\bar{M}_n/\bar{a};\, \bar{\zeta}_k)}{(\bar{\zeta}_k - 1/r)(w - \bar{\zeta}_k)} \right.$$
$$\left. + \frac{\lambda(w)^{2n}}{w - 1/r} \bar{N}_n(w)\bar{\rho}(w) \right] \frac{\mathrm{d}w}{w - \zeta} = 0.$$

Now recall that $N_n(\zeta)$ is analytic outside the unit circle, and that is goes to a constant as $\zeta \to \infty$ and therefore the residue of $N_n(w)/[(w-\zeta)(w-1/r)]$ as $w \to \infty$ is zero and the integral on the unit circle can be evaluated in terms of the residues at the points ζ and $1/r$, both outside the unit circle. Taking into account (3.3*e*) and (3.4*b*) one finally obtains for $|\zeta| > 1$:

$$N_n(\zeta) = \begin{pmatrix} r - 1/\zeta \\ -R_+/\Delta_n \end{pmatrix} + \sum_{k=1}^{J} \frac{(\zeta - 1/r)\bar{C}_k \lambda(\bar{\zeta}_k)^{2n} \bar{N}_n(\bar{\zeta}_k)}{(\bar{\zeta}_k - 1/r)(\zeta - \bar{\zeta}_k)} + \frac{1}{2\pi \mathrm{i}} \int \frac{(\zeta - 1/r)\lambda(w)^{2n}}{w - 1/r} \bar{N}_n(w)\bar{\rho}(w) \frac{\mathrm{d}w}{w - \zeta}. \tag{3.11b}$$

Equation (3.11*b*) is the second equation that will provide the solution of the inverse problem. Note that, as in the IST for IDNLS with decaying boundary conditions, the equations of the inverse problem depend on Δ_n, which in general is unknown. In that context, the problem is easily solved by the introduction of modified eigenfunctions whose asymptotic values do not depend on the potentials. The same does not seem to be straightforward here, however. To circumvent this problem, we note that the potential could be reconstructed, for instance, by means of the large-ζ expansion of $N_n(\zeta)$. According to (3.3*a*), from the asymptotics of the first and second component of (3.11*b*) one obtains, respectively

$$1/\Delta_n = 1 + \sum_{k=1}^{J} \frac{1}{r\bar{\zeta}_k - 1} \bar{C}_k [\lambda(\bar{\zeta}_k)]^{2n} \bar{N}_n^{(1)}(\bar{\zeta}_k) - \frac{1}{2\pi \mathrm{i}} \int_{|\zeta'|=1} \lambda(\zeta')^{2n} \bar{N}_n^{(1)}(\zeta')\bar{\rho}(\zeta') \frac{\mathrm{d}\zeta'}{r\zeta' - 1}, \tag{3.12a}$$

$$-\frac{R_n}{\Delta_n} = -\frac{R_+}{\Delta_n} + \sum_{k=1}^{J} \frac{1}{\bar{\zeta}_k - 1/r} \bar{C}_k [\lambda(\bar{\zeta}_k)]^{2n} \bar{N}_n^{(2)}(\bar{\zeta}_k) - \frac{1}{2\pi \mathrm{i}} \int_{|\zeta'|=1} \lambda(\zeta')^{2n} \bar{N}_n^{(2)}(\zeta')\bar{\rho}(\zeta') \frac{\mathrm{d}\zeta'}{\zeta' - 1/r}. \tag{3.12b}$$

Equations (3.11) and (3.12*a*) are a system of five *linear* algebraic-integral equations for the five unknowns $N_n^{(1)}(\zeta)$, $N_n^{(2)}(\zeta)$, $\bar{N}_n^{(1)}(\zeta)$, $\bar{N}_n^{(1)}(\zeta)$ and $1/\Delta_n$ in terms of scattering coefficients $\{\rho(\zeta), \bar{\rho}(\zeta) : |\zeta| = 1\}$, $\{\zeta_k : |\zeta_k| > 1, C_k\}_{k=1}^J$ and $\{\bar{\zeta}_k : |\bar{\zeta}_k| < 1, \bar{C}_k\}_{k=1}^J$, since $0 < r < 1$ is given and $\lambda(\zeta)$ is a known function. If this system admits a (unique) solution (once appropriately closed by evaluating (3.11*a*) at $\zeta = \bar{\zeta}_j$ and (3.11*b*) at $\zeta = \zeta_j$ for all $j = 1, \ldots, J$), then (3.12*b*) allows one to reconstruct the potential R_n.

It should be also noted that (3.12*a*) yields $1/\Delta_n$ in terms of $\bar{N}_n^{(1)}(\zeta)$ and scattering data. Since $1/\Delta_n$ appears linearly in (3.11*a*) and (3.11*b*), by substitution of (3.12*a*) into (3.11*a*) and (3.11*b*) we obtain a system of liner equations for $N_n(\zeta)$ and $\bar{N}_n(\zeta)$ only.

4. Gel'fand–Levitan–Marchenko equations

It is also possible to provide a reconstruction of the potentials by developing the discrete analog of Gel'fand–Levitan–Marchenko (GLM) integral equations. Let us consider triangular representations for the eigenfunctions $N_n(\zeta)$ and $\bar{N}_n(\zeta)$:

$$N_n(\zeta) = \begin{pmatrix} r - 1/\zeta \\ -R_+/\Delta_n \end{pmatrix} + \sum_{j=n}^{\infty} \frac{r\zeta - 1}{\zeta - r} \lambda(\zeta)^{2(n-j)} K_{n,j} \qquad |\zeta| > 1 \tag{4.1a}$$

$$\bar{N}_n(\zeta) = \begin{pmatrix} Q_+/\Delta_n \\ \zeta - r \end{pmatrix} + \sum_{j=n}^{\infty} \frac{\zeta - r}{r\zeta - 1} \lambda(\zeta)^{2(j-n)} \bar{K}_{n,j} \qquad |\zeta| < 1, \tag{4.1b}$$

where $K_{n,j}$ and $\bar{K}_{n,j}$ are two-component ζ-independent vectors. Note that these representations are compatible with the asymptotic behavior of the eigenfunctions as $n \to +\infty$, since $\Delta_n \to 1$ in that limit, and they are also compatible with the ζ-asymptotics provided $K_{n,j}$ and $\bar{K}_{n,j}$ satisfy certain constraints. In fact, since $\lambda^2(\zeta)$ decays as $\zeta \to 0$ and $1/\lambda^2(\zeta)$ decays as $\zeta \to \infty$, one has

$$N_n(\zeta) \sim \begin{pmatrix} r \\ -R_+/\Delta_n \end{pmatrix} + rK_{n,n} \qquad \zeta \to \infty, \qquad N_n(1/r) = \begin{pmatrix} 0 \\ -R_+/\Delta_n \end{pmatrix}$$

$$\bar{N}_n(\zeta) \sim \begin{pmatrix} Q_+/\Delta_n \\ -r \end{pmatrix} + r\bar{K}_{n,n} \qquad \zeta \to 0, \qquad \bar{N}_n(r) = \begin{pmatrix} Q_+/\Delta_n \\ 0 \end{pmatrix}$$

and comparing with (2.63) and (2.64) we obtain

$$rK_{n,n} = \begin{pmatrix} r(1/\Delta_n - 1) \\ (R_+ - R_n)/\Delta_n \end{pmatrix}, \qquad r\bar{K}_{n,n} = \begin{pmatrix} (Q_n - Q_+)/\Delta_n \\ -r(1/\Delta_n - 1) \end{pmatrix}. \tag{4.2}$$

These formulas provide the reconstruction of Δ_n and Q_n, R_n in terms of the kernels $K_{n,n}$ and $\bar{K}_{n,n}$ of the triangular representations for the eigenfunctions. Note also that the choice of power expansion in $\lambda^2(\zeta)$ in the triangular representations (4.1) allows us to admit kernels $K_{n,j}$ and $\bar{K}_{n,j}$ that are not necessarily strongly decaying at infinity. Moreover, the symmetry relation (2.67) for the eigenfunctions corresponds to

$$\bar{K}_{n,j} = -\boldsymbol{\sigma}_1 K_{n,j} \qquad \forall n, j \in \mathbb{Z}.$$

We now apply the operator $\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \mathrm{d}\zeta \, \lambda(\zeta)^{2(n-m-1)}$ for $m \geqslant n$ to the first equation of the inverse problem, equation (3.11*a*), and then substitute the triangular representations (4.1) to obtain

$$\begin{aligned}
&\sum_{j=n}^{\infty} \bar{K}_{n,j} \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta - r}{r\zeta - 1} \lambda(\zeta)^{2(j-m-1)} \, \mathrm{d}\zeta \\
&\quad = \sum_{k=1}^{J} \frac{C_k (r\zeta_k - 1)\lambda(\zeta_k)^{-2n}}{(\zeta_k - r)^2} \begin{pmatrix} r - 1/\zeta_k \\ -R_+/\Delta_n \end{pmatrix} \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta - r}{\zeta - \zeta_k} \lambda(\zeta)^{2(n-m-1)} \, \mathrm{d}\zeta \\
&\qquad + \sum_{j=n}^{\infty} K_{n,j} \sum_{k=1}^{J} \frac{C_k}{\zeta_k - r} \lambda(\zeta_k)^{-2j} \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta - r}{\zeta - \zeta_k} \lambda(\zeta)^{2(n-m-1)} \, \mathrm{d}\zeta \\
&\qquad - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{\mathrm{d}w}{w - r} \lambda(w)^{-2n} \rho(w) \begin{pmatrix} r - 1/w \\ -R_+/\Delta_n \end{pmatrix} \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \lambda(\zeta)^{2(n-m-1)} \frac{\zeta - r}{w - \zeta} \, \mathrm{d}\zeta \\
&\qquad - \sum_{j=n}^{\infty} K_{n,j} \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \mathrm{d}w \frac{rw - 1}{(w - r)^2} \lambda(w)^{-2j} \rho(w) \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \lambda(\zeta)^{2(n-m-1)} \frac{\zeta - r}{w - \zeta} \, \mathrm{d}\zeta.
\end{aligned}$$

Evaluating the integrals with respect to ζ, we finally obtain

$$\bar{K}_{n,m} = \delta_{n,m} G_{1,n} + F_{1,m} + \sum_{j=n}^{\infty} K_{n,j} [\delta_{n,m} G_{2,j} + F_{2,m+j-n}], \tag{4.3}$$

where

$$G_{1,n} = \sum_{k=1}^{J} \frac{C_k \lambda(\zeta_k)^{-2(n+1)}}{r\zeta_k - 1} \begin{pmatrix} r - 1/\zeta_k \\ -R_+/\Delta_n \end{pmatrix} + \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{\lambda(w)^{-2(n+1)}}{rw - 1} \rho(w) \begin{pmatrix} r - 1/w \\ -R_+/\Delta_n \end{pmatrix} \mathrm{d}w \tag{4.4a}$$

$$G_{2,n} = \sum_{k=1}^{J} \frac{C_k \lambda(\zeta_k)^{-2(n+1)}}{\zeta_k - r} + \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{\lambda(w)^{-2(n+1)}}{w - r} \rho(w)\, \mathrm{d}w \tag{4.4b}$$

$$F_{1,m} = \sum_{k=1}^{J} C_k \lambda(\zeta_k)^{-2(m+1)} \begin{pmatrix} r - 1/\zeta_k \\ -R_+/\Delta_n \end{pmatrix} + \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} (rw - 1)\rho(w)\lambda(w)^{-2(m+1)} \begin{pmatrix} r - 1/w \\ -R_+/\Delta_n \end{pmatrix} \mathrm{d}w \tag{4.4c}$$

$$F_{2,m} = \sum_{k=1}^{J} \frac{C_k (r\zeta_k - 1)}{\zeta_k - r} \lambda(\zeta_k)^{-2(m+1)} + \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{rw - 1}{w - r} \rho(w)\lambda(w)^{-2(m+1)}\, \mathrm{d}w. \tag{4.4d}$$

In a similar way, applying the operator $\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \mathrm{d}\zeta\, \lambda(\zeta)^{2(m-n-1)}$ for $m \geqslant n$ to (3.11b) yields

$$K_{n,m} = \delta_{n,m} \bar{G}_{1,n} + \bar{F}_{1,m} + \sum_{j=n}^{\infty} \bar{K}_{n,j} [\delta_{n,m} \bar{G}_{2,j} + \bar{F}_{2,m+j-n}], \tag{4.5}$$

where

$$\bar{F}_{1,m} = \sum_{k=1}^{J} \bar{C}_k \lambda(\bar{\zeta}_k)^{2(m-1)} \begin{pmatrix} Q_+/\Delta_n \\ \bar{\zeta}_k - r \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \lambda(w)^{2(m-1)} \bar{\rho}(w) \begin{pmatrix} Q_+/\Delta_n \\ w - r \end{pmatrix} \mathrm{d}w \tag{4.6a}$$

$$\bar{F}_{2,m} = \sum_{k=1}^{J} \bar{C}_k \frac{\bar{\zeta}_k - r}{r\bar{\zeta}_k - 1} \lambda(\bar{\zeta}_k)^{2(m-1)} - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{w - r}{rw - 1} \lambda(w)^{2(m-1)} \bar{\rho}(w)\, \mathrm{d}w \tag{4.6b}$$

$$\bar{G}_{1,n} = r \sum_{k=1}^{J} \frac{\bar{C}_k \lambda(\bar{\zeta}_k)^{2n}}{\bar{\zeta}_k - 1/r} \begin{pmatrix} Q_+/\Delta_n \\ \bar{\zeta}_k - r \end{pmatrix} - r \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{\lambda(w)^{2n}}{w - 1/r} \bar{\rho}(w) \begin{pmatrix} Q_+/\Delta_n \\ w - r \end{pmatrix} \mathrm{d}w \tag{4.6c}$$

$$\bar{G}_{2,n} = \sum_{k=1}^{J} \frac{\bar{C}_k \lambda(\bar{\zeta}_k)^{2n}}{(\bar{\zeta}_k - 1/r)^2} - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(w - r)\lambda(w)^{2n}}{(w - 1/r)^2} \bar{\rho}(w)\, \mathrm{d}w \tag{4.6d}$$

Together with (4.2), equations (4.3) and (4.5) provide a closed system and allow in principle to reconstruct the potentials.

5. Time evolution and conserved quantities

Having determined how to reconstruct the potential from the scattering data in section 3, our last task in the implementation of the IST is to determine the time evolution of the scattering data. Accordingly, from now on we will write explicitly the time dependence of the potentials $Q_n(\tau)$ and $R_n(\tau)$ and scattering eigenfunctions $\phi_n(z, \tau)$, $\bar{\phi}_n(z, \tau)$, $\psi_n(z, \tau)$, $\bar{\psi}_n(z, \tau)$ etc.

Let $\lim_{n\to\pm\infty} Q_n = Q_\pm(\tau)$ and $\lim_{n\to\pm\infty} R_n = R_\pm(\tau)$ be the boundary data as a function of time. The value of $Q_\pm(\tau)$ and $R_\pm(\tau)$ is uniquely determined from the initial data via (1.4). Indeed, interchanging the order of limit and derivative, (1.4) and (2.4) imply

$$\lim_{n\to\pm\infty} \frac{\mathrm{d}Q_n}{\mathrm{d}\tau} = \frac{\mathrm{d}Q_\pm}{\mathrm{d}\tau} = 2\mathrm{i}Q_o^2 Q_\pm, \qquad \lim_{n\to\pm\infty} \frac{d(R_n Q_n)}{\mathrm{d}\tau} = \frac{\mathrm{d}\big(Q_o^2\big)}{\mathrm{d}\tau} = 0. \tag{5.1}$$

Hence we obtain immediately

$$Q_\pm(\tau) = Q_o\,\mathrm{e}^{\mathrm{i}\theta_\pm(\tau)}, \tag{5.2a}$$

where

$$\theta_\pm(\tau) = \theta_\pm(0) + 2Q_o^2\tau. \tag{5.2b}$$

Equation (5.2) determines the time evolution of the asymptotic phase of the potential as $n \to \pm\infty$. Moreover, since $R_n(\tau) = Q_n^*(\tau)$, we have $R_\pm(\tau) = Q_\pm^*(\tau) = Q_o^2\,\mathrm{e}^{-\mathrm{i}\theta_\pm(\tau)}$.

Let us now determine the time evolution of the eigenfunctions. The time dependence of the solutions of the Lax pair is specified by (2.1*b*). The asymptotic behavior of the eigenfunctions $\phi_n(z,\tau)$, $\bar\phi_n(z,\tau)$, $\psi_n(z,\tau)$ and $\bar\psi_n(z,\tau)$, however, is given at all times τ by (2.2*b*). Note that from (2.1*b*) and (2.4) it follows that

$$\mathbf{M}_n(z,\tau) \sim \begin{pmatrix} \mathrm{i}(Q_o^2-\omega_0) & -\mathrm{i}(z-1/z)Q_\pm(\tau) \\ -\mathrm{i}(z-1/z)R_\pm(\tau) & -\mathrm{i}(Q_o^2-\omega_0) \end{pmatrix} \qquad n \to \pm\infty,$$

where we have introduced the short-hand notation $\omega_0 = \frac{1}{2}(z-1/z)^2$. Using this asymptotic behavior, we then obtain a system of equations which fixes, at large space infinities, the time dependence of the asymptotic values of the eigenfunctions. Indeed, using the asymptotic form of the scattering problem as $n \to \pm\infty$,

$$Q_\pm(\tau)v_n^{(2)} \simeq v_{n+1}^{(1)} - z v_n^{(1)}, \qquad R_\pm(\tau)v_n^{(1)} \simeq v_{n+1}^{(2)} - v_n^{(2)}/z, \tag{5.3}$$

we obtain

$$\frac{\partial}{\partial\tau}v_n^{(1)} \sim \mathrm{i}\left(Q_o^2 + \frac{z^2-1/z^2}{2}\right)v_n^{(1)} - \mathrm{i}(z-1/z)v_{n+1}^{(1)}, \tag{5.4a}$$

$$\frac{\partial}{\partial\tau}v_n^{(2)} \sim -\mathrm{i}(z-1/z)v_{n+1}^{(2)} - \mathrm{i}\left(Q_o^2 - \frac{z^2-1/z^2}{2}\right)v_n^{(2)}, \tag{5.4b}$$

as $n \to \pm\infty$. In order to satisfy (5.4*b*), we introduce modified eigenfunctions to be solutions of the time-differential equation (2.2*b*):

$$\tilde\phi_n(\tau) = \mathrm{e}^{\mathrm{i}\omega_\infty^{(1)}\tau}\phi_n(\tau), \qquad \tilde\psi_n(\tau) = \mathrm{e}^{\mathrm{i}\omega_\infty^{(2)}\tau}\psi_n(\tau), \tag{5.5}$$

$$\tilde{\bar\phi}_n(\tau) = \mathrm{e}^{\mathrm{i}\omega_\infty^{(2)}\tau}\bar\phi_n(\tau), \qquad \tilde{\bar\psi}_n(\tau) = \mathrm{e}^{\mathrm{i}\omega_\infty^{(1)}\tau}\bar\psi_n(\tau). \tag{5.6}$$

One has

$$\frac{\partial\tilde\phi_n}{\partial\tau} = \mathrm{i}\omega_\infty^{(1)}\tilde\phi_n + \mathrm{e}^{\mathrm{i}\omega_\infty^{(1)}\tau}\frac{\partial\phi_n}{\partial\tau}, \tag{5.7}$$

with similar equations for the other three modified eigenfunctions.

Since $Q_\pm(\tau) = Q_o\,\mathrm{e}^{\mathrm{i}\theta_\pm(\tau)}$, from (2.9) we have, as $n \to -\infty$

$$\phi_n(\tau) \simeq \begin{pmatrix} Q_-(\tau) \\ \lambda r - z \end{pmatrix}\lambda^n r^n, \qquad \frac{\partial\phi_n(\tau)}{\partial\tau} \simeq \begin{pmatrix} \mathrm{i}\dot\theta_-(\tau)Q_-(\tau) \\ 0 \end{pmatrix}\lambda^n r^n.$$

Requiring that the components of $\tilde{\phi}_n(z, \tau)$ satisfy (5.4) asymptotically as $|n| \to \infty$ we then obtain, from the second component of (5.7),

$$\omega_\infty^{(1)} = -Q_o^2 + \frac{z^2 - 1/z^2}{2} - \lambda r(z - 1/z). \tag{5.8}$$

It then follows that the time evolution of the scattering eigenfunction $\phi_n(z, t)$ is given by

$$\frac{\partial \phi_n}{\partial \tau} = \big(\mathbf{M}_n - \mathrm{i}\omega_\infty^{(1)}\mathbf{I}\big)\phi_n, \tag{5.9}$$

where $\mathbf{I}$ is the 2×2 identity matrix as before. In a similar way one obtains that

$$\omega_\infty^{(2)} = Q_o^2 + \frac{z^2 - 1/z^2}{2} - r(z - 1/z)/\lambda.$$

Introducing the matrix

$$\mathbf{\Omega}(z) = \mathrm{diag}\big(\mathrm{i}\omega_\infty^{(1)}(z), \mathrm{i}\omega_\infty^{(2)}(z)\big),$$

we can also write the time evolution equations for the matrix eigenfunctions $\mathbf{\Phi}_n(z, \tau)$ and $\mathbf{\Psi}_n(z, \tau)$ defined in (2.16) as

$$\begin{aligned} \frac{\partial \mathbf{\Phi}_n(z, \tau)}{\partial \tau} &= \mathbf{M}_n(z, \tau)\mathbf{\Phi}_n(z, \tau) - \mathbf{\Phi}_n(z, \tau)\mathbf{\Omega}(z), \\ \frac{\partial \mathbf{\Psi}_n(z, \tau)}{\partial \tau} &= \mathbf{M}_n(z, \tau)\mathbf{\Psi}_n - \mathbf{\Psi}_n(z, \tau)\mathbf{\Omega}(z). \end{aligned} \tag{5.10}$$

Note that, as a consequence of (5.2), we obtain that the phase difference is time independent:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\theta_+(\tau) - \theta_-(\tau)] = 0.$$

(This was to be expected, since the asymptotic values of the amplitude of the potential as $n \to \pm\infty$ coincide.) From the scattering equation (2.17) we then obtain

$$\frac{\partial \mathbf{\Phi}_n(z, \tau)}{\partial \tau} = \frac{\partial \mathbf{\Psi}_n(z, \tau)}{\partial \tau}\mathbf{T}(z, \tau) + \mathbf{\Psi}_n(z, \tau)\frac{\partial \mathbf{T}(z, \tau)}{\partial \tau}.$$

That is, taking into account (5.10),

$$\frac{\partial \mathbf{T}(z, \tau)}{\partial \tau} = [\mathbf{\Omega}(z), \mathbf{T}(z, \tau)].$$

Componentwise, from (2.17)

$$\frac{\partial a(z, \tau)}{\partial \tau} = \frac{\partial \bar{a}(z, \tau)}{\partial \tau} = 0, \tag{5.11a}$$

$$\begin{aligned} \frac{\partial b(z, \tau)}{\partial \tau} &= \mathrm{i}\big[\omega_\infty^{(2)}(z) - \omega_\infty^{(1)}(z)\big]b(z, \tau) \equiv \mathrm{i}\big[2Q_o^2 + r(z - 1/z)(\lambda - 1/\lambda)\big]b(z, \tau), \\ \frac{\partial \bar{b}(z, \tau)}{\partial \tau} &= -\mathrm{i}\big[\omega_\infty^{(2)}(z) - \omega_\infty^{(1)}(z)\big]b(z, \tau) \equiv -\mathrm{i}\big[2Q_o^2 + r(z - 1/z)(\lambda - 1/\lambda)\big]\bar{b}(z, \tau). \end{aligned} \tag{5.11b}$$

We conclude that $a(z, \tau)$ and $\bar{a}(z, \tau)$ are time independent, while

$$b(z, \tau) = b(z, 0)\,\mathrm{e}^{2\mathrm{i}Q_o^2\tau + \mathrm{i}\mu(z)\tau} \tag{5.12a}$$

$$\bar{b}(z, \tau) = \bar{b}(z, 0)\,\mathrm{e}^{-2\mathrm{i}Q_o^2\tau - \mathrm{i}\mu(z)\tau}, \tag{5.12b}$$

where the function

$$\mu(z) = r(z - 1/z)(\lambda - 1/\lambda) \tag{5.13}$$

expresses the discrete analog of the linear dispersion relation of the continuous case. Note that, using (2.23), the above dispersion relation can be written in terms of the uniformization variable ζ as

$$\mu(\zeta) = r^2 \frac{(\zeta - 2/r + 1/\zeta)(\zeta - 2r + 1/\zeta)}{(\zeta - r)(1/\zeta - r)}. \tag{5.14}$$

Similar arguments can be used to obtain the time dependence of the norming constants for the discrete eigenvalues. Namely, if z_k is a discrete eigenvalue, from (2.21) one has

$$\frac{\partial}{\partial\tau}\phi_n(z_k,\tau) = \dot{b}_k\psi_n(z_k,\tau) + b_k\frac{\partial}{\partial\tau}\psi_n(z_k,\tau).$$

Recalling (2.2b) we also have

$$\mathbf{M}_n(z_k,\tau)\phi_n(z_k,\tau) - \mathrm{i}\omega_\infty^{(1)}(z_k)\phi_n(z_k)$$
$$= \dot{b}_k\psi_n(z_k,\tau) + b_k\mathbf{M}_n(z_k,\tau)\psi_n(z_k,\tau) - \mathrm{i}b_k\omega_\infty^{(2)}(z_k)\psi_n(z_k,\tau). \tag{5.15}$$

(Note that the operator $\mathbf{M}_n$ depends on z.) Therefore we have $\dot{b}_k = \mathrm{i}\big[\omega_\infty^{(2)}(z_k) - \omega_\infty^{(1)}(z_k)\big]b_k$, which yields

$$b_k(\tau) = b_k(0)\,\mathrm{e}^{2\mathrm{i}Q_o^2\tau + \mathrm{i}\mu(z_k)\tau}, \tag{5.16a}$$

where $\mu(z)$ is again given by (5.13). Similarly, one obtains

$$\bar{b}_k(\tau) = \bar{b}_k(0)\,\mathrm{e}^{-2\mathrm{i}Q_o^2\tau - \mathrm{i}\mu(\bar{z}_k)\tau}. \tag{5.16b}$$

The time evolution of the modified scattering matrix $\mathbf{S}(\zeta,\tau)$ is the same as $\mathbf{T}(z,\tau)$ (cf (2.33)).

Next, we show how to obtain an infinity of conserved quantities for (1.4). Recall that the scattering coefficient $a(\zeta)$ is time independent. Since $a(\zeta)$ is analytic outside the unit circle $|\zeta| = 1$ and tends to 1 as $\zeta \to \infty$, it admits a Laurent series expansion whose coefficients are the constant of the motion as well. Substituting the expansions for the eigenfunctions (cf (A.5)) into the Wronskian representation (2.35) it follows that the Laurent expansion for the function $r(\zeta + 1/\zeta - 2r)a(\zeta)$ is given by

$$r(\zeta + 1/\zeta - 2r)a(\zeta) = -\Delta_n\left\{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{M_n^{(1),k}N_n^{(2),j}}{\zeta^{k+j}} - \sum_{j=0}^{\infty}\sum_{k=-1}^{\infty}\frac{M_n^{(2),k}N_n^{(1),j}}{\zeta^{k+j}}\right\}.$$

Renaming the indices yields

$$r(\zeta + 1/\zeta - 2r)a(\zeta) = -\Delta_n\left\{\sum_{\ell=0}^{\infty}\frac{1}{\zeta_\ell}\sum_{j=0}^{\ell}M_n^{(1),\ell-j}N_n^{(2),j} - \sum_{\ell=-1}^{\infty}\frac{1}{\zeta_\ell}\sum_{j=0}^{\ell+1}M_n^{(2),\ell-j}N_n^{(1),j}\right\}$$

and therefore for any $\ell \in \mathbb{N}$, the $1/\zeta^\ell$ coefficient of the Laurent series expansion for $r(\zeta + 1/\zeta - 2r)a(\zeta)$ is given by

$$I_\ell = \Delta_n\left[\sum_{j=0}^{\ell}\left(M_n^{(1),\ell-j}N_n^{(2),j} - M_n^{(2),\ell-j}N_n^{(1),j}\right) - M_n^{(2),-1}N_n^{(1),\ell+1}\right]. \tag{5.17}$$

Since $a(\zeta)$ is constant in time, (5.17) are an infinite set of conserved quantities for the IDNLS equation for $\ell = -1, 0, 1, \ldots$. The first few of them are

$$I_{-1} = \Delta_n N_n^{(1),0}M_n^{(2),-1}$$
$$I_0 = \Delta_n\big[-N_n^{(1),0}M_n^{(2),0} + M_n^{(1),0}N_n^{(2),0} - N_n^{(1),1}M_n^{(2),-1}\big]$$
$$\ldots$$

etc. Substituting the explicit expressions for the coefficients of the Laurent series expansion of the eigenfunctions derived in the appendix yields

$$I_{-1} = r, \qquad I_0 = -\sum_{n=-\infty}^{\infty} \left[R_n Q_{n-1} - Q_o^2\right], \qquad \ldots .$$

A second set of conserved quantities can be obtained in a similar way from the coefficients of the Laurent expansion for $\bar{a}(\zeta)$, i.e.

$$J_\ell = \Delta_n \left[\sum_{j=0}^{\ell} \left(\bar{M}_n^{(2),\ell-j} \bar{N}_n^{(1),j} - \bar{M}_n^{(1),\ell-j} \bar{N}_n^{(2),j}\right) - \bar{M}_n^{(1),-1} \bar{N}_n^{(2),\ell+1}\right]. \tag{5.18}$$

Recalling symmetries (2.67) and (2.75), these conserved quantities can be written as

$$J_{-1} = r, \qquad J_0 = -\sum_{n=-\infty}^{\infty} \left[Q_n R_{n-1} - Q_o^2\right], \qquad \ldots .$$

Also, by taking into account the τ-dependence of the scattering coefficients (2.17), it follows that the determinant (2.19) of the scattering matrix $\mathbf{T}(z)$ is a constant of the motion as well, that is

$$c_\infty(\tau) = \prod_{j=-\infty}^{\infty} \frac{1 - Q_j(\tau) R_j(\tau)}{1 - Q_o^2} \equiv \prod_{j=-\infty}^{\infty} \frac{1 - Q_j(0) R_j(0)}{1 - Q_o^2}. \tag{5.19}$$

The system of equations (1.4) is a Hamiltonian system, with coordinates $Q_n(\tau)$ and momenta $R_n(\tau)$ respectively. The Hamiltonian is given by (cf [16, 22])

$$H = \sum_{n=-\infty}^{\infty} \left[R_n(Q_{n+1} + Q_{n-1}) - 2Q_o^2\right] + 2\log\left[1 - \left(R_n Q_n - Q_o^2\right)\right]. \tag{5.20}$$

Finally, note that motion constants are also given in terms of the scattering data by the trace formula (2.83). In fact, recalling that $a(\zeta)$ and $\bar{a}(\zeta)$, as well as their zeros $\zeta_k, \bar{\zeta}_k$ (discrete eigenvalues) are time independent, the coefficients of the expansions of, say, $\bar{a}_(\zeta)$ as $\zeta \to 0$ in (2.83)

$$K_n = \oint_{|w|=1} \frac{\log[1 - |\lambda(w)|^2 |\bar{\rho}(w)|^2]}{w^n} \, \mathrm{d}w, \qquad n \in \mathbb{Z} \tag{5.21}$$

provide an infinite set of conserved quantities, assuming all of these integrals are convergent.

6. One-soliton solution

Pure soliton solutions are obtained when the scattering data comprise proper eigenvalues and the reflection coefficients vanish identically on the unit circle $|\zeta| = 1$. As is well known, in this situation the algebraic-integral system that linearizes the inverse problem reduces to a purely algebraic system of equations and can be solved exactly.

Let us consider the linear system (3.11) and (3.12a) for the case of one-soliton (reflectionless, and with just one pair of eigenvalues $\bar{\zeta}_1, \zeta_1 = 1/\bar{\zeta}_1^*$ and associated norming constants $\bar{C}_1, C_1$ respectively)

$$N_n^{(1)}(\zeta_1) + A_n \bar{N}_n^{(1)}(\bar{\zeta}_1) = 1/\zeta_1 - r \equiv r - \bar{\zeta}_1^*, \tag{6.1a}$$

$$N_n^{(2)}(\zeta_1) + A_n \bar{N}_n^{(2)}(\bar{\zeta}_1) + R_+/\Delta_n = 0, \tag{6.1b}$$

$$\bar{N}_n^{(1)}(\bar{\zeta}_1) + B_n N_n^{(1)}(\zeta_1) - Q_+/\Delta_n = 0, \tag{6.1c}$$

$$\bar{N}_n^{(2)}(\bar{\zeta}_1) + B_n N_n^{(2)}(\zeta_1) = \bar{\zeta}_1 - r, \tag{6.1d}$$

$$1/\Delta_n + C_n \bar{N}_n^{(1)}(\bar{\zeta}_1) = 1, \tag{6.1e}$$

and, from (3.12*b*),

$$R_n = R_+ - D_n \Delta_n \bar{N}_n^{(2)}(\bar{\zeta}_1), \tag{6.2}$$

where for brevity we have defined the functions

$$A_n = \frac{\zeta_1 - 1/r}{(\bar{\zeta}_1 - \zeta_1)(\bar{\zeta}_1 - 1/r)} \bar{C}_1(\lambda^2(\bar{\zeta}_1))^n \equiv \frac{r - \bar{\zeta}_1^*}{r(|\bar{\zeta}_1|^2 - 1)(\bar{\zeta}_1 - 1/r)} \bar{C}_1(\lambda^2(\bar{\zeta}_1))^n, \tag{6.3a}$$

$$B_n = \frac{\bar{\zeta}_1 - r}{(\zeta_1 - \bar{\zeta}_1)(\zeta_1 - r)} C_1(\lambda^2(\zeta_1))^{-n} \equiv \frac{(\bar{\zeta}_1^*)^2(\bar{\zeta}_1 - r)}{r(1 - |\bar{\zeta}_1|^2)(1/r - \bar{\zeta}_1^*)} C_1(\lambda^2(\bar{\zeta}_1))^n, \tag{6.3b}$$

$$C_n = -\frac{1/r}{\bar{\zeta}_1 - 1/r} \bar{C}_1(\lambda^2(\bar{\zeta}_1))^n \equiv \frac{|\bar{\zeta}_1|^2 - 1}{\bar{\zeta}_1^* - r} A_n, \tag{6.3c}$$

$$D_n = \frac{1}{\bar{\zeta}_1 - 1/r} \bar{C}_1(\lambda^2(\bar{\zeta}_1))^n \equiv -r \frac{|\bar{\zeta}_1|^2 - 1}{\bar{\zeta}_1^* - r} A_n. \tag{6.3d}$$

Note that in (6.3*b*) we have used the symmetries $\lambda(1/z^*) = (1/\lambda(z))^*$ and $\zeta_1 = 1/\bar{\zeta}_1^*$ ($|\zeta_1| > 1$), and the fact that both $\lambda^2(\zeta_1)$ and $\lambda^2(\bar{\zeta}_1)$ are real, to conclude that $\lambda^2(\zeta_1) = 1/\lambda^2(\bar{\zeta}_1)$. Indeed, note that $\lambda^2(\bar{\zeta}_1) = \bar{\zeta}_1(\bar{\zeta}_1 - r)(\bar{\zeta}_1^* - 1/r)/[r|\bar{\zeta}_1 - 1/r|^2]$. Therefore $\operatorname{Im}\lambda^2(\bar{\zeta}_1) = (|\bar{\zeta}_1|^2 + 1 - 2\operatorname{Re}\bar{\zeta}_1/r)\operatorname{Im}\bar{\zeta}_1$. The term in brackets is identically zero, since $\bar{\zeta}_1$ is on the circle of center $1/r$ and radius Q_o/r. We conclude that both $\lambda^2(\bar{\zeta}_1)$ and $\lambda^2(\zeta_1)$ appearing in (6.3) are real.

One can solve the linear algebraic system (6.1), obtain $1/\Delta_n$ and $\bar{N}_n^{(2)}(\bar{\zeta}_1)$, and then substitute into (6.2) to reconstruct the potential. The coefficients of the linear system are all expressed in terms of A_n and B_n. Also, note that from (2.24) it follows

$$\frac{\lambda^2(\zeta_1)}{\lambda^2(\bar{\zeta}_1)} = 1/\lambda^4(\bar{\zeta}_1) = \frac{r^2}{|\bar{\zeta}_1|^2} \frac{|\bar{\zeta}_1 - 1/r|^2}{|\bar{\zeta}_1 - r|^2} \equiv \frac{Q_o^2}{|\bar{\zeta}_1|^2|\bar{\zeta}_1 - r|^2}, \tag{6.4}$$

where we used that $\bar{\zeta}_1$ is on the circle on center $1/r$ and radius Q_o/r (cf (2.76) and figure 3). Consequently, from the symmetry (3.4) in the norming constants it follows that

$$A_n B_n = \frac{|C_1|^2|\bar{\zeta}_1|^2}{(|\bar{\zeta}_1|^2 - 1)^2} \left(\frac{|\bar{\zeta}_1|^2}{Q_o^2}|\bar{\zeta}_1 - r|^2\right)^{n+1}. \tag{6.5}$$

Note that (6.5) implies that $A_n B_n > 0\ \forall n \in \mathbb{Z}$. From the above linear system (6.1) one then obtains

$$1/\Delta_n = \frac{1 - |\bar{\zeta}_1|^2 A_n B_n}{1 + A_n Q_+(|\bar{\zeta}_1|^2 - 1)/(\bar{\zeta}_1^* - r) - A_n B_n} \tag{6.6a}$$

and

$$\bar{N}_n^{(2)}(\bar{\zeta}_1) = \frac{(\bar{\zeta}_1 - r)(1 + Q_+ C_n - A_n B_n) + R_+ B_n(1 + |\bar{\zeta}_1|^2 A_n B_n)}{(1 - A_n B_n)(1 + Q_+ C_n - A_n B_n)}. \tag{6.6b}$$

Then, as shown in the appendix, by defining $A_n B_n = x_n$, from (6.2) we obtain the reconstruction of the potential as

$$R_n = R_+ \left[1 + \frac{r}{Q_o}(1 - |\bar{\zeta}_1|^2)\, e^{i\phi_1} \frac{x_n}{1 + |\bar{\zeta}_1| x_n}\right], \tag{6.7}$$

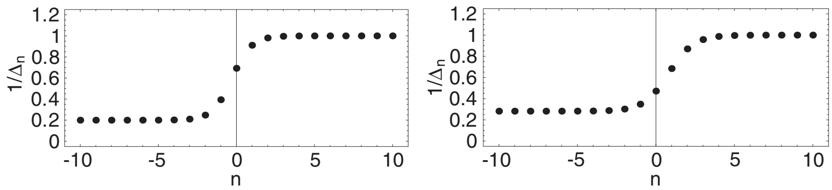

Figure 4. The value of $1/\Delta_n$ as a function of n resulting from two different choices of discrete eigenvalue. Left: $\bar{\zeta}_1 = (1 - Q_o)/r$, with $Q_o = 2/3, \theta_+ = 0$ and norming constant $\bar{C}_1 = 1/2$. Right: $\bar{\zeta}_1 = (1 - Q_o\,\mathrm{e}^{\mathrm{i}\pi/12})/r$, with $Q_o = 2/3, \theta_+ = 0$ and $\bar{C}_1 = 2\,\mathrm{e}^{\mathrm{i}\pi/12}$.

where $\phi_1 = \arg(\bar{\zeta}_1 - r)$, and x_n is a positive quantity for all $n \in \mathbb{Z}$, given by

$$x_n = |\bar{C}_1|\frac{|\bar{\zeta}_1|}{1 - |\bar{\zeta}_1|^2}(\lambda^2(\bar{\zeta}_1))^n = \mathrm{d}\lambda^{2n}(\bar{\zeta}_1) > 0, \qquad d = |\bar{C}_1\bar{\zeta}_1|/(1 - |\bar{\zeta}_1|^2).$$

Then, we can write

$$R_n = R_+\left[1 + \frac{a\lambda^{2n}(\bar{\zeta}_1)}{1 + d|\bar{\zeta}_1|\lambda^{2n}(\bar{\zeta}_1)}\right], \qquad a = r\,\mathrm{e}^{\mathrm{i}\phi_1}|\bar{C}_1\bar{\zeta}_1|/Q_o. \tag{6.8}$$

Note that since $|\lambda(\bar{\zeta}_1)| < 1$, then $R_n \to R_+$ as $n \to +\infty$, and as $n \to -\infty$ one has $R_n \to R_+a/(d|\bar{\zeta}_1|) \equiv R_-$, where the last identity follows from (2.84).

The time dependence in (6.7) is completely determined by that of R_+ and $|\bar{C}_1|$, for which we have, respectively, $R_+(\tau) = R_+(0)\,\mathrm{e}^{-2\mathrm{i}Q_o^2\tau}$ and

$$|\bar{C}_1(\tau)| = |\bar{C}_1(0)|\exp[\mathrm{Im}\,\mu(\bar{\zeta}_1)\tau], \tag{6.9}$$

where $\mu(\zeta)$ is given by (5.14). It is illuminating to look at the time dependence in the original variables z and λ. According to (3.4a) and (5.16), it is given by

$$\bar{C}_k(\tau) = \bar{C}_k(0)\exp\left[-2\mathrm{i}Q_o^2\tau - \mathrm{i}\mu(\bar{z}_k)\tau\right],$$

and

$$\mu(\bar{z}_k) = r(\bar{z}_k - 1/\bar{z}_k)(\lambda(\bar{z}_k) - 1/\lambda(\bar{z}_k)).$$

Now we can use the fact that points of the discrete spectrum are such that $|\bar{z}_k| = 1$ and $|\mathrm{Re}\,\bar{z}_k| > r$, and that for such points

$$\xi(\bar{z}_k) = (\bar{z}_k + 1/\bar{z}_k)/2r = (\bar{z}_k + \bar{z}_k^*)/2r \equiv \mathrm{Re}\,\bar{z}_k/r > 1,$$

and therefore

$$\lambda(\bar{z}_k) = \xi(\bar{z}_k) - \sqrt{\xi(\bar{z}_k)^2 - 1} \in \mathbb{R}.$$

Note also that

$$\lambda(\bar{z}_k) - 1/\lambda(\bar{z}_k) = -2\sqrt{\xi(\bar{z}_k)^2 - 1},$$

and therefore we have

$$\mu(\bar{z}_k) = -2r(\bar{z}_k - 1/\bar{z}_k)\sqrt{(\mathrm{Re}\,\bar{z}_k)^2/r^2 - 1} = -4\mathrm{i}\,\mathrm{Im}\,\bar{z}_k\sqrt{(\mathrm{Re}\,\bar{z}_k)^2 - r^2}$$

(and it is actually purely imaginary), so that finally

$$\bar{C}_k(\tau) = \bar{C}_k(0)\exp\left[-2\mathrm{i}Q_o^2\tau - 4\tau\,\mathrm{Im}\,\bar{z}_k\sqrt{(\mathrm{Re}\,\bar{z}_k)^2 - r^2}\right].$$

The value of $1/\Delta_n$ as a function n is shown in figure 4 for two different choices of discrete eigenvalue, which produce respectively to a stationary (black) dark soliton and a moving (gray) dark soliton. The shape of the two corresponding soliton solutions is shown in figures 5 and 6.

It is worth mentioning that the Casorati determinant form of dark solitons was recently obtained in [24].

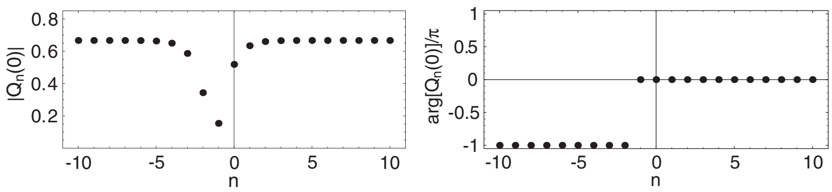

Figure 5. The amplitude (left) and argument (right) of the black dark-soliton solution generated by the choice of discrete eigenvalue and norming constant in the left part of figure 4.

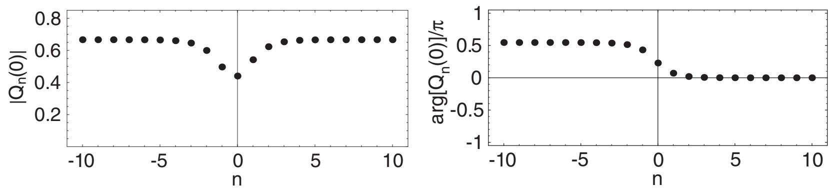

Figure 6. The amplitude (left) and argument (right) of the grey dark-soliton solution generated by the choice of discrete eigenvalue and norming constant in the right part of figure 4.

7. Small amplitude limit and linearization

Unlike the case with decaying boundary conditions, the term 'small amplitude' here does *not* refer to the solution itself, but rather to the difference between the solution and the uniform background. Here we discuss the small-amplitude limit of the IST and we compare it to the solution obtained by directly linearizing the IDNLS equation (1.4) around a uniform background.

7.1. Small amplitude limit from the inverse problem

To obtain the small-amplitude limit of the IST, consider the equations of the inverse problem (3.11) in the absence of solitons, namely

$$\bar{N}_n(\zeta) = \begin{pmatrix} Q_+/\Delta_n \\ \zeta - r \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\zeta - r)(1/\lambda^2(w))^n}{(w-\zeta)(w-r)} N_n(w)\rho(w)\,\mathrm{d}w,$$

$$N_n(\zeta) = \begin{pmatrix} r - 1/\zeta \\ -R_+/\Delta_n \end{pmatrix} + \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\zeta - 1/r)(\lambda^2(w))^n}{(w-\zeta)(w-1/r)} \bar{N}_n(w)\bar{\rho}(w)\,\mathrm{d}w,$$

$$\frac{1}{\Delta_n} = 1 - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\lambda^2(w))^n}{rw-1} \bar{N}_n^{(1)}(w)\bar{\rho}(w)\,\mathrm{d}w,$$

$$R_n = R_+ + \frac{r\Delta_n}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\lambda^2(w))^n}{rw-1} \bar{N}_n^{(2)}(w)\bar{\rho}(w)\,\mathrm{d}w.$$

We can solve these equations iteratively:

$$\bar{N}_n(\zeta) = \begin{pmatrix} Q_+/\Delta_n \\ \zeta - r \end{pmatrix} - \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \frac{(\zeta - r)(1/\lambda^2(w))^n}{(w-\zeta)(w-r)} \left\{ \begin{pmatrix} r - 1/w \\ -R_+/\Delta_n \end{pmatrix} + \frac{1}{2\pi \mathrm{i}} \int_{|w'|=1} \frac{(w - 1/r)(\lambda^2(w'))^n}{(w'-w)(w'-1/r)} \begin{pmatrix} Q_+/\Delta_n \\ w' - r \end{pmatrix} \bar{\rho}(w')\,\mathrm{d}w' + \cdots \right\} \rho(w)\,\mathrm{d}w,$$

$$N_n(\zeta) = \begin{pmatrix} r - 1/\zeta \\ -R_+/\Delta_n \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \frac{(\zeta - 1/r)(\lambda^2(w))^n}{(w-\zeta)(w-1/r)} \left\{ \begin{pmatrix} Q_+/\Delta_n \\ w - r \end{pmatrix} \right.$$
$$\left. - \frac{1}{2\pi\mathrm{i}} \int_{|w'|=1} \frac{(w-r)(1/\lambda^2(w'))^n}{(w'-w)(w'-r)} \begin{pmatrix} r - 1/w' \\ -R_+/\Delta_n \end{pmatrix} \rho(w')\,\mathrm{d}w' + \cdots \right\} \bar\rho(w)\,\mathrm{d}w,$$

$$\frac{1}{\Delta_n} = 1 - \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \frac{(\lambda^2(w))^n}{rw-1}$$
$$\times \left[\frac{Q_+}{\Delta_n} - \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \frac{(w-r)(1/\lambda^2(w'))^n}{(w'-w)(w'-r)} (r - 1/w')\rho(w')\,\mathrm{d}w' + \cdots \right] \bar\rho(w)\,\mathrm{d}w,$$

$$R_n = R_+ + \frac{\Delta_n r}{2\pi\mathrm{i}} \int_{|w|=1} \frac{(\lambda^2(w))^n}{rw-1}$$
$$\times \left[(w-r) + \frac{1}{2\pi\mathrm{i}} \frac{R_+}{\Delta_n} \int_{|w'|=1} \frac{(w-r)(1/\lambda^2(w'))^n}{(w'-w)(w'-r)} \rho(w')\,\mathrm{d}w' + \cdots \right] \bar\rho(w)\,\mathrm{d}w.$$

In particular, at leading order in $\rho(\zeta)$ and $\bar\rho(\zeta)$, one has

$$\Delta_n \sim 1 + \frac{Q_+}{2\pi\mathrm{i}} \int_{|w|=1} \bar\rho(w) \frac{(\lambda^2(w))^n}{rw-1}\,\mathrm{d}w$$

and

$$R_n \sim R_+ + \frac{r}{2\pi\mathrm{i}} \int_{|w|=1} \bar\rho(w) \frac{(\lambda^2(w))^n}{rw-1} (w-r)\,\mathrm{d}w.$$

If we explicitly introduce the time dependence as follows from (5.2) and (5.12), this gives

$$R_n(\tau) \sim \mathrm{e}^{-2\mathrm{i}Q_o^2\tau} \left[R_+(0) + \frac{r}{2\pi\mathrm{i}} \int_{|w|=1} \bar\rho(w,0)\,\mathrm{e}^{-\mathrm{i}\mu(w)\tau} \frac{(\lambda^2(w))^n}{rw-1} (w-r)\,\mathrm{d}w \right], \tag{7.1}$$

where $\mu(w)$ is given by (5.13). Note that in (7.1) λ^2 plays the role of a periodic Fourier variable, which suggests that one should perform a change of variable and express the dispersion relation in terms of λ as well. From $r(\lambda + 1/\lambda) = z + 1/z$ it follows that $z = \eta \pm \sqrt{\eta^2 - 1}$ and $1/z = \eta \mp \sqrt{\eta^2 - 1}$, where $\eta = r(\lambda + 1/\lambda)/2$ and

$$z - 1/z = \pm\sqrt{r^2(\lambda + 1/\lambda)^2 - 4}. \tag{7.2}$$

Therefore, we can write the dispersion relation as

$$\mu(\lambda) = \pm r(\lambda - 1/\lambda)\sqrt{r^2(\lambda + 1/\lambda)^2 - 4}. \tag{7.3}$$

From equation (2.24) we have

$$\lambda^2(w) = w\frac{w-r}{rw-1} \tag{7.4}$$

and therefore we can rewrite (7.1) as follows:

$$R_n(\tau) \sim \mathrm{e}^{-2\mathrm{i}Q_o^2\tau} \left[R_+(0) + \frac{r}{2\pi\mathrm{i}} \int_{|w|=1} (\lambda^2(w))^{n+1} \bar\rho(w,0)\,\mathrm{e}^{-\mathrm{i}\mu(w)\tau} \frac{\mathrm{d}w}{w} \right]. \tag{7.5}$$

Moreover, recall that $w = \lambda(w)/z(w)$, and therefore

$$\frac{\mathrm{d}w}{w} = \frac{z}{\lambda} \frac{z - \lambda\,\mathrm{d}z/\mathrm{d}\lambda}{z^2}\,\mathrm{d}\lambda \equiv \left[\frac{1}{\lambda} - \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\lambda} \right] \mathrm{d}\lambda.$$

In order to obtain the expression of $\mathrm{d}z/\mathrm{d}\lambda$, we use $r(\lambda + 1/\lambda) = z + 1/z$, which gives

$$\frac{\mathrm{d}z}{\mathrm{d}\lambda} = r\frac{1 - 1/\lambda^2}{1 - 1/z^2},$$

and finally

$$\frac{\mathrm{d}w}{w} = \left[1 - r\frac{\lambda - 1/\lambda}{z - 1/z}\right]\frac{\mathrm{d}\lambda}{\lambda} \equiv \left[1 - \frac{\mu(\lambda)}{r^2(\lambda + 1/\lambda)^2 - 4}\right]\frac{\mathrm{d}\lambda}{\lambda},$$

where in the last formula we used (7.2). Substituting into (7.5) yields

$$R_n(\tau) \sim \mathrm{e}^{-2\mathrm{i}Q_o^2\tau}\left\{R_+(0) + \frac{r}{2\pi\mathrm{i}}\int_{|\lambda|=1}\left[1 - \frac{\mu(\lambda)}{r^2(\lambda + 1/\lambda)^2 - 4}\right](\lambda^2)^{n+1}\bar{\rho}(\lambda, 0)\,\mathrm{e}^{-\mathrm{i}\mu(\lambda)\tau}\frac{\mathrm{d}\lambda}{\lambda}\right\}, \tag{7.6}$$

where $\mu(\lambda)$ is given by (7.3). As far as the contour of integration is concerned, this follows from (7.4), knowing that w is on the unit circle. We already showed that $|w| = 1$ if and only if $|\lambda| = 1$, and in fact if $w = \mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta \leqslant 2\pi$ from (7.4) it follows

$$\lambda^2 = \mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{e}^{\mathrm{i}\theta} - r}{r\,\mathrm{e}^{\mathrm{i}\theta} - 1} \equiv -\mathrm{e}^{2\mathrm{i}\theta}\frac{q}{q^*}, \qquad q = 1 - r\,\mathrm{e}^{-\mathrm{i}\theta}$$

and then as θ spans the unit circle, so does λ. Finally, to obtain a more traditional Fourier representation, we perform one more change of variables and define $y = \lambda^2$. This leads to

$$R_n(\tau) \sim \mathrm{e}^{-2\mathrm{i}Q_o^2\tau}\left\{R_+(0) + \frac{r}{4\pi\mathrm{i}}\int_{|y|=1}\left[1 - \frac{\mu(y)}{r^2(y + 1/y + 2) - 4}\right]y^n\bar{\rho}(\sqrt{y}, 0)\,\mathrm{e}^{-\mathrm{i}\mu(y)\tau}\,\mathrm{d}y\right.$$
$$\left. + \frac{r}{4\pi\mathrm{i}}\int_{|y|=1}\left[1 + \frac{\mu(y)}{r^2(y + 1/y + 2) - 4}\right]y^n\bar{\rho}(-\sqrt{y}, 0)\,\mathrm{e}^{\mathrm{i}\mu(y)\tau}\mathrm{d}y\right\}, \tag{7.7}$$

where from (7.3) it follows

$$\mu^2(y) = r^2(y + 1/y - 2)[r^2(y + 1/y + 2) - 4]. \tag{7.8}$$

7.2. Linearization and small-amplitude limit via discrete Fourier transform

We now consider the solution of the linearized IDNLS equation. Recall that $Q_n \to Q_\pm = Q_o\,\mathrm{e}^{\mathrm{i}\theta_\pm(\tau)}$ as $n \to \pm\infty$, with $\theta_\pm(\tau) = \theta_\pm(0) + 2\mathrm{i}Q_o^2\tau$. We then consider the 'normalized' discrete NLS equation for the rescaled field $\tilde{Q}_n = Q_n\,\mathrm{e}^{-2\mathrm{i}Q_o^2\tau}$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{Q}_n = (\tilde{Q}_{n+1} + \tilde{Q}_{n-1})(1 - |\tilde{Q}_n|^2) - 2(1 - Q_o^2)\tilde{Q}_n$$

and define

$$\tilde{Q}_n(\tau) = \mathrm{e}^{\mathrm{i}\theta_+(0)}(Q_o + u_n(\tau)).$$

If u_n is small (that is, $|u_n| \ll Q_o$), neglecting terms quadratic in u_n one obtains the following equation:

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}\tau} = r^2(u_{n+1} + u_{n-1} - 2u_n) - 2Q_o^2(u_n + u_n^*). \tag{7.9}$$

We then seek for solutions of (7.9) in the form

$$\hat{u}(y, \tau) = \sum_{n=-\infty}^{\infty} y^{-n}u_n(\tau), \qquad u_n(\tau) = \frac{1}{2\pi\mathrm{i}}\int_{|y|=1} y^{n-1}\hat{u}(y, \tau)\,\mathrm{d}y. \tag{7.10}$$

Note that

$$u_n^*(\tau) = \frac{1}{2\pi \mathrm{i}} \int_{|y|=1} y^{n-1} \hat{u}^*(1/y, \tau)\, \mathrm{d}y$$

and substituting into (7.9) yields the following coupled system of ordinary differential equations:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{u}(y,\tau) = r^2(y + 1/y - 2)\hat{u}(y,\tau) - 2Q_o^2(\hat{u}(y,\tau) + \hat{u}^*(1/y,\tau))$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{u}^*(1/y,\tau) = r^2(y + 1/y - 2)\hat{u}^*(1/y,\tau) - 2Q_o^2(\hat{u}(y,\tau) + \hat{u}^*(1/y,\tau))$$

or, calling $a = \hat{u}(y,\tau)$ and $b = \hat{u}^*(1/y,\tau)$,

$$\mathrm{i}\frac{\mathrm{d}a}{\mathrm{d}\tau} = r^2(y + 1/y - 2)a - 2Q_o^2(a + b) \tag{7.11a}$$

$$-\mathrm{i}\frac{\mathrm{d}b}{\mathrm{d}\tau} = r^2(y + 1/y - 2)b - 2Q_o^2(a + b) \tag{7.11b}$$

with the constraint $a(y,\tau) = b^*(1/y,\tau)$. The above system implies

$$\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} = -r^2(y + 1/y - 2)[r^2(y + 1/y + 2) - 4]a,$$

whose general solution is given by

$$a(y,\tau) = A_1(y)\,\mathrm{e}^{\mathrm{i}\mu(y)\tau} + A_2(y)\,\mathrm{e}^{-\mathrm{i}\mu(y)\tau}, \tag{7.12}$$

where

$$\mu(y) = r\sqrt{2 - y + 1/y}\sqrt{4 - r^2(y + 1/y + 2)}, \tag{7.13}$$

and $A_1(y)$ and $A_2(y)$ are arbitrary functions. Similarly,

$$b(y,\tau) = B_1(y)\,\mathrm{e}^{-\mathrm{i}\mu(y)\tau} + B_2(y)\,\mathrm{e}^{\mathrm{i}\mu(y)\tau} \tag{7.14}$$

and from the symmetry $a(y,\tau) = b^*(1/y,\tau)$ it follows[5]

$$B_j^*(1/y) = A_j(y), \qquad j = 1, 2, \tag{7.15}$$

where we have used that

$$\mu^*(1/y) = \mu(y). \tag{7.16}$$

In order for (7.12) and (7.14) to satisfy system (7.11), $A_1(y)$ and $A_2(y)$ must satisfy a symmetry condition. In fact, if we substitute (7.12) and (7.14) into (7.11) and make use of symmetry (7.15), we obtain

$$A_2^*(1/y) = \frac{r^2(y + 1/y - 2) + \mu(y) - 2Q_o^2}{2Q_o^2} A_1(y). \tag{7.17}$$

Then, we can determine the arbitrary functions $A_1(y)$ and $A_2(y)$ in terms of the initial data, as the following linear system:

$$A_1(y) + A_2(y) = \sum_{n=-\infty}^{\infty} y^{-n} u_n(0), \tag{7.18a}$$

[5] With this choice of signs in (7.13), since $|y| = 1$ the argument of each square root is real and positive, and therefore $\mu^*(y) = \mu(y)$; moreover, μ is symmetric for the exchange $y \to 1/y$.

$$A_1^*(1/y) + A_2^*(1/y) = \sum_{n=-\infty}^{\infty} y^{-n} u_n^*(0), \tag{7.18b}$$

which can be solved using (7.17), obtaining

$$A_1(y) = \frac{2Q_o^4}{\mu(y)\left[r^2(y+1/y-2)+\mu(y)-2Q_o^2\right]} \times \left[\sum_{n=-\infty}^{\infty} y^{-n} u_n(0) + \frac{r^2(y+1/y-2)+\mu(y)-2Q_o^2}{2Q_o^2} \sum_{n=-\infty}^{\infty} y^{-n} u_n^*(0)\right]. \tag{7.19}$$

From (7.12) and the second part of (7.10) we then obtain

$$u_n(\tau) = \frac{1}{2\pi\mathrm{i}} \int_{|y|=1} y^{n-1} [A_1(y)\,\mathrm{e}^{\mathrm{i}\mu(y)\tau} + A_2(y)\,\mathrm{e}^{-\mathrm{i}\mu(y)\tau}]\,\mathrm{d}y, \tag{7.20}$$

and consequently

$$R_n(\tau) = \mathrm{e}^{-2\mathrm{i}Q_o^2\tau}\left\{R_+(0) - \frac{\mathrm{e}^{-\mathrm{i}\theta_+(0)}}{2\pi\mathrm{i}} \int_{|y|=1} y^{n-1}[A_1^*(1/y)\,\mathrm{e}^{-\mathrm{i}\mu(y)\tau} + A_2^*(1/y)\,\mathrm{e}^{\mathrm{i}\mu(y)\tau}]\,\mathrm{d}y\right\}. \tag{7.21}$$

We now compare this solution with the expression obtained from the small-amplitude limit of the inverse problem. Comparing (7.21) with (7.7) yields

$$A_1^*(1/y) = -\frac{\mathrm{e}^{\mathrm{i}\theta_+(0)}r}{2}\left[1 - \frac{\mu(y)}{r^2(y+1/y+2)-4}\right] y\bar{\rho}(\sqrt{y}, 0), \tag{7.22a}$$

$$A_2^*(1/y) = -\frac{\mathrm{e}^{\mathrm{i}\theta_+(0)}r}{2}\left[1 + \frac{\mu(y)}{r^2(y+1/y+2)-4}\right] y\bar{\rho}(-\sqrt{y}, 0). \tag{7.22b}$$

As we show in the appendix, the symmetry (2.73c) for the reflection coefficients corresponds to (7.17) for the functions $A_1(y)$ and $A_2(y)$ for any value of y on the unit circle.

8. Continuum limit

It is instructive to see how the solution of the discrete problem tends to the solution of the continuous case in the limit $h \to 0$, where h is the lattice spacing. To study the correspondence between the discrete to the continuous case, recall that

$$Q_n = q_n h, \qquad Q_o = hq_0, \qquad r^2 = 1 - h^2 q_0^2, \qquad z = \mathrm{e}^{\mathrm{i}kh}. \tag{8.1}$$

Then $k = -\mathrm{i}[\log|z| + \mathrm{i}\arg z]/h$, and real values of k in the interval $[-\pi/h, \pi/h]$ span the entire unit circle for z. In particular, the portion of the unit circle with $|\mathrm{Re}\, z| < r \equiv \sqrt{1-h^2q_0^2}$, corresponding to the continuous spectrum (see below), is mapped into two disjoint segments of the interval $[-\pi/h, \pi/h]$, namely, $[-\tilde{k}_0, -k_0] \cup [k_0, \tilde{k}_0]$ (cf figure 7), where

$$k_0 = \frac{1}{h}\arctan\frac{hq_0}{\sqrt{1-h^2q_0^2}}, \qquad \tilde{k}_0 = \frac{\pi}{h} - k_0.$$

When $h \to 0$, one has $k_0 \to q_0$ and $\tilde{k}_0 \to \infty$ so in the continuous limit one is left with only the two branch points $\pm q_0$ instead of the four $\pm z_0$ and $\pm z_0^*$. Moreover,

$$r = \sqrt{1-h^2q_0^2} \sim 1 - \frac{h^2q_0^2}{2} + O(h^4) \qquad z \sim 1 + \mathrm{i}kh + O(h^2). \tag{8.2}$$

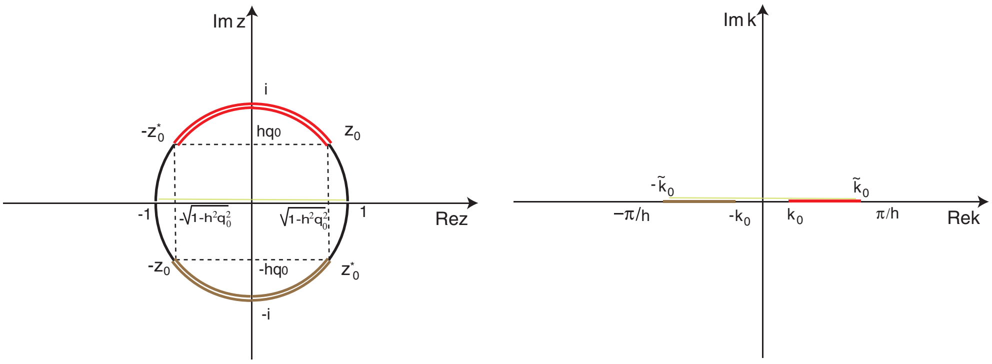

Figure 7. The transformation for the spectral parameter $z = \mathrm{e}^{\mathrm{i}kh}$. Note that $k_0 = \frac{1}{h}\arctan\frac{hq_0}{\sqrt{1-h^2q_0^2}}$ and $\tilde{k}_0 = \frac{\pi}{h} - \frac{1}{h}\arctan\frac{hq_0}{\sqrt{1-h^2q_0^2}}$. The continuous case is obtained in the limit $h \to 0$.

(This figure is in colour only in the electronic version)

Thus, from (2.10) we obtain

$$\lambda = \frac{\cos(kh)}{\sqrt{1-h^2q_0^2}} \pm \sqrt{\frac{\cos^2(kh)}{1-h^2q_0^2} - 1} \sim 1 \pm \mathrm{i}h\sqrt{k^2-q_0^2} + O(h^2), \tag{8.3}$$

where the signs $\pm$ correspond to the upper/lower sheet of the Riemann surface, and the exterior/interior of the unit circle when λ is expressed in terms of the uniformization variable ζ. From the definition of ζ (2.23) it then follows

$$\zeta = \frac{\lambda}{z} = \mathrm{e}^{-\mathrm{i}kh}\left[\frac{\cos(kh)}{\sqrt{1-h^2q_0^2}} \pm \sqrt{\frac{\cos^2(kh)}{1-h^2q_0^2} - 1}\right] \sim 1 + \mathrm{i}h\big[\pm\sqrt{k^2-q_0^2} - k\big] + O(h^2). \tag{8.4}$$

Here, however, it is not obvious what the role of the signs $\pm$ is, apart from the fact that if we choose k_j to be the real value of the continuous spectral parameter corresponding to a discrete eigenvalue ζ_j, then we expect $|k_j| < q_0$ and the two values of $\zeta_j = \zeta(k_j)$ correspond to ζ_j and $\bar{\zeta}_j$, with the first one outside the unit circle, and the second one inside. Therefore, if we define, in analogy with the continuous case,

$$\nu_j = \sqrt{q_0^2 - k_j^2} > 0, \qquad -q_0 < k_j < q_0, \tag{8.5}$$

one has

$$\bar{\zeta}_j = 1 - \mathrm{i}h(k_j - \mathrm{i}\nu_j) + O(h^2), \qquad \zeta_j = 1 - \mathrm{i}h(k_j + \mathrm{i}\nu_j) + O(h^2) \tag{8.6}$$

such that

$$|\bar{\zeta}_j|^2 = 1 - 2h\nu_j + O(h^2), \qquad |\bar{\zeta}_j| = 1 - h\nu_j + O(h^2), \tag{8.7}$$

whereas

$$|\zeta_j|^2 = 1 + 2h\nu_j + O(h^2), \qquad |\zeta_j| = 1 + h\nu_j + O(h^2). \tag{8.8}$$

Note also that from (A.22) it follows that $\tan\phi_1 = \tan\psi_1 + O(h)$ and by the way the angles are defined, one expects

$$\phi_1 = \pi - \psi_1 + O(h). \tag{8.9}$$

Moreover, $\bar{\zeta}_1 - 1/r = Q_o/r\, \mathrm{e}^{\mathrm{i}\psi_1}$, which at order h gives

$$-\mathrm{i}(k_1 - \mathrm{i}\nu_1) = q_0\, \mathrm{e}^{\mathrm{i}\psi_1}, \tag{8.10}$$

i.e., since $k_1^2 + \nu_1^2 = q_0$,

$$\psi_1 = -\frac{\pi}{2} - \arg\alpha_1 \quad \leftrightarrow \quad \phi_1 = \arg\alpha_1 - \pi/2, \tag{8.11}$$

where $\alpha_1 = k_1 + \mathrm{i}\nu_1$. Note also that we have

$$\lambda^2(\bar{\zeta}_j) \sim 1 - 2h\nu_j \qquad \text{(real and smaller than 1)} \tag{8.12}$$

and $|\bar{\zeta}_j|/(1 - |\bar{\zeta}_j|^2) \sim 1/2h\nu_j$. Therefore, 'rescaling' the norming constant as follows:

$$\bar{C}_1 = h\bar{c}_1 \tag{8.13}$$

and writing from (8.12) $\lambda^2(\bar{\zeta}_j) \sim \mathrm{e}^{-2h\nu_j}$, we obtain

$$x_n \sim \frac{|\bar{c}_1|}{2\nu_j}\,\mathrm{e}^{-2\nu_j hn} \to \frac{|\bar{c}_1|}{2\nu_j}\,\mathrm{e}^{-2\nu_j x} \qquad \text{as} \quad h \to 0, \quad nh \to x. \tag{8.14}$$

Then from (A.20) it is straightforward to obtain

$$\Delta_n \to 1 \qquad \text{as} \quad h \to 0 \tag{8.15}$$

and (6.7) gives the solution corresponding to (1.2).

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Appendix

Here we present several relations which are useful in the development of the direct and inverse scattering transforms and we provide the proof of several claims in the text.

Evaluation of $|\lambda|$. The evaluation of $|\lambda|$ as given in (2.10) is an important issue. Let us first consider the case of real ξ. If $\xi \in \mathbb{R}$ and $|\xi| > 1$, then $\lambda \in \mathbb{R}$ and either $|\lambda| > 1$ or $|\lambda| < 1$ (cf figure 1.) On the other hand, if $\xi \in \mathbb{R}$ and $|\xi| < 1$, then $\lambda = \xi \pm \mathrm{i}\sqrt{1-\xi^2}$ and $|\lambda| = 1$. Now, note that in terms of the original variable $z = x + \mathrm{i}y$:

$$\mathrm{Re}\,\xi = \frac{x}{2r}\left(1 + \frac{1}{x^2+y^2}\right), \qquad \mathrm{Im}\,\xi = \frac{y}{2r}\left(1 - \frac{1}{x^2+y^2}\right)$$

and $\xi \in \mathbb{R}$ corresponds to either $y \equiv \mathrm{Im}\,z = 0$ or $|z| = 1$. If $\mathrm{Im}\,z = 0$, then the condition $-1 < \xi < 1$ corresponds to $-2r < x + 1/x < 2r$, which, since $r^2 - 1 < 0$, is never satisfied. Therefore, we are left with $|z| = 1$ and $-2r < 2x < 2r$, i.e. $|\mathrm{Re}\,z| < r$. Hence, all points on the circle $|z| = 1$ with $|\mathrm{Re}\,z| < r$, are such that $-1 < \xi < 1$ and therefore $|\lambda| = 1$. It is possible to show that this condition is also necessary, i.e. that $|\lambda| = 1 \to \xi \in \mathbb{R}$ and $-1 < \xi < 1$. Note that z can also be expressed in terms of ξ as follows:

$$z = r\big(\xi \pm \sqrt{\xi^2 - 1/r^2}\big), \qquad 1/z = r\big(\xi \mp \sqrt{\xi^2 - 1/r^2}\big), \tag{A.1}$$

with branch points at $\xi = \pm 1/r$.

Proof of remark 1. We intend to show that $(\lambda r - z)^2 + Q_o^2 = 0$ iff $z = \pm z_0$ or $z = \pm z_0^*$. This can be done more easily in terms of the uniform variable ζ. Taking into account (2.8) and (2.23) one has

$$(\lambda r - z)^2 + Q_o^2 = -(\lambda r - z)(1/\lambda r - 1/z) + Q_o^2 = -2r^2 + r(\lambda/z + z/\lambda) \equiv r(\zeta + 1/\zeta - 2r) \tag{A.2}$$

and, since we proved that the continuous spectrum in the ζ-plane is mapped onto the (punctured) unit circle, we see that for $|\zeta| = 1$ one has $(\lambda(\zeta)r - z(\zeta))^2 + Q_o^2 = 0$ if and only if $\zeta + \zeta^* = 2r$, i.e. if and only if $\zeta = r \pm \mathrm{i}Q_o$. This shows that the Wronskians (2.15) vanish only at ζ_0, ζ_0^*, which are image, in the ζ-plane, of the branch points $z = \pm z_0, \pm z_0^*$.

Location of the discrete eigenvalues in the z-plane. Consider a discrete eigenvalue z_k and suppose $\phi_n(z_k) = \left(\phi_n^{(1)}(z_k)\phi_n^{(2)}(z_k)\right)$ is a bound state, i.e. it decays fast as $|n| \to \infty$. One can easily show from the scattering problem that

$$\left|\phi_{n+1}^{(1)}(z_k)\right|^2 - \left|\phi_{n+1}^{(2)}(z_k)\right|^2 = (1 - Q_n R_n)\left[\left|\phi_n^{(1)}(z_k)\right|^2 - \left|\phi_n^{(2)}(z_k)\right|^2\right] + (|z_k|^2 - 1)\left[\left|\phi_{n+1}^{(2)}(z_k)\right|^2 + (1 - Q_n R_n)\left|\phi_n^{(1)}(z_k)\right|^2\right]. \tag{A.3}$$

From the definition of Δ_n in (2.38) it follows that $1 - Q_n R_n = \left(1 - Q_o^2\right)\Delta_n/\Delta_{n+1} \equiv r^2\Delta_n/\Delta_{n+1}$, and multiplying both members of (A.3) by $r^{-2n-2}\Delta_{n+1}$ it follows that

$$r^{-2n-2}\Delta_{n+1}\left[\left|\phi_{n+1}^{(1)}(z_k)\right|^2 - \left|\phi_{n+1}^{(2)}(z_k)\right|^2\right] = r^{-2n}\Delta_n\left[\left|\phi_n^{(1)}(z_k)\right|^2 - \left|\phi_n^{(2)}(z_k)\right|^2\right] + (|z_k|^2 - 1)\left[r^{-2n-2}\Delta_{n+1}\left|\phi_{n+1}^{(2)}(z_k)\right|^2 + r^{-2n}\Delta_n\left|\phi_n^{(1)}(z_k)\right|^2\right].$$

Summing over all n one obtains

$$(|z_k|^2 - 1)\sum_{n=-\infty}^{+\infty}\frac{\Delta_n}{r^{2n}}\left[\left|\phi_n^{(1)}(z_k)\right|^2 + \left|\phi_n^{(2)}(z_k)\right|^2\right] = 0.$$

Noting that the expression in brackets is strictly positive, as well as Δ_n, we conclude that $|z_k| = 1$.

Proof of remark 2. From (2.24) it follows $|\lambda|^2 \leqslant 1$ iff $|r\zeta - 1| \geqslant |\zeta||\zeta - r|$, i.e.

$$|\lambda|^2 \leqslant 1 \quad\Leftrightarrow\quad (|\zeta|^2 - 1)(|\zeta|^2 + 1 - r(\zeta + \zeta^*)) \leqslant 0$$

or equivalently

$$|\lambda|^2 \leqslant 1 \quad\Leftrightarrow\quad (|\zeta|^2 - 1)\left(|\zeta - r|^2 + Q_o^2\right) \leqslant 0$$

and we conclude that (2.25*a*) holds. Similarly, again from (2.24) it follows

$$|z|^2 \geqslant 1 \quad\Leftrightarrow\quad |\zeta - r| \geqslant |\zeta||r\zeta - 1|.$$

Squaring both sides one obtains

$$|z|^2 \geqslant 1 \quad\Leftrightarrow\quad (|\zeta|^2 - 1)[|\zeta|^2 + 1 - (\zeta + \zeta^*)/r] \leqslant 0.$$

Note further that $r^2 = 1 - Q_o^2$ and therefore the term in square bracket is $|\zeta|^2 + 1 - 1/r(\zeta + \zeta^*) = |\zeta - 1/r|^2 - Q_o^2/r^2$. We therefore conclude that the sign of $|z|^2 - 1$ depends on the sheet of the complex z-plane and on whether the point is inside or outside the larger circle $|\zeta - 1/r| = Q_o/r$. Thus (2.25*b*) holds.

Relevant values of the mapping (z, λ) *to* ζ. Recalling that $\xi = (z + 1/z)/2r$ and $\zeta = \lambda/z$, we can calculate explicitly the image of various distinguished points in the z-plane. The results are shown in table 1.

Note that since $0 < r, Q_o < 1$, one has $0 < 1/r - Q_o/r < 1$ and $1 < 1/r < 1/r + Q_o/r < 2/r$ and therefore all points on sheet I are mapped into points that are outside the unit circle $|\zeta| = 1$, and all points on sheet II are mapped inside. Note also that (2.24) shows that any given value of ζ is in general the image of two different points on either one or the other z-plane.

Green's functions. To find the explicit expression for Green's functions introduced in section 2.3, let us write them in terms of the discrete Fourier transform, i.e.

$$\mathbf{G}_n^{\pm}(\zeta) = \frac{1}{2\pi\mathrm{i}}\oint_{|p|=1} p^{n-1}\hat{\mathbf{G}}^{\pm}(p)\,\mathrm{d}p$$

Table 1. Distinguished points in both sheets of the complex z-plane and their images.

z	ξ	λ	ζ
		Sheet I: $\lambda = \xi + \sqrt{\xi^2 - 1}$.	
$z_0 = r + \mathrm{i}Q_o$	1	1	$z_0^* = \zeta_0^*$
i	0	$\pm\mathrm{i}$ (above/below the cut)	± 1
$-z_0^* = -r + \mathrm{i}Q_o$	-1	-1	$z_0 = \zeta_0$
-1	$-1/r$	$-1/r - Q_o/r$	$1/r + Q_o/r$
$-z_0 = -r - \mathrm{i}Q_o$	-1	-1	$z_0^* = \zeta_0^*$
-i	0	$\pm\mathrm{i}$ (above/below the cut)	∓ 1
$z_0^* = r - \mathrm{i}Q_o$	1	1	$z_0 = \zeta_0$
1	$1/r$	$1/r + Q_o/r$	$1/r + Q_o/r$
0	$\sim 1/2zr$	$\sim 2\xi$	∞
∞	$\sim z/2r$	$\sim 2\xi$	$1/r$
		Sheet II: $\lambda = \xi - \sqrt{\xi^2 - 1}$.	
$z_0 = r + \mathrm{i}Q_o$	1	1	$z_0^* = \zeta_0^*$
i	0	$\mp\mathrm{i}$ (above/below the cut)	∓ 1
$-z_0^* = -r + \mathrm{i}Q_o$	1	-1	$z_0 = \zeta_0$
-1	$-1/r$	$-1/r + Q_o/r$	$1/r - Q_o/r$
$-z_0 = -r - \mathrm{i}Q_o$	-1	-1	$z_0^* = \zeta_0^*$
$-\mathrm{i}$	0	$\mp\mathrm{i}$ (above/below the cut)	± 1
$z_0^* = r - \mathrm{i}Q_o$	1	1	$z_0 = \zeta_0$
1	$1/r$	$1/r - Q_o/r$	$1/r - Q_o/r$
$z = 0$	$\sim 1/2zr$	$\sim\xi\left[1 - \left(1 - \frac{1}{2}\xi^{-2} + \cdots\right)\right]$	r
∞	$\sim z/2r$	$\sim\xi\left[1 - \left(1 - \frac{1}{2}\xi^{-2} + \cdots\right)\right]$	0

which then satisfy, according to (2.44),

$$\frac{1}{2\pi\mathrm{i}}\oint_{|p|=1} p^{n-1}\left[p\hat{\mathbf{G}}^{\pm}(p) - \frac{1}{r}\begin{pmatrix} z/\lambda & Q_{\pm}/\lambda^2 \\ R_{\pm} & 1/(z\lambda)\end{pmatrix}\hat{\mathbf{G}}^{\pm}(p)\right]\mathrm{d}p$$
$$= \frac{1}{r}\begin{pmatrix} 1/\lambda^2 & 0 \\ 0 & 1\end{pmatrix}\frac{1}{2\pi\mathrm{i}}\oint_{|p|=1} p^{n-1}\,\mathrm{d}p.$$

Therefore one has

$$\hat{\mathbf{G}}^{\pm}(p) = \frac{1}{r(p-1)(p-1/\lambda^2)}\begin{pmatrix} [p - 1/(rz\lambda)]/\lambda^2 & Q_{\pm}/(r\lambda^2) \\ R_{\pm}/(r\lambda^2) & p - z/(r\lambda), \end{pmatrix}$$

and consequently

$$\mathbf{G}_n^{\pm}(\zeta) = \frac{1}{2\pi\mathrm{i}}\oint_{|p|=1} p^{n-1}\frac{1}{r(p-1)(p-1/\lambda^2)}\begin{pmatrix} [p - 1/(rz\lambda)]/\lambda^2 & Q_{\pm}/(r\lambda^2) \\ R_{\pm}/(r\lambda^2) & p - z/(r\lambda)\end{pmatrix}.$$

The integrals above depend only whether the poles $1, 1/\lambda^2$ are located inside or outside the contour of integration. However, when $|\lambda| = 1$, both poles are on the contour and one has to consider contours that are perturbed away from $|p| = 1$ to avoid the singularities. In particular, for the upper sign we consider a contour C^{out} enclosing $p = 0$ and $p = 1, 1/\lambda^2$ and for the lower sign a contour C^{in} enclosing $p = 0$ but neither $p = 1$ nor $p = 1/\lambda^2$. The residue theorem gives

$$\frac{1}{2\pi\mathrm{i}}\oint_{C^{\mathrm{out}}}\frac{f(p)}{p - b_j}p^{n-1}\,\mathrm{d}p = \begin{cases}(b_j)^{n-1}f(b_j) & n \geqslant 1 \\ 0 & n \leqslant 0\end{cases}$$

and

$$\frac{1}{2\pi \mathrm{i}} \oint_{C^{\mathrm{in}}} \frac{f(p)}{p - b_j} p^{n-1}\, \mathrm{d}p = \begin{cases} 0 & n \geqslant 1 \\ -(b_j)^{n-1} f(b_j) & n \leqslant 0 \end{cases}$$

for $b_1 = 1, b_2 = 1/\lambda^2$ for any function $f(p)$ which is regular at b_j, we obtain two Green's functions

$$\mathbf{G}_n^{\mathrm{out}}(\zeta) = \theta(n-1) \frac{1}{r(1 - 1/\lambda^2)} \left\{ \begin{pmatrix} [1 - 1/(rz\lambda)]/\lambda^2 & Q_-/(r\lambda^2) \\ R_-/(r\lambda^2) & 1 - z/(r\lambda) \end{pmatrix} - \lambda^{-2(n-1)} \begin{pmatrix} [1/\lambda^2 - 1/(rz\lambda)]/\lambda^2 & Q_-/(r\lambda^2) \\ R_-/(r\lambda^2) & 1/\lambda^2 - z/(r\lambda) \end{pmatrix} \right\}$$

$$\mathbf{G}_n^{\mathrm{in}}(\zeta) = -\theta(-n) \frac{1}{r(1 - 1/\lambda^2)} \left\{ \begin{pmatrix} [1 - 1/(rz\lambda)]/\lambda^2 & Q_+/(r\lambda^2) \\ R_+/(r\lambda^2) & 1 - z/(r\lambda) \end{pmatrix} - \lambda^{-2(n-1)} \begin{pmatrix} [\lambda^{-2} - 1/(rz\lambda)]/\lambda^2 & Q_+/(r\lambda^2) \\ R_+/(r\lambda^2) & 1/\lambda^2 - z/(r\lambda) \end{pmatrix} \right\}.$$

Next, using (2.24) we finally obtain (2.45).

Neumann series. Consider the summation equation (2.62) for $\tilde{M}_n(\zeta)$, which we write in the form

$$\tilde{M}_n(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=-\infty}^{n-1} (\mathbf{DW})_j(\zeta) \tilde{M}_j(\zeta),$$

where $\mathbf{D}_j = \mathrm{diag}(1, \lambda(\zeta)^{2(j+1-n)})$ and $\mathbf{W}_n(\zeta)$ is the energy-dependent potential matrix defined in (2.58). A solution of the above equation can be sought for in the form of a Neumann series

$$\tilde{M}_n(\zeta) = \sum_{k=0}^{\infty} \gamma_n^{(k)}(\zeta),$$

where

$$\gamma_n^{(0)}(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \gamma_n^{(k+1)}(\zeta) = \sum_{j=-\infty}^{n-1} (\mathbf{DW})_j(\zeta) \gamma_j^{(k)}(\zeta) \qquad k \geqslant 0.$$

If the potentials f_n, g_n, h_n in (2.60) are ℓ_1, one can establish a bound on the $\gamma_n^{(k)}$ such that the series representation converges absolutely and uniformly in n and uniformly in ζ in the region $|\zeta| \geqslant 1$. In fact, we prove by induction on k that for $|\zeta| \geqslant 1$:

$$\left\| \gamma_n^{(k)}(\zeta) \right\| \leqslant \sum_{j=-\infty}^{n-1} \frac{\|\mathbf{W}_j(\zeta)\|^k}{k!} \leqslant \sum_{j=-\infty}^{n-1} \frac{\|\mathbf{W}_j\|^k}{k!}, \tag{A.4}$$

where $\|\cdot\|$ denotes any matrix norm and $\|\mathbf{W}_j\|$ results from bounding each element of the energy-dependent matrix potential $\mathbf{W}_j(\zeta)$ with respect to ζ, and hence its norm. Note that the entries of $\mathbf{W}_j$ then only depends on the functions f_n, g_n, h_n, which are summable by assumption. Recall that for $|\zeta| \geqslant 1$ one has $|\lambda(\zeta)|^{2(j+1-n)} \leqslant 1$ for any $j \leqslant n$ and therefore $\|\mathbf{D}_j(\zeta)\| \leqslant 1$ as well. Using the recursive definition for $\gamma_n^{(k)}$ we then obtain

$$\begin{aligned} \left\| \gamma_n^{(k+1)}(\zeta) \right\| &\leqslant \sum_{j=-\infty}^{n-1} \|\mathbf{D}_j(\zeta)\| \|\mathbf{W}_j(\zeta)\| \left\| \gamma_j^{(k)}(\zeta) \right\| \\ &\leqslant \sum_{j=-\infty}^{n-1} \|\mathbf{W}_j\| \sum_{m=-\infty}^{j-1} \frac{\|\mathbf{W}_m\|^k}{k!} \leqslant \sum_{j=-\infty}^{n-1} \frac{\|\mathbf{W}_j\|^{k+1}}{(k+1)!} \end{aligned}$$

and in the last inequality a summation by parts formula (cf, for instance, [16]) was used. The bounds in (A.4) are absolutely and uniformly summable in k if $\|\mathbf{W}\|_1 \equiv \sum_{j=-\infty}^{\infty} \|\mathbf{W}_j\| < \infty$ and this completes the proof.

WKB expansion for the eigenfunctions. The coefficients of the Laurent series for the eigenfunctions can be obtained by means of a WKB expansion. Let us write the large-ζ expansion of the eigenfunctions $M_n(\zeta)$ and $N_n(\zeta)$ as

$$M_n^{(1)}(\zeta) = \sum_{k=0}^{\infty} M_n^{(1),k}/\zeta^k, \qquad M_n^{(2)}(\zeta) = \sum_{k=-1}^{\infty} M_n^{(2),k}/\zeta^k \tag{A.5a}$$

$$N_n^{(1)}(\zeta) = \sum_{k=0}^{\infty} N_n^{(1),k}/\zeta^k, \qquad N_n^{(2)}(\zeta) = \sum_{k=0}^{\infty} N_n^{(2),k}/\zeta^k. \tag{A.5b}$$

Note that from (2.24) it follows

$$1/\lambda^2 = r/\zeta - Q_o^2 \sum_{j=0}^{\infty} r^j/\zeta^{j+2}, \qquad \lambda^2 = \frac{\zeta}{r} + Q_o^2 \sum_{j=0}^{\infty} 1/(r^{j+2}\zeta^j) \tag{A.6a}$$

$$1/(\lambda z) = r - Q_o^2 \sum_{j=0}^{\infty} r^j/\zeta^{j+1}, \qquad \lambda z = \frac{1}{r} + Q_o^2 \sum_{j=1}^{\infty} 1/(r^{j+1}\zeta^j). \tag{A.6b}$$

As a consequence, substituting (A.5) and (A.6) into the scattering problem (2.31) we obtain

$$r\sum_{k=0}^{\infty} M_{n+1}^{(1),k}/\zeta^k = \sum_{k=1}^{\infty} M_n^{(1),k-1}/\zeta^k + rQ_n \sum_{k=0}^{\infty} M_n^{(2),k-1}/\zeta^k - Q_o^2 Q_n \sum_{k=1}^{\infty} 1/\zeta^k \sum_{j=0}^{k-1} r^j M_n^{(2),k-j-2} \tag{A.7a}$$

$$r\sum_{k=-1}^{\infty} M_{n+1}^{(2),k}/\zeta^k = R_n \sum_{k=0}^{\infty} M_n^{(1),k}/\zeta^k + r\sum_{k=-1}^{\infty} M_n^{(2),k}/\zeta^k - Q_o^2 \sum_{k=0}^{\infty} 1/\zeta^k \sum_{j=0}^{k} r^j M_n^{(2),k-j-1}. \tag{A.7b}$$

Equating the coefficients of the different powers of ζ gives a coupled set of difference equations. In particular, $k = -1$ yields

$$M_n^{(2),-1} = 1 \tag{A.8a}$$

(taking into account the boundary conditions) and $k = 0$ in (A.7*a*) gives

$$M_n^{(1),0} = Q_{n-1}, \tag{A.8b}$$

i.e., the first of (2.63*a*). Then the equations (A.7) can be solved iteratively, anchoring the iteration from (A.8). In fact, one has

$$r\Delta M_n^{(2),k} = R_n M_n^{(1),k} - Q_o^2 \sum_{j=0}^{k} r^j M_n^{(2),k-j-1} \qquad k = 0, 1, \ldots$$

$$rM_{n+1}^{(1),k} = M_n^{(1),k-1} + rQ_n M_n^{(2),k-1} - Q_o^2 Q_n \sum_{j=0}^{k-1} r^j M_n^{(2),k-j-2} \qquad k = 1, 2, \ldots,$$

where Δ denotes the shift operator, i.e. $\Delta f_n = f_{n+1} - f_n$. Then, we can write

$$rM_n^{(2),k} = \sum_{j=-\infty}^{n-1} \left[R_j M_j^{(1),k} - R_- M_{-\infty}^{(1),k}\right] - Q_o^2 \sum_{\ell=0}^{k} r^\ell \sum_{j=-\infty}^{n-1} \left[M_j^{(2),k-\ell-1} - M_{-\infty}^{(2),k-\ell-1}\right] \qquad k = 0, 1, \ldots \tag{A.9a}$$

$$rM_{n+1}^{(1),k} = M_n^{(1),k-1} + rQ_n M_n^{(2),k-1} - Q_o^2 Q_n \sum_{j=0}^{k-1} r^j M_n^{(2),k-j-2} \qquad k = 1, 2, \ldots, \tag{A.9b}$$

where the terms in square brackets are subtracted out so that the corresponding series are convergent. For example, we can easily obtain

$$rM_n^{(2),0} = \sum_{j=-\infty}^{n-1} \left[R_j Q_{j-1} - Q_o^2\right], \qquad rM_{n+1}^{(1),1} = Q_{n-1} + Q_n \sum_{j=-\infty}^{n-1} \left[R_j Q_{j-1} - Q_o^2\right] - Q_o^2 Q_n \tag{A.10}$$

and so on. Similarly, substituting (A.5) and (A.6) into the scattering problem (2.31*b*) for $N_n(\zeta)$, at leading order yields

$$R_n N_n^{(1),0} = -rN_n^{(2),0} \qquad r^2 N_{n+1}^{(1),0} = N_n^{(1),0} + rQ_n N_n^{(2),0}.$$

Substituting the first one into the second one, we then get the difference equation

$$N_{n+1}^{(1),0} = \frac{1 - Q_n R_n}{r^2} N_n^{(1),0},$$

whose solution can be written as

$$N_n^{(1),0} = \alpha/\Delta_n,$$

where Δ_n is defined by (2.38) and α is an arbitrary constant. If the limits $n \to +\infty$ and $\zeta \to \infty$ commute, we expect $N_n^{(1),0} \sim r$ as $n \to +\infty$, which, since $1/\Delta_n \to 1$ as $n \to +\infty$, fixes $\alpha = r$. Then we obtain the second of (2.63*a*). For the other coefficients of the Laurent expansion (A.5*b*) for $k \geqslant 1$ one has the equations

$$r^2 N_{n+1}^{(1),k} = N_n^{(1),k} + Q_o^2 \sum_{j=0}^{k-1} N_n^{(1),j} r^{-k+j} + rQ_n N_n^{(2),k}$$

$$r^2 N_{n+1}^{(2),k-1} = R_n N_n^{(1),k} + Q_o^2 R_n \sum_{j=0}^{k-1} N_n^{(1),j} r^{-k+j} + rN_n^{(2),k}.$$

We multiply the second equation for Q_n and subtract it from the first one, thus obtaining

$$r^2 N_{n+1}^{(1),k} = Q_n r^2 N_{n+1}^{(2),k-1} + (1 - Q_n R_n) N_n^{(1),k} + Q_o^2 \sum_{j=0}^{k-1} N_n^{(1),j} r^{-k+j} (1 - Q_n R_n)$$

$$rN_n^{(2),k} = r^2 N_{n+1}^{(2),k-1} - R_n N_n^{(1),k} - Q_o^2 R_n \sum_{j=0}^{k-1} N_n^{(1),j} r^{-k+j-1}$$

which can be solved iteratively. Let us introduce for all $k \geqslant 1$ the functions $\varphi_n^{(j),k} = \Delta_n N_n^{(j),k}$. Taking into account the definition of Δ_n, we see that $(1 - Q_n R_n)\Delta_{n+1} = r^2 \Delta_n$ and therefore multiplying the first equation by Δ_{n+1} yields

$$\Delta\varphi_n^{(1),k} = Q_n \varphi_{n+1}^{(2),k-1} + Q_o^2 \sum_{j=0}^{k-1} r^{-k+j} \varphi_n^{(1),j} \qquad k = 1, 2, \ldots$$

which can be explicitly solved

$$\Delta_n N_n^{(1),k} = -\sum_{j=n}^{\infty}\left[Q_j \Delta_{j+1} N_{j+1}^{(2),k-1} - Q_+ N_\infty^{(2),k-1}\right] - Q_o^2 \sum_{\ell=0}^{k-1} r^{\ell-k} \sum_{j=n}^{\infty}\left[\Delta_j N_j^{(1),\ell} - N_\infty^{(1),\ell}\right], \tag{A.11a}$$

and together with

$$r N_n^{(2),k} = r^2 N_{n+1}^{(2),k-1} - R_n N_n^{(1),k} - Q_o^2 R_n \sum_{j=0}^{k-1} N_n^{(1),j} r^{-k+j-1} \tag{A.11b}$$

complete the recursion. Similarly, from the behavior of (2.31a) as $\zeta \to 0$ one obtains (2.63b).

We now discuss the asymptotic behavior of eigenfunctions and scattering data at points r and $1/r$. The scattering problem (2.31b) for $\bar{M}_n$ about $\zeta \to r$ becomes

$$r\bar{M}_{n+1}(\zeta) \sim \begin{pmatrix} -(\zeta-r)\big/Q_o^2 & Q_n \\ -r(\zeta-r)R_n\big/Q_o^2 & r \end{pmatrix} \bar{M}_n(\zeta) \qquad \zeta \to r$$

and if we assume

$$\bar{M}_n(\zeta) \sim \begin{pmatrix} \bar{M}_n^{(1),0} + (\zeta-r)\bar{M}_n^{(1),1} + \cdots \\ \bar{M}_n^{(2),0} + (\zeta-r)\bar{M}_n^{(2),1} + \cdots \end{pmatrix}$$

we obtain the first of (2.64), where the choice of the constant is compatible with the limit as $n \to -\infty$ corresponding to $\left(-Q_o^2/r, -R_-\right)^T \equiv (r - 1/r, -R_-)^T$. The same expansion as $\zeta \to r$ for $\bar{N}_n(\zeta)$ yields the second of (2.64). The expansions for $M_n(\zeta)$ and $N_n(\zeta)$ about $\zeta = 1/r$ are obtained in a similar way.

Proof of remark 3. Let us first prove part (a). From remark 1 it follows that if $z \in \mathcal{C}$, then $\xi \in]-1, 1[$ and therefore $\lambda = \xi \pm \mathrm{i}\sqrt{1-\xi^2}$. Then $|\lambda| = 1$, and hence $|\zeta| = |\lambda|/|z| = 1$. Conversely, take ζ such that $|\zeta| = 1$; then from (2.24) it follows that $|z(\zeta)|^4 = 1$. Moreover, from remark 2 one also has $|\lambda(\zeta)| = 1$, and therefore one has $-1 < \xi < 1$ which, recalling the definition of $\xi = (z + 1/z)/2r$, and since $|z| = 1 \Leftrightarrow 1/z = z^*$, implies that $-r < \operatorname{Re} z < r$, i.e. $z \in \mathcal{C}$.

We now prove part (b). Suppose that $z \in \mathcal{D}$. Then $\xi \in]-1/r, -1[\cup]1, 1/r[$ on either sheet, and consequently $\lambda = \xi \pm \sqrt{\xi^2 - 1} \in \mathbb{R}$. The claim is that such z is mapped onto the circle $|\zeta - 1/r| = Q_o/r$ in the ζ-plane, i.e. onto the locus

$$|\zeta|^2 - 2\operatorname{Re}\zeta/r = -1. \tag{A.12a}$$

Observe that for $z \in \mathcal{D}$ one has $|\zeta|^2 = |\lambda|^2 = \lambda^2$ and $\operatorname{Re}\zeta = \operatorname{Re}(\lambda z^*)/|z|^2 \equiv \lambda \operatorname{Re} z$ and therefore, in terms of λ, z, the locus (A.12a) becomes

$$\lambda(\lambda - 2\operatorname{Re} z/r) = -1. \tag{A.12b}$$

Taking into account (A.1) and recalling that for $z \in \mathcal{D}$ it is $\xi \in]-1/r, -1[\cup]1, 1/r[$, it follows that $\operatorname{Re} z = r\xi$ and

$$\lambda - 2\operatorname{Re} z/r = -\left(\xi \mp \sqrt{\xi^2 - 1}\right)$$

hence, recalling (2.10), one can check that (A.12b) is indeed satisfied. Conversely, if one takes an arbitrary ζ on the circle $|\zeta - 1/r| = Q_o/r$, then $|\zeta|^2 = 2\operatorname{Re}\zeta/r - 1$ and from (2.24) it follows that $|z(\zeta)|^4 = 1$. Then, taking into account the result of (a), one necessarily has $z \in \mathcal{D}$.

Derivation of (2.82). Since the diagonal terms of $\mathbf{T}$ vanish when evaluated at a discrete eigenvalue, equations (2.81) can be written as

$$\frac{\bar{a}'(\bar{\zeta}_k)}{\bar{b}(\bar{\zeta}_k)} = \frac{1}{z}\frac{dz}{d\zeta}\bigg|_{\zeta=\bar{\zeta}_k} \sum_{j=-\infty}^{\infty} \left(\mathbf{\Psi}_{j+1}^{-1}\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{\Psi}_j\right)_{21}(\bar{\zeta}_k) \tag{A.13a}$$

$$\frac{a'(\zeta_k)}{b(\zeta_k)} = \frac{1}{z}\frac{dz}{d\zeta}\bigg|_{\zeta=\zeta_k} \sum_{j=-\infty}^{\infty} \left(\mathbf{\Psi}_{j+1}^{-1}\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{\Psi}_j\right)_{12}(\zeta_k), \tag{A.13b}$$

where we used that $T_{12}(\bar{\zeta}_k) = \bar{b}(\bar{\zeta}_k)$ and $T_{21}(\zeta_k) = b(\zeta_k)$. Now note that from (2.24) it follows

$$\frac{1}{z}\frac{dz}{d\zeta} = -\frac{r}{2}\frac{\zeta^2 - 2r\zeta + 1}{\zeta(\zeta - r)(r\zeta - 1)}$$

and, from (2.15*b*) and (2.38),

$$\mathbf{\Psi}_j^{-1} = \frac{1}{W(\bar{\psi}_j, \psi_j)}\begin{pmatrix} \psi_j^{(2)} & -\psi_j^{(1)} \\ -\bar{\psi}_j^{(2)} & \bar{\psi}_j^{(1)} \end{pmatrix} \equiv -\frac{\Delta_j \zeta r^{-2j}}{r(\zeta^2 - 2r\zeta + 1)}\begin{pmatrix} \psi_j^{(2)} & -\psi_j^{(1)} \\ -\bar{\psi}_j^{(2)} & \bar{\psi}_j^{(1)} \end{pmatrix}.$$

Therefore, calculating $\mathbf{\Psi}_{j+1}^{-1}\mathbf{Z}\boldsymbol{\sigma}_3\mathbf{\Psi}_j$ and substituting into (A.13) we obtain

$$\frac{\bar{a}'(\bar{\zeta}_k)}{\bar{b}(\bar{\zeta}_k)} = -\frac{1}{2}\frac{1}{(\bar{\zeta}_k - r)(r\bar{\zeta}_k - 1)} \sum_{j=-\infty}^{\infty} r^{-2j}\Delta_j \times \left[z(\bar{\zeta}_k)\bar{\psi}_{j-1}^{(1)}(\bar{\zeta}_k)\bar{\psi}_j^{(2)}(\bar{\zeta}_k) + \bar{\psi}_j^{(1)}(\bar{\zeta}_k)\bar{\psi}_{j-1}^{(2)}(\bar{\zeta}_k)/z(\bar{\zeta}_k)\right] \tag{A.14a}$$

$$\frac{a'(\zeta_k)}{b(\zeta_k)} = \frac{1}{2}\frac{1}{(\zeta_k - r)(r\zeta_k - 1)} \sum_{j=-\infty}^{\infty} r^{-2j}\Delta_j \times \left[z(\zeta_k)\psi_j^{(2)}(\zeta_k)\psi_{j-1}^{(1)}(\zeta_k) + \psi_j^{(1)}(\zeta_k)\psi_{j-1}^{(2)}(\zeta_k)/z(\zeta_k)\right]. \tag{A.14b}$$

Now observe that since the discrete eigenvalues in the z-plane are located on the unit circle, in correspondence of any z_k it is $z_k^* = 1/z_k$ and therefore the discrete eigenfunctions satisfy an additional symmetry:

$$\psi_n(\zeta_k) = -\frac{r\lambda(\zeta_k) - z(\zeta_k)}{Q_+}\sigma_1\psi_n^*(\zeta_k) \tag{A.15a}$$

$$\bar{\psi}_n(\bar{\zeta}_k) = \frac{r\lambda(\bar{\zeta}_k) - z(\bar{\zeta}_k)}{R_+}\sigma_1\bar{\psi}_n^*(\bar{\zeta}_k) \tag{A.15b}$$

which allows us to express the second component of the discrete eigenfunctions in (A.14) in terms of the first one and obtain (2.82).

Reconstruction of the one-soliton solution (6.7). Let us introduce the short-hand notation $A_n B_n = x_n^2$ for (6.5), where

$$x_n = \frac{|\bar{C}_1|}{Q_o}\frac{|\bar{\zeta}_1 - r|}{1 - |\bar{\zeta}_1|^2}(\lambda^2(\bar{\zeta}_1))^n.$$

Note that $x_n \geqslant 0$, since we showed that $\lambda^2(\bar{\zeta}_1) \in \mathbb{R}$ and $|\bar{\zeta}_1| < 1$. The condition $|\bar{\zeta}_1 - 1/r| = Q_o/r$ is equivalent to

$$|\bar{\zeta}_1 - r| = Q_o|\bar{\zeta}_1| \tag{A.16}$$

and therefore we can also write

$$x_n = |\bar{C}_1|\frac{|\bar{\zeta}_1|}{1 - |\bar{\zeta}_1|^2}(\lambda^2(\bar{\zeta}_1))^n. \tag{A.17}$$

Note also that

$$Q_+C_n = Q_+\frac{|\bar{\zeta}_1|^2-1}{\bar{\zeta}_1^*-r}A_n \equiv -\bar{C}_1\,\mathrm{e}^{\mathrm{i}(\theta_+-\psi_1)}(\lambda^2(\bar{\zeta}_1))^n.$$

Then, since from (3.7) one has $\bar{C}_1 = \pm|\bar{C}_1|\,\mathrm{e}^{\mathrm{i}(\psi_1-\theta_+)}$, it follows that Q_+C_n is also real. Correspondingly, from (A.17) it follows that $Q_+C_n = \mp|\bar{C}_1|(\lambda^2(\bar{\zeta}_1))^n \equiv -\beta x_n$, where

$$\beta = (1-|\bar{\zeta}_1|^2)/|\bar{\zeta}_1|. \tag{A.18}$$

Note that $\beta > 0$, since $|\bar{\zeta}_1| < 1$. Combining these results, we can write

$$\Delta_n = \frac{1\mp\beta x_n - x_n^2}{1-|\bar{\zeta}_1|^2 x_n^2}. \tag{A.19}$$

Now note that $\lambda^2(\bar{\zeta}_1) < 1$ and therefore $x_n \to 0$ as $n \to +\infty$ and $x_n \to \infty$ as $n \to -\infty$, which means that the denominator becomes zero, and therefore Δ_n becomes singular, unless there is a cancelation with the numerator. In any case, one has $\Delta_n \to 1$ as $n \to +\infty$ and $\Delta_n \to 1/|\bar{\zeta}_1|^2 > 1$ as $n \to -\infty$. Let us factorize both numerator and denominator of (A.19)

$$\Delta_n = \frac{(\beta_1 - x_n)(\beta_2 + x_n)}{(1-|\bar{\zeta}_1|x_n)(1+|\bar{\zeta}_1|x_n)}.$$

Note that $\beta_1\beta_2 = 1$, which means that one of the two roots is greater than 1 and the other one is smaller than 1. Also, they have the same sign. Moreover, $\beta_1 - \beta_2 = \mp\beta \lessgtr 0$, so for the upper sign it is $\beta_2 > \beta_1$ and for the lower sign $\beta_1 > \beta_2$. Explicitly,

$$\Delta_n = \frac{(1/|\bar{\zeta}_1| \pm x_n)(|\bar{\zeta}_1| \mp x_n)}{(1-|\bar{\zeta}_1|x_n)(1+|\bar{\zeta}_1|x_n)} \equiv \frac{|\bar{\zeta}_1| \mp x_n}{|\bar{\zeta}_1|(1\mp|\bar{\zeta}_1|x_n)}. \tag{A.20}$$

Of these two expressions for Δ_n, the one corresponding to the upper sign is singular, while the one corresponding to the lower sign is regular for all n. Correspondingly, we obtain for the potential the two expressions

$$R_n = R_+\left[1 \mp \frac{r}{Q_o^2}\frac{1-|\bar{\zeta}_1|^2}{|\bar{\zeta}_1|}(\bar{\zeta}_1 - r)\frac{x_n}{1\mp|\bar{\zeta}_1|x_n}\right]. \tag{A.21}$$

Again, the expression corresponding to the lower (upper) sign is regular (singular). This can be further simplified by noting that, from (A.16), we also have

$$\frac{r}{Q_o^2}\frac{1-|\bar{\zeta}_1|^2}{|\bar{\zeta}_1|}(\bar{\zeta}_1 - r) \equiv \frac{r}{Q_o}(1-|\bar{\zeta}_1|^2)\,\mathrm{e}^{\mathrm{i}\phi_1},$$

where $\phi_1 = \arg(\bar{\zeta}_1 - r)$ and it can be expressed in terms of ψ_1 as follows:

$$\tan\phi_1 = \frac{\sin\psi_1}{Q_o + \cos\psi_1}. \tag{A.22}$$

We therefore finally obtain, for the expression corresponding to the lower sign, (6.7).

Equivalence of different representations of the linear limit. Here, we show that the symmetry (2.73*c*) for the reflection coefficients corresponds to (7.17) for the functions $A_1(y)$ and $A_2(y)$ for any value of y on the unit circle. Taking into account (2.73*c*), equations (7.17) and (7.22) yield

$$1 + \frac{\mu(y)}{r^2(y+1/y+2)-4} = -\frac{r^2(y+1/y-2)+\mu(y)-2Q_o^2}{2Q_o^2}\left[1 - \frac{\mu(y)}{r^2(y+1/y+2)-4}\right]. \tag{A.23}$$

Moreover, from definition (7.3) of $\mu^2(y)$ it follows that

$$\left[1+\frac{\mu(y)}{r^2(y+1/y+2)-4}\right]\left[1-\frac{\mu(y)}{r^2(y+1/y+2)-4}\right]=-\frac{4Q_o^2}{r^2(y+1/y+2)-4}$$

and therefore (A.23) is equivalent to

$$\left[1+\frac{\mu(y)}{r^2(y+1/y+2)-4}\right]^2=-2\frac{r^2(y+1/y-2)+\mu(y)-2Q_o^2}{r^2(y+1/y+2)-4},$$

which is identically satisfied.

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