Discrete Vector Solitons: Composite Solitons, Yang–Baxter Maps and Computation

By M. J. Ablowitz, B. Prinari, and A. D. Trubatch

Collisions of solitons for an integrable discretization of the coupled nonlinear Schrödinger equation are investigated. By a generalization of Manakov's well-known formulas for the polarization shift of interacting vector solitons, it is shown that the multisoliton interaction process is equivalent to a sequence pairwise interactions and, moreover, the net result of the interaction is independent of the order in which such collisions occur. Further, the order-invariance is shown to be related to the fact that the map that determines the interaction of two such solitons satisfies the Yang–Baxter relation. The associated matrix factorization problem is discussed in detail and the notion of fundamental and composite solitons is elucidated. Moreover, it is shown that, in analogy with the continuous case, collisions of fundamental solitons can be described by explicit fractional linear transformations of a complex-valued scalar polarization state. Because the parameters controlling the energy switching between the two components exhibit nontrivial information transformation, they can, in principle, be used to implement logic operations.

1. Introduction

In recent years, the vector nonlinear Schrödinger equation (VNLS)

$$i\mathbf{q}_t = \mathbf{q}_{xx} + 2\,\|\mathbf{q}\|^2\,\mathbf{q}, \tag{1}$$

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where $\mathbf{q}$ is a vector-valued function of $(x, t) \in \mathbb{R}^2$, has emerged as an important model for the propagation of pulses in nonlinear optical fibers and waveguides (cf., e.g., [15, 22]). Mathematically speaking, VNLS is interesting because it is an integrable system in the sense that it is solvable by the inverse scattering transform (IST) and has exact soliton solutions. The soliton solutions of VNLS are of particular interest because they have potential applications in long distance telecommunications (cf., e.g., [22]) and all-optical computing (cf., e.g., [12]).

In this paper, we consider the related semi-discrete equation

$$i\frac{d}{d\tau}\mathbf{Q}_n = \mathbf{Q}_{n-1} - 2\mathbf{Q}_n + \mathbf{Q}_{n+1} + \|\mathbf{Q}_n\|^2(\mathbf{Q}_{n-1} + \mathbf{Q}_{n+1}), \tag{2}$$

where $\mathbf{Q}_n$ is a vector-valued function of the discrete spatial variable $n \in \mathbb{Z}$ and the continuous time-like variable $\tau \in \mathbb{R}$. The relation between integrable discrete vector NLS (IDVNLS) (2) and VNLS (1) is made more explicit by the change of variables

$$\mathbf{Q}_n = h\mathbf{q}_n = h\mathbf{q}(nh), \quad \tau = h^{-2}t, \tag{3}$$

which transforms Equation (2) to

$$i\frac{d}{dt}\mathbf{q}_n = \frac{\mathbf{q}_{n-1} - 2\mathbf{q}_n + \mathbf{q}_{n+1}}{h^2} + \|\mathbf{q}_n\|^2(\mathbf{q}_{n-1} + \mathbf{q}_{n+1}). \tag{4}$$

In the form (4), IDVNLS converges to the PDE (1) in the limit $nh \to x$, $h \to 0$ (i.e., the limit in which h, the spatial separation between the nodes, vanishes and the lattice is replaced by the continuum).

Like VNLS itself, the discretization (2) is completely integrable in the sense that it can be solved via the IST and it has exact soliton solutions [5, 19]. This is the reason why we refer to (2) as IDVNLS. In this paper, we describe both the soliton solutions of IDVNLS and the dynamics of the soliton interaction. We observe that the discrete vector-soliton interaction defines a Yang–Baxter map and can be represented in terms of fractional linear transformations on scalar-valued polarization states. In addition, we show how to implement logic operations with the discrete vector solitons.

It has been 30 years since Manakov [14] first derived the vector-soliton solutions of VNLS. While Manakov provided some physical motivation for his equation as an asymptotic model for electromagnetic waves in a waveguide, he did not provide a derivation of VNLS from equations of physics. Rather, he was mostly concerned with the mathematical solution of VNLS by the IST, the characterization of the soliton solutions, and the basics of soliton interaction. To be precise, Manakov only considered the two-component system (sometimes referred to as coupled NLS), but the extension of his method to an arbitrary number of components is straightforward. In fact, Manakov's construction of the IST for VNLS was an extension of earlier work by Zakharov and

Shabat [23], who solved the scalar nonlinear Schrödinger equation (NLS) via the IST.

Shortly after Manakov's investigation of VNLS, Ablowitz and Ladik found a discretization of the scalar NLS that could also be solved by the IST [2, 3]. We refer to this spatially discrete equation,

$$i\frac{d}{d\tau}Q_n = Q_{n-1} - 2Q_n + Q_{n+1} + |Q_n|^2(Q_{n-1} + Q_n), \tag{5}$$

as the integrable discrete nonlinear Schrödinger equation (IDNLS). Under a change of variables of the type (3), IDNLS becomes

$$i\frac{d}{dt}q_n = \frac{q_{n-1} + q_{n-1} - 2q_n}{h^2} + |q_n|^2(q_{n+1} + q_{n-1}), \tag{6}$$

which converges to the scalar NLS in the continuum limit (i.e., $nh \to x, h \to 0$).

Remarkably, the IST for the discrete equation (5) can be carried out in full detail and provides a discrete analog of the IST for NLS (cf., [5]). Moreover, the soliton solutions of IDNLS reproduce the dynamics of the soliton solutions of NLS. In fact, in the continuum limit, the key formulas in the IST of IDNLS converge to the corresponding formulas in the IST of NLS. In particular, soliton solutions of IDNLS converge to the soliton solutions of NLS. However, as described in the Appendix, IDNLS has additional soliton dynamics that have no counterpart in the corresponding PDE (i.e., NLS).

While the vector (VNLS) and discrete (IDNLS) variants of NLS have been widely studied for many years, the investigation of the discrete, vector system (2) is much more recent. Based on analogy with IDNLS, numerical evidence and the existence of a restricted class of multisoliton solutions constructed via Hirota's method, the system (2) was proposed as an integrable discretization of VNLS [4]. In fact, the IST for the discrete, vector system (2) employs a special case of a more general, discrete, block-matrix scattering problem studied in [9, 8] that is associated with the IST for related spatially discrete vector and matrix equations [4, 19]. A discussion of the soliton solutions of IDVNLS from the point of view of the IST is presented in Section 2.1.

To formulate the IST specifically for two-component IDVNLS (2), one must introduce an extra symmetry into the discrete, block-matrix scattering problem [5, 18]. As a consequence of this additional symmetry, IDVNLS has "composite" soliton solutions. In fact, there are both "fundamental" soliton solutions that are the counterpart of the vector solitons of VNLS and "composite" soliton solutions that have no counterpart in the continuous limit. Moreover, in general, composite solitons also have no counterpart in the solutions of IDNLS. The fundamental solitons are described in Section 2.2 and the composite solitons are described in Section 2.3.

When the vector solitons of IDVNLS emerge from a collision with one another, each soliton retains its characteristic shape and velocity. Indeed, this coherent particle-like behavior is the phenomenological definition of solitons, as distinct from ordinary solitary waves. On the other hand, the soliton interactions induce shifts in the polarizations and envelope peaks of the individual solitons. While the collision-induced shift of the soliton peak is also observed in the interaction of scalar solitons (e.g., for scalar NLS, IDNLS, and KdV), the collision-induced shifts of the soliton polarization vectors constitute the distinctive feature of *vector*–soliton interaction.

The technique of Zakharov and Shabat [23] and Manakov [14] for the determination of the phase shifts in continuous nonlinear Schrödinger systems can be adapted to determine the phase shifts in a generic IDVNLS J-soliton interaction [5]. In the case of a two-soliton interaction, the equations governing the phase shifts yield an explicit formula for the collision-induced polarization shifts. We review these results in Sections 3.1 (J-soliton interaction) and 3.2 (two-soliton interaction).

Generally speaking, it is well known that scalar multiple-soliton interactions are equivalent to the net effect of pairwise soliton collisions. Moreover, the total net phase shift on each soliton is independent of the order of these interactions. For NLS and IDNLS, this reduction of multiple soliton interactions to iterated pairwise interactions can be determined by a cursory examination of the formulas that describe the cumulative effect of multiple scalar-soliton interactions. In contrast, Manakov [14] observed that, for the vector-soliton interaction, it is not apparent that the net collision-induced shifts in the polarizations and the offsets of the envelope peaks of the individual solitons are independent of the order of the soliton interactions. Nor is it clear that the total shift for each individual soliton is equivalent to the net result of the shifts due to pairwise interaction with the remaining $J - 1$ solitons. Indeed, this has only been investigated recently [6, 20, 13]. In Section 3.3, we show that the J-soliton interaction in IDVNLS is also, in fact, equivalent to the order-independent combination of the pairwise soliton interactions.

Vector and matrix soliton interactions can be recast as Yang–Baxter maps. In [17, 21, 11] the soliton interactions of the matrix KdV equation were described from this point of view. The same problem for the VNLS equation is addressed in [6]. In Section 4, we show that the soliton interaction of IDVNLS also generates a Yang–Baxter map.

In the two-component case, the polarization state of a vector soliton can be described by a single complex scalar. Moreover, the polarization shifts induced by soliton collisions can be conveniently described by fractional linear transformations (FLTs) on these scalars. The FLTs that describe the interactions of solitons in the Mankov equation were given in [12]. In Section 5, we derive the FLTs that describe polarization shifts for the solitons of the two-component IDVNLS.

The value of a logical "bit" can be encoded in the scalar polarization state of a vector soliton. Given such an encoding, one can identify vector-soliton interactions with logical operations. With this insight, Jakubowsky et al. [12] suggested that Manakov solitons (i.e., the solitons of two-component VNLS) could be used for computation. Subsequently Steiglitz [16] showed that it is possible to construct both the "not-and" or NAND and the bit-reproducing FANOUT logic gates by means of controlled Manakov-soliton interactions. Moreover, he described a mechanism for arranging the output of one gate to become the input of a subsequent gate. With such a construction, time-controlled interactions of Manakov spatial solitons are computationally universal in the sense of Turing equivalence. In Section 6, we show that the same result holds for the spatially discrete vector solitons of two-component IDVNLS.

2. Discrete vector solitons from the IST perspective

2.1. Solitons and eigenvalues

In this section, we give a brief overview of the theory of discrete vector solitons from the point of view of the IST. A detailed account can be found in [5].

For the purpose of the IST, IDVNLS (2) is written as a *matrix* equation

$$i\frac{d}{d\tau}\hat{\mathbf{Q}}_n = \hat{\mathbf{Q}}_{n-1} - 2\hat{\mathbf{Q}}_n + \hat{\mathbf{Q}}_{n+1} + \hat{\mathbf{Q}}_n\hat{\mathbf{Q}}_n^H(\hat{\mathbf{Q}}_{n-1} + \hat{\mathbf{Q}}_{n+1}), \tag{7}$$

where the superscript H denotes the Hermitian (conjugate transpose) and, in the two-component case,

$$\hat{\mathbf{Q}}_n = \begin{pmatrix} Q_n^{(1)} & Q_n^{(2)} \\ (-1)^n Q_n^{(2)*} & (-1)^{n+1} Q_n^{(1)*} \end{pmatrix},$$

so that

$$\hat{\mathbf{Q}}_n\hat{\mathbf{Q}}_n^H = \left|Q_n^{(1)}\right|^2 + \left|Q_n^{(2)}\right|^2\mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix. There is a recurrence relation that generates the appropriate matrix, $\hat{\mathbf{Q}}_n$, for the N-component version of IDVNLS (cf., [19]), but we consider only the two-component case here.

The key prerequisite to the IST solution of IDVNLS is the establishment of a relation between the matrix evolution equation (7) and the difference equation

$$\mathbf{v}_{n+1} = \begin{pmatrix} z\mathbf{I} & \hat{\mathbf{Q}}_n \\ -\hat{\mathbf{Q}}_n^H & z^{-1}\mathbf{I} \end{pmatrix}\mathbf{v}_n, \tag{8}$$

For a given $\hat{\mathbf{Q}}_n$, and at fixed τ, this difference equation is a (generalized) eigenvalue problem in the spectral parameter z. A discrete eigenvalue z_j is a (complex) value of z for which (8) has a solution, $\mathbf{v}_n$, that vanishes as $n \to \pm\infty$. Importantly, the eigenvalues remain fixed as $\hat{\mathbf{Q}}_n(\tau)$ evolves according to the evolution equation (7). Moreover, for each eigenvalue there is an associated 2×2 matrix, referred to as the "norming constant," whose time evolution depends only on the (fixed) eigenvalue. Specifically, it can be shown that, as $\hat{\mathbf{Q}}_n$ evolves according to (7),

$$\mathbf{C}_j(\tau) = e^{i(z_j^2 + z_j^{-2} - 2)\tau}\mathbf{C}_j(0),$$

where $\mathbf{C}_j(\tau)$ is the norming constant associated with the eigenvalue z_j. We note that the eigenvalue problem (8) is a special case of the more general discrete block-matrix problem considered in [9, 8, 10, 19] in relation to more general, semi-discrete, NLS-like evolution equations.

Due to symmetries in the block-matrix scattering problem (8), the eigenvalues appear in octets. Moreover, the norming constants associated with the eigenvalues of each octet satisfy symmetry relations. Specifically, these eigenvalue/norming constant octets are of the form

$$\left\{(\pm z_j, \mathbf{C}_j),\ \left(\pm 1/z_j^*, \bar{\mathbf{C}}_j\right),\ (\pm i/z_j, \hat{\mathbf{C}}_j),\ \left(\pm i z_j^*, \tilde{\mathbf{C}}_j\right)\right\}.$$

where: (i) without loss of generality, we assume that $|z_j| > 1$ and $-\frac{\pi}{2} < \arg z_j \leq \frac{\pi}{2}$; (ii) the symmetries among the norming constants can be expressed as

$$\mathbf{C}_j = \begin{pmatrix} \gamma_j^{(1)} & \delta_j^{(2)} \\ \gamma_j^{(2)} & -\delta_j^{(1)} \end{pmatrix}, \qquad \bar{\mathbf{C}}_j = z_j^{*-2}\begin{pmatrix} \gamma_j^{(1)*} & \gamma_j^{(2)*} \\ \delta_j^{(2)*} & -\delta_j^{(1)*} \end{pmatrix},$$

$$\hat{\mathbf{C}}_j = -z_j^{-2}\begin{pmatrix} \delta_j^{(1)} & \delta_j^{(2)} \\ \gamma_j^{(2)} & -\gamma_j^{(1)} \end{pmatrix}, \qquad \tilde{\mathbf{C}}_j = \begin{pmatrix} \delta_j^{(1)*} & \gamma_j^{(2)*} \\ \delta_j^{(2)*} & -\gamma_j^{(1)*} \end{pmatrix}.$$

Note that, for convenience (as explained below), we introduced the vectors

$$\boldsymbol{\gamma}_j = \begin{pmatrix} \gamma_j^{(1)} \\ \gamma_j^{(2)} \end{pmatrix}, \quad \boldsymbol{\delta}_j = \begin{pmatrix} \delta_j^{(1)} \\ \delta_j^{(2)} \end{pmatrix},$$

that, together, constitute the norming constant, $\mathbf{C}_j$, associated with the eigenvalue z_j. It is clear from the symmetries among the norming constants that the jth octet is completely (and explicitly) determined by the "primary" eigenvalue/norming-constant pair $\{z_j, \mathbf{C}_j\}$ where, by assumption, $|z_j| > 1$ and $-\frac{\pi}{2} < \arg z_j \leq \frac{\pi}{2}$.

As is typical of soliton equations solvable by the IST, the solitons of IDVNLS are identified with eigenvalues of (8). In a generic solution of IDVNLS defined on the doubly infinite lattice such that $\mathbf{Q}_n \to 0$ sufficiently rapidly as $n \to \pm\infty$, there is a one-to-one correspondence between the eigenvalue octets and the solitons. That is, for each octet there is a corresponding soliton and vice versa. Moreover, the characteristic amplitude and velocity of each soliton are determined by the corresponding primary eigenvalue of the associated octet. The time-invariance of the amplitude and velocity of each soliton is a manifestation of the time-invariance of the associated eigenvalue octet. To construct a J-solution of IDVNLS, one can posit data consisting of J distinct eigenvalue octets and then solve the inverse problem of the IST to obtain the J-soliton solution formula. In this case, the inverse problem reduces to a system of linear algebraic equations (cf., [5]).

As noted in the introduction, unlike other integrable nonlinear Schödinger systems, IDVNLS has both "fundamental" and "composite" soliton solutions. Both types of solution are single solitons in the sense that each type is associated with a single octet of eigenvalues. We discuss these two types of solitons in the two subsequent sections.

2.2. Fundamental solitons

To obtain a single fundamental soliton, we consider scattering data that consist of a single eigenvalue/norming constant octet such that $\boldsymbol{\delta}_1 = \mathbf{0}$ and $\gamma_1 \neq \mathbf{0}$. If we write $z_1 = e^{a+ib}$, where, by our previous assumption, $a > 0$ and $-\frac{\pi}{2} < b \leq \frac{\pi}{2}$, the fundamental soliton can be written as

$$\begin{pmatrix} Q_n^{(1)}(\tau) \\ Q_n^{(2)}(\tau) \end{pmatrix} = \mathbf{p} \sinh(2a) e^{i(2b(n+1)+2\omega\tau)} \operatorname{sech}[2a(n+1) - 2v\tau - d], \tag{9}$$

where

$$v = -\sinh 2a \sin 2b \quad \omega = 1 - \cosh 2a \cos 2b, \tag{10}$$

and

$$\mathbf{p} = -\frac{\gamma_1^*(0)}{\|\gamma_1(0)\|} \quad e^d = \frac{\|\gamma_1(0)\|}{\sinh 2a}. \tag{11}$$

According to (9)–(11), a fundamental soliton consists of a traveling sech envelope with a complex modulation multiplied by a fixed vector, **p**, that is referred to as the "polarization" of the soliton. The complex-modulated sech envelope is the typical form of the soliton in nonlinear Schrödinger systems.

We note that the fundamental soliton is equivalent to the scalar one-soliton solution of IDNLS (5) multiplied by the polarization vector. In this aspect, the fundamental soliton is analogous to the vector-soliton solution of VNLS, which is the scalar one-soliton solution of NLS multiplied by a (complex) unimodular polarization vector. Thus, the fundamental soliton of IDVNLS is the spatially discrete counterpart of the vector-soliton solution of VNLS.

In fact, the fundamental soliton converges to the soliton of VNLS. Under the change of variables (3) where, in addition,

$$z_1 = e^{-ik_1h} \qquad \gamma_1 = h\mathbf{c}_1$$

the expression (9)–(11) becomes the one-soliton solution of (6), i.e.,

$$\mathbf{q}_n(t) = \mathbf{p}e^{-i(2\xi nh - 2wt)}\frac{\sinh(2\eta h)}{h}\operatorname{sech}(2\eta nh - 2vt - d),$$

where

$$v = \frac{\sinh(2\eta h)\sin(2\xi h)}{h^2}, \qquad w = \frac{1 - \cosh(2\eta h)\cos(2\xi h)}{h^2},$$
$$\mathbf{p} = -\frac{\mathbf{c}_1^*(0)}{\|\mathbf{c}_1(0)\|}, \qquad e^d = \frac{h\|\mathbf{c}_1(0)\|}{\sinh(2\eta h)},$$

and $k_1 = \xi + i\eta$. In this form, the fundamental soliton solution converges in the continuum limit ($h \to 0$, $nh \to x$) to the one-soliton solution of VNLS that is associated with the eigenvalue pair $\{k_1, k_1^*\}$.

The case $\boldsymbol{\delta}_1 \neq \mathbf{0}$, $\gamma_1 = \mathbf{0}$ is equivalent to (9)–(11). To see this, we first note that the eigenvalues z_1 and $\pm iz_1^*$ (where the sign is chosen so that the second eigenvalue is in the region Im $z > 0$ in the complex z-plane) are the two members of the eigenvalue octet in the region $|z| > 1$, Im $z > 0$. In fact, there is no intrinsic property that determines which of these two eigenvalues ought to be regarded as the primary eigenvalue. The symmetry in the associated norming constants shows that choice of $\boldsymbol{\delta}_1 \neq \mathbf{0}$, $\gamma_1 = \mathbf{0}$ versus $\gamma_1 \neq \mathbf{0}$, $\boldsymbol{\delta}_1 = \mathbf{0}$ is equivalent to the choice of primary eigenvalue.

A fundamental soliton is characterized by three complex parameters: the discrete eigenvalue $z_1 = e^{a+ib}$ that fixes the envelope amplitude and frequency of complex modulation (and, hence, the velocity) of the soliton; and the two components of the vector $\gamma_1(0)$ that determines the polarization and the location of the sech envelope peak at $\tau = 0$.

The dynamics of a single vector soliton are essentially governed by the scalar IDNLS because the polarization remains fixed and IDVNLS reduces to the scalar evolution equation. The vector nature of the system manifests itself in the interaction dynamics of solitons whose polarizations are not parallel or orthogonal. The dynamics of vector-soliton interaction are discussed in Section 3.

2.3. *Composite solitons*

2.3.1. *Coalescence of fundamental solitons.* We refer to the solutions of IDVNLS that correspond to a single octet in which both $\gamma_1 \neq \mathbf{0}$, and $\boldsymbol{\delta}_1 \neq \mathbf{0}$, as "composite" solitons. We use the term composite because these solutions may be obtained by the coalescence of two fundamental soliton octets. That is, we consider scattering data composed of two fundamental soliton octets, such that $\boldsymbol{\gamma}_j \neq \mathbf{0}$ and $\boldsymbol{\delta}_j = \mathbf{0}$ for $j = 1, 2$. Moreover, the octets are such that the primary eigenvalue of the second octet is of the form $z_2 = \pm i z_1^* + \epsilon$ (where the sign is chosen so that the second eigenvalue is in the region $\operatorname{Im} z > 0$ in the complex z-plane). In the coalescence limit (i.e., $\epsilon \to 0$), the scattering data become a single octet (with z_1 as the primary eigenvalue) in which $\boldsymbol{\gamma}_1$ is unchanged, $\boldsymbol{\delta}_1 = \boldsymbol{\gamma}_2$ and the potential, $\mathbf{Q}_n(\tau)$, converges to a composite-soliton solution of VNLS.

In contrast to the fundamental solitons considered in the preceding section, composite solitons do not have a continuous counterpart (i.e., a corresponding solution in the PDE limit). Composite solitons contain a term of the form $(-1)^n$, which induces oscillations on the grid scale. Therefore, in the continuum limit, these oscillations become infinitely rapid and the solution breaks down. We conclude that composite solitons represent states of IDVNLS that do not occur in the continuum limit, VNLS.

The explicit formula of a general composite soliton is not illuminating. However, the composite soliton itself consists of a localized traveling envelope with both temporal and spatial oscillations, as well as a complex spatial modulation. The oscillating envelope travels with a constant velocity equal to that of a fundamental soliton associated with the same eigenvalue octet (as given by (10)). To shed light on the structure of composite solitons, we consider in more detail two special cases of the composite soliton, namely the "orthogonal" case, $\boldsymbol{\gamma}_1 \cdot \boldsymbol{\delta}_1 = 0$, and the "parallel" case, $W(\boldsymbol{\gamma}_1^*, \boldsymbol{\delta}_1) = 0$. In both cases, the solution formula provides additional insight into the nature of the composite soliton.

2.3.2. *Orthogonal composite soliton.* When $\boldsymbol{\gamma}_1 \cdot \boldsymbol{\delta}_1 = 0$, a single octet with primary eigenvalue $z_1 = e^{a+ib}$ corresponds to the solution

$$\begin{pmatrix} Q_n^{(1)}(\tau) \\ Q_n^{(2)}(\tau) \end{pmatrix} = \left[(-1)^n e^{-i(\zeta(n,\tau)-4\tau)} \sin \mu \mathbf{p}^{\perp} - e^{i\zeta(n,\tau)} \cos \mu \mathbf{p}\right] \times \sinh(2a) \operatorname{sech}[\theta(n,\tau) - \hat{d}\,],$$

where

$$\theta(n,\tau) = 2a(n+1) + 2\sinh(2a)\sin(2b)\tau, \tag{12}$$

$$\zeta(n,\tau) = 2b(n+1) + 2[1 - \cosh(2a)\cos(2b)]\tau, \tag{13}$$

and

$$\cos\mu = \frac{\|\gamma_1(0)\|}{\left(\|\gamma_1(0)\|^2 + \|\boldsymbol{\delta}_1(0)\|^2\right)^{\frac{1}{2}}}, \qquad \sin\mu = \frac{\|\boldsymbol{\delta}_1(0)\|}{\left(\|\gamma_1(0)\|^2 + \|\boldsymbol{\delta}_1(0)\|^2\right)^{\frac{1}{2}}},$$

$$\mathbf{p} = \frac{\gamma_1(0)^*}{\|\gamma_1(0)\|} \quad \mathbf{p}^\perp = \frac{\boldsymbol{\delta}_1(0)}{\|\boldsymbol{\delta}_1(0)\|},$$

$$e^{\hat{d}} = \frac{\left(\|\gamma_1(0)\|^2 + \|\boldsymbol{\delta}_1(0)\|^2\right)^{1/2}}{\sinh 2a}.$$

This solution is the (weighted) sum of two fundamental-soliton-like solutions. The two constituent sech envelopes have mutually orthogonal polarizations because, by construction, $\mathbf{p}^* \cdot \mathbf{p}^\perp = 0$. Moreover, one of the envelopes has oscillation on the grid scale ($(-1)^n$) while the other does not. Hence, there is no continuum limit of this solution to VNLS.

2.3.3. *Parallel composite soliton.* When $W(\gamma_1^*, \boldsymbol{\delta}_1) = 0$, the composite soliton formula can be written as

$$\begin{pmatrix} Q_n^{(1)}(\tau) \\ Q_n^{(2)}(\tau) \end{pmatrix} = \mathbf{p}\left[2(-1)^n e^{-i(\zeta(n,\tau)-4\tau+\rho)} \frac{e^{-\theta(n,\tau)} + A^- e^{-3\theta(n,\tau)}}{1 + B(n,\tau)e^{-2\theta(n,\tau)} + \tilde{B}e^{-4\theta(n,\tau)}} \right.$$
$$\left. - 2e^{i\zeta(n,\tau)} \frac{e^{-\theta(n,\tau)} + A^+ e^{-3\theta(n,\tau)}}{1 + B(n,\tau)e^{-2\theta(n,\tau)} + \tilde{B}e^{-4\theta(n,\tau)}}\right],$$

where, $\theta(n, \tau)$ and $\zeta(n, \tau)$ are given by (12) and (13) and

$$\begin{aligned}
\mathbf{p} &= \frac{\gamma_1^*(0)}{\|\gamma_1(0)\|} \\
A^- &= (\alpha e^{-ib} - \beta^* e^{ib})(\alpha e^{ib} + \beta^* e^{-ib})\|\gamma_1(0)\|^2 \|\boldsymbol{\delta}_1(0)\| \\
A^+ &= (\alpha e^{ib} - \beta e^{-ib})(\alpha e^{-ib} + \beta e^{ib})\|\gamma_1(0)\| \|\boldsymbol{\delta}_1(0)\|^2 \\
\tilde{B} &= (\alpha^2 - |\beta|^2)^2 \|\gamma_1(0)\| \|\boldsymbol{\delta}_1(0)\| \\
B(n,\tau) &= \alpha^2\left(\|\gamma_1(0)\|^2 + \|\boldsymbol{\delta}_1(0)\|^2\right) + 2(-1)^n \|\gamma_1(0)\| \|\boldsymbol{\delta}_1(0)\| \\
&\quad \times \mathrm{Re}\left(\beta^{*\,2} e^{-i(\zeta(n,\tau)-\rho)}\right),
\end{aligned}$$

with

$$e^{i\rho} = \frac{\gamma_1^*(0) \cdot \boldsymbol{\delta}_1(0)}{\|\gamma_1(0)\| \|\boldsymbol{\delta}_1(0)\|}, \quad \alpha = \frac{1}{\sinh(2a)},$$

$$\beta = \frac{1}{\cosh 2a \cos 2b - i \sinh 2a \sin 2b}.$$

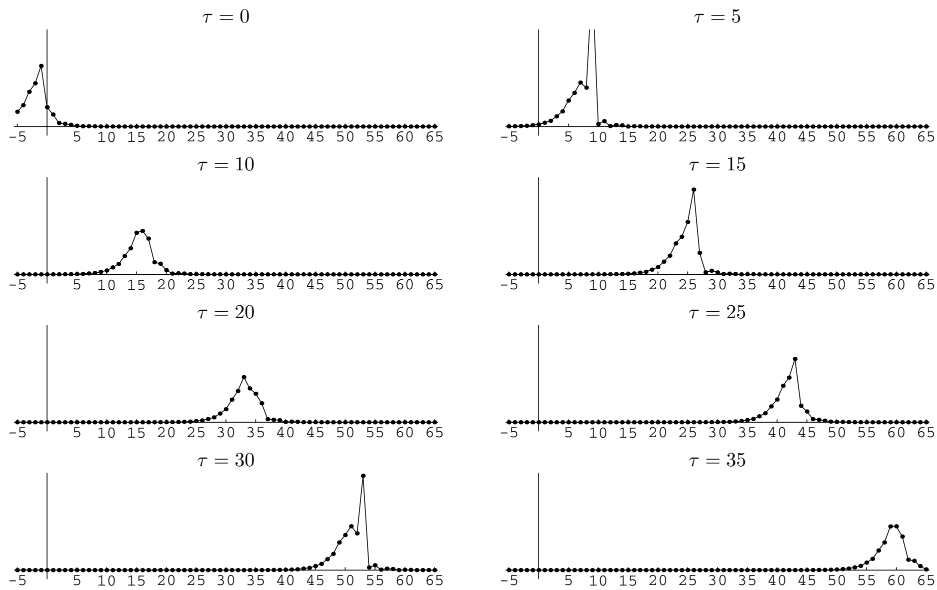

Figure 1. "Parallel" composite soliton solution of IDVNLS with $a = 0.25$, $b = 0.5$, $\boldsymbol{\gamma}_1(0) = (0.1, 0)^T$, $\boldsymbol{\delta}_1(0) = (10, 0)^T$. Note that the vertical axis of the plot is $|Q_n^{(1)}(\tau)|$ because, without loss of generality, $\gamma_1^{(2)} = \delta_1^{(2)} = 0$ and, therefore, $|Q_n^{(2)}(\tau)| = 0$ for this solution.

In this form, one can see a number of interesting properties of the solution

- The solution is a (complex) scalar function multiplied by a polarization vector **p**. Therefore, the scalar function (inside the square brackets) is a solution of IDNLS whose scattering data comprise an octet of eigenvalues.
- If $B(n, \tau)$ is treated as independent of n, each of the ratios in square brackets can be rewritten as a sum of two sech envelopes in the phase variable $\theta(n, \tau)$. In fact, the n-dependence of $B(n, \tau)$ is bounded and this introduces a spatial distortion in the sech envelopes.
- The τ-dependence of $B(n, \tau)$ is periodic and bounded. This introduces temporal oscillations into the envelopes.

In the parallel case, the composite soliton is, in general, a localized traveling envelope with temporal and spatial oscillations (see Figure 1). The formula above shows that the shape of the overall envelope is the result of the superposition of simpler constituents, namely modulated sech envelopes. In fact, the solution can be written as

$$\begin{pmatrix} Q_n^{(1)}(\tau) \\ Q_n^{(2)}(\tau) \end{pmatrix} =$$

$$\mathbf{p}\big\{e^{\theta_+(n,\tau)}\big[-e^{i\zeta(n,\tau)}A_n^+ + (-1)^n e^{-i(\zeta(n,\tau)-4\tau+\rho)}\tilde{A}_n^+\big]\operatorname{sech}[\theta(n,\tau)+\theta_+(n,\tau)]$$
$$+ e^{\theta_-(n,\tau)}\big[-e^{i\zeta(n,\tau)}A_n^- + (-1)^n e^{-i(\zeta(n,\tau)-4\tau+\rho)}\tilde{A}_n^-\big]$$
$$\times \operatorname{sech}[\theta(n,\tau)+\theta_-(n,\tau)]\big\},$$

where

$$A_n^\pm = \|\gamma_1(0)\|\left[\frac{1}{2} \pm \frac{B(n,\tau)-2G(n,\tau)}{2\sqrt{B(n,t)^2-4\tilde{B}}}\right]$$

$$\tilde{A}_n^\pm = \|\boldsymbol{\delta}_1(0)\|\left[\frac{1}{2} \pm \frac{B(n,\tau)-2\tilde{G}(n,\tau)}{2\sqrt{B(n,\tau)^2-4\tilde{B}}}\right]$$

$$\theta_\pm(n,\tau) = -\frac{1}{2}\log\frac{B(n,\tau) \pm \sqrt{B(n,\tau)^2-4\tilde{B}}}{2}$$

$$G(n,\tau) = (-1)^n\beta^*(\beta^* - \alpha e^{-2ib})(\gamma_1(0)\cdot\boldsymbol{\delta}_1(0))\, e^{-2i\zeta(n,\tau)}$$
$$+\alpha(\alpha - \beta e^{-2ib})\|\boldsymbol{\delta}_1(0)\|^2$$
$$\tilde{G}(n,\tau) = (-1)^n\beta(\beta - \alpha e^{2ib})(\gamma_1(0)\cdot\boldsymbol{\delta}_1(0))^* e^{2i\zeta(n,\tau)}$$
$$+\alpha(\alpha - \beta^* e^{2ib})\|\gamma_1(0)\|^2$$

and all other quantities are defined at the beginning of Section 2.3.3. Note that $\tilde{G}(n,\tau)$ is $G(n,\tau)$ complex conjugate with $\gamma_1(0)$ and $\boldsymbol{\delta}_1(0)$ exchanged. Note also that $B(n,\tau) \geq 0$, $\tilde{B} \geq 0$, and also $B(n,\tau)^2 - 4\tilde{B} \geq 0$. Indeed, $B(n,\tau) \geq 2\sqrt{\tilde{B}}$ if and only if

$$\frac{\|\gamma_1(0)\|^2 + \|\boldsymbol{\delta}_1(0)\|^2}{2|\gamma_1(0)\cdot\boldsymbol{\delta}_1(0)|}$$
$$\geq \alpha^{-2}\left\{(\alpha^2 - |\beta|^2) - \frac{1}{(\omega^2+v^2)^2}[(\omega^2 - v^2)\cos 2\zeta(n,\tau) + 2v\omega\sin 2\zeta(n,\tau)]\right\},$$

where we introduced

$$v = -\sinh 2a \sin 2b, \quad \omega = \cosh 2a \cos 2b.$$

The left-hand side is always greater than or equal to 1, while the right-hand side is always smaller than or equal to 1, which proves the statement. Therefore,

$\theta_{\pm}(n, \tau)$ are real, bounded functions depending on n and τ only via $(-1)^n$ (which one can get rid of by considering even and odd grid points separately) and $\cos 2\zeta(n, \tau)$, $\sin 2\zeta(n, \tau)$.

3. Dynamics of discrete vector-soliton interactions

3.1. J-soliton interaction

In a generic J-soliton solution, the solitons have unequal envelope velocities. Therefore, in the long-time limits (i.e., $\tau \to \pm\infty$) the solution is of the form

$$\mathbf{Q}_n^{\pm} \sim \sum_{j=1}^{J} \mathbf{Q}_{n,j}^{\pm},$$

where $\mathbf{Q}_{n,j}^{\pm}$ denotes a one-soliton solution of IDVNLS whose amplitude and velocity are determined by an eigenvalue octet with the primary eigenvalue z_j. The superscript "$\pm$" distinguishes between the forward long-time limit (i.e., $\tau \to +\infty$, denoted by the superscript $+$) and the backward long-time limit (i.e., $\tau \to -\infty$, denoted by $-$). Moreover, because their velocities are distinct, the individual solitons are spatially well separated in the long-term limits.

Without loss of generality, we assume that velocities of the individual solitons are ordered such that

$$v_1 < v_2 < \cdots < v_J.$$

Then, in the backward long-time limit, the solitons are arranged from left to right in the order

$$J\ (J-1)\ (J-2)\ \cdots\ 2\ 1,$$

while, in the forward long-time limit ($\tau \to +\infty$), the solitons are reversed, i.e., the solitons are in the order

$$1\ 2\ \cdots\ (J-1)\ J.$$

That is, in the passage from the backward long-time limit to the forward long-time limit, the solitons reverse their spatial order. In this process the solitons pass through one another, thereby inducing phase shifts.

When the solitons are spatially well separated (i.e., in the long-time limits), the phase of the jth soliton is associated with a 2×2 matrix $\mathbf{S}_j^{\pm}$ where, as before, the superscript distinguishes between the forward and backward long-time limits. It can be shown [5] that these $2J$ matrices satisfy the J equations

$$\mathbf{S}_j^+ = \prod_{\substack{\ell=j+1 \\ \text{right}}}^{J} \mathbf{c}_\ell^+(z_j) \prod_{\substack{\ell=1 \\ \text{right}}}^{j-1} \left[\mathbf{c}_\ell^-(z_j)\right]^{-1} \mathbf{S}_j^- \prod_{\substack{\ell=j+1 \\ \text{right}}}^{J} \mathbf{a}_\ell^-(z_j) \prod_{\substack{\ell=1 \\ \text{right}}}^{j-1} \left[\mathbf{a}_\ell^+(z_j)\right]^{-1}, \tag{14}$$

where: (i) $j = 1, \ldots, J$; the notation "right" indicates that the matrix with index ℓ is to the right of the matrix with index $\ell - 1$; (ii) we define $\prod_\alpha^\beta = 1$ for $\alpha > \beta$; and (iii) the "transmission coefficients," $\mathbf{c}_j^\pm(z)$ and $\mathbf{a}_j^\pm(z)$, are as described below.

In the case of fundamental solitons, the transmission coefficients are of the form

$$\mathbf{a}_j^\pm(z) \equiv \mathbf{a}_j(z) = \begin{pmatrix} \frac{z^2 - z_j^2}{z^2 - z_j^{*-2}} & 0 \\ 0 & \frac{z^2 + z_j^{*2}}{z^2 + z_j^{-2}} \end{pmatrix} \tag{15}$$

$$\mathbf{c}_j^\pm(z) = \frac{z_j^{*-2} + z^{-2}}{z_j^2 + z^{-2}} \left[\mathbf{I} + \frac{\left(z_j^{*-2} - z_j^2\right)\left(z_j^{*-2} + z_j^{-2}\right)}{\left(z^2 - z_j^{*-2}\right)\left(z_j^{*-2} + z^{-2}\right)} \frac{1}{\|\mathbf{s}_j\|^2} \left(\mathbf{s}_j^\pm\right)^* \left(\mathbf{s}_j^\pm\right)^T \right], \tag{16}$$

where $z_j = e^{a_j + i b_j}$ is the primary eigenvalue associated with the jth soliton and

$$\mathbf{S}_j = (\mathbf{s}_j \quad 0),$$

with $\mathbf{S}_j$ a 2×2 matrix and $\mathbf{s}_j = (s_j^{(1)}, s_j^{(2)})^T$ a two-component column vector. Note that, for fundamental solitons, the coefficients $\mathbf{a}_j(z)$ are unchanged by the passage from the backward to the forward long-time limit, but the coefficients $\mathbf{c}_j^\pm(z)$ are distinguished by their dependence, respectively, on $\mathbf{s}_j^\pm$. The polarization and the offset of the envelope peak (i.e., the phase) of the jth soliton in the long-time limits are given by

$$\mathbf{p}_j^\pm = \frac{\left(\mathbf{s}_j^\pm\right)^*}{\|\mathbf{s}_j^\pm\|}, \quad e^{d_j^\pm} = \frac{\|\mathbf{s}_j^\pm\|}{\sinh(2a_j)}. \tag{17}$$

That is, $\mathbf{Q}_{n,j}^\pm$ is given by (9) and (10) with $\mathbf{p} = \mathbf{p}_j^\pm$, $d = d_j^\pm$, $a = a_j$, and $b = b_j$.

To compute the effect of soliton collisions on the soliton phases, we use (14) to determine the phases after the collisions (i.e., the $\mathbf{S}_j^+$ with $j = 1, \ldots, J$) in terms of the phases before the collisions (i.e., the $\mathbf{S}_j^-$ with $j = 1, \ldots, J$). The calculation proceeds as follows: First, with $j = J$, Equation (14) determines $\mathbf{S}_J^+$ in terms of the $\mathbf{S}_j^-$ for $j = 1, \ldots, J$ (note that it depends on $\mathbf{p}_j^- = (\mathbf{s}_j^-)^*/\|\mathbf{s}_j^-\|$ for all $j = 1, \ldots, J$ through the $\mathbf{c}_j^-$). Second, with $j = J - 1$, Equation (14) determines $\mathbf{S}_{J-1}^+$ in terms of $\mathbf{S}_j^-$ for $j = 1, \ldots, J$ and $\mathbf{S}_J^+$. More generally, for a fixed j, (14) depends on $\mathbf{S}_\ell^+$ only for $\ell > j$. Hence, one can proceed iteratively (decrementing j) to determine $\mathbf{S}_j^+$ for $j = 1, \ldots, J$.

3.2. Two-soliton interaction

We now consider the interaction of two fundamental solitons in more detail. In the two-soliton case, the system defined by Equation (14) with $J = 2$, $j = 1$ and $J = 2$, $j = 2$ yields the relations

$$\mathbf{p}_1^+ = \frac{1}{\chi}\frac{\left(z_1^2 - z_2^{*-2}\right)\left(z_2^2 + z_1^{-2}\right)}{\left(z_1^2 - z_2^2\right)\left(z_2^{*-2} + z_1^{-2}\right)}\left(\mathbf{p}_1^- + \frac{\left(z_2^{*-2} - z_2^2\right)\left(z_2^2 + z_2^{*2}\right)}{\left(z_2^2 - z_1^2\right)\left(z_2^2 + z_1^{-2}\right)}(\mathbf{p}_2^{-*}\cdot\mathbf{p}_1^-)\mathbf{p}_2^-\right) \tag{18}$$

$$\mathbf{p}_2^+ = \frac{1}{\chi}\frac{\left(z_1^2 - z_2^{*-2}\right)\left(\bar{z}_1^2 + z_2^{*2}\right)}{\left(\bar{z}_1^2 - z_2^{*-2}\right)\left(z_1^2 + z_2^{*2}\right)}\left(\mathbf{p}_2^- + \frac{\left(z_1^2 - z_1^{*-2}\right)\left(z_1^{*-2} + z_1^{-2}\right)}{\left(z_1^{*-2} - z_2^{*-2}\right)\left(z_1^{*-2} + z_2^{*2}\right)}(\mathbf{p}_1^{-*}\cdot\mathbf{p}_2^-)\mathbf{p}_1^-\right) \tag{19}$$

where

$$\chi = \left|\frac{\left(z_1^2 - z_2^{*-2}\right)\left(z_1^{*-2} + z_2^{*2}\right)}{\left(z_1^{*-2} - z_2^{*-2}\right)\left(z_1^2 + z_2^{*2}\right)}\right|^2 \times\left[1 + \frac{\left(z_1^2 - z_1^{*-2}\right)\left(z_2^2 - z_2^{*-2}\right)\left(z_1^{*-2} + z_1^{-2}\right)\left(z_2^{*2} + z_2^2\right)}{\left(z_2^2 - z_1^2\right)\left(z_1^{*-2} - z_2^{*-2}\right)\left(z_2^2 + z_1^{-2}\right)\left(z_1^{*-2} + z_2^{*2}\right)}\left|\mathbf{p}_1^{-*}\cdot\mathbf{p}_2^-\right|^2\right]. \tag{20}$$

Equations (18) and (19), give, explicitly, the polarization shift in IDVNLS and are the analog of Manakov's formulas for the polarization shift in the VNLS soliton interaction. Indeed, in the continuous limit, (i.e., with the change of variables $z_j = e^{-ik_jh}$ and the limit $h \to 0$), the formulas (18) and (19) converge to Manakov's formula (cf. [14]). Like the continuous (i.e., Manakov) case, the magnitudes of each component of the soliton polarizations are changed by the interaction except when the polarizations are either parallel or orthogonal prior to the soliton interaction.

The shift in envelope peak of each soliton can also be determined explicitly from (14). In a two-soliton interaction, the shifts are given by

$$e^{d_2^+ - d_2^-} = \frac{\|\mathbf{s}_2^+\|}{\|\mathbf{s}_2^-\|} = \chi \qquad e^{d_1^+ - d_1^-} = \frac{\|\mathbf{s}_1^+\|}{\|\mathbf{s}_1^-\|} = \frac{1}{\chi}.$$

In contrast to the scalar soliton interaction—where the shift of the envelope peak depends only on the (fixed) eigenvalues of the interacting solitons—in the vector-soliton interaction, the shift of the envelope is nontrivially dependent on the polarizations.

3.3. Order-independence of the J-soliton interaction

For three or more solitons, the order of interactions (collisions) is not unique in the passage from the backward to the forward long-time limits. For instance, in a three-soliton interaction, the spatial ordering of the solitons can proceed as

$$3\ 2\ 1 \to 2\ 3\ 1 \to 2\ 1\ 3 \to 1\ 2\ 3, \tag{21}$$

as τ increases. Alternatively, the spatial ordering can proceed as

$$3\ 2\ 1 \to 3\ 1\ 2 \to 1\ 3\ 2 \to 1\ 2\ 3, \tag{22}$$

In the first case the second and third soliton pass through one another before the second and the first interact, while the order is reversed in the second case. The order of the soliton interaction depends on the relative distances between the solitons and their relative velocities at large negative τ, when the solitons are well separated. The number of possible interaction paths increases rapidly with the number of solitons. In this section, we show that, at least for the interaction of fundamental solitons, the net phase shift of each soliton is independent of the order of the soliton interactions. Moreover, we show that, for each soliton, the net phase shift is equivalent to the net result of pairwise interactions with the other solitons.

In the following it is convenient to introduce some new notation. We denote the matrix that determines the polarization and offset of soliton j after the interaction with soliton ℓ as $\mathbf{S}_{\{j,\ell\}}$. Moreover, we drop the minus sign superscript for the backward long-time limit. For example, in the two-soliton interaction,

$$\mathbf{S}_1^- \to \mathbf{S}_1, \quad \mathbf{S}_2^- \to \mathbf{S}_2, \quad \mathbf{S}_1^+ \to \mathbf{S}_{\{1,2\}}, \quad \mathbf{S}_2^+ \to \mathbf{S}_{\{2,1\}}.$$

In this notation, the phase shifts of jth soliton and ℓth solitons (with $v_j > v_\ell$), due to the pairwise interaction between them, are given by

$$\mathbf{S}_{\{j,\ell\}} = \mathbf{c}_\ell^{-1}(z_j)\ \mathbf{S}_j \mathbf{a}_\ell^{-1}(z_j) \tag{23}$$

$$\mathbf{S}_{\{\ell,j\}} = \mathbf{c}_j(z_\ell)\ \mathbf{S}_\ell \mathbf{a}_j(z_\ell), \tag{24}$$

which follows from (14) with $J = 2$.

Now, if we assume that the total phase shift is determined by the pairwise interactions of the solitons, then we can use (23) and (24) iteratively to compute the phase shifts for soliton 3 in a three-soliton interaction. If the soliton interaction proceeds according to (21), the iteration yields

$$\mathbf{S}_{\{\{3,2\},1\}} = \mathbf{c}_1^{-1}(z_3)\mathbf{c}_2^{-1}(z_3)\ \mathbf{S}_3 \mathbf{a}_1^{-1}(z_3)\mathbf{a}_2^{-1}(z_3), \tag{25}$$

where the subscript $\{\{3, 2\}, 1\}$ is interpreted as follows: the leftmost index indicates that the matrix refers to the state of soliton 3; the nested pairs are then read form the "inside" out. The pair $\{3, 2\}$ indicates that soliton 3 first interacted with soliton 2 (in its original, i.e., initial state); at the next (outer) level of nested braces, the second argument is 1, which indicates that the $\{3, 2\}$ soliton subsequently interacts with soliton 1 (which is in its initial state when the interaction occurs). Note that, when the three-soliton interaction takes place in this order, the interaction between 1 and 2 is the last pairwise interaction and has no effect on the final state of soliton 3.

In fact, Equation (25) is equivalent to (14), with $J = 3$ and $j = 3$, rewritten in the notation of this section. We conclude that, if the interaction proceeds according to (21), total phase shift of the fastest soliton is indeed the net result of iterated pairwise interactions with the remaining two solitons. This calculation extends immediately to the fastest soliton in a J-soliton interaction in which the fastest soliton passes through all the other $J-1$ solitons before any of the other solitons interact.

On the other hand, if the three-soliton interaction proceeds according to (22), the iteration of (23) and (24) yields

$$\mathbf{S}_{\{\{3,\{1,2\}\},\{2,1\}\}} = \mathbf{c}_{\{2,1\}}^{-1}(z_3)\mathbf{c}_{\{1,2\}}^{-1}(z_3)\ \mathbf{S}_3\mathbf{a}_2^{-1}(z_3)\mathbf{a}_1^{-1}(z_3) \tag{26}$$

where the subscripts are read as above. That is, again, the leftmost index 3 indicates that the matrix refers to the state of soliton 3. However, in this case, the two innermost pairs in the subscript on the left-hand side are $\{1, 2\}$ and $\{2, 1\}$, which indicates that the interaction between solitons 1 and 2 occurs before either of these solitons collides with soliton 3. Instead, the first interaction for soliton 3 is the interaction with soliton $\{1, 2\}$ (i.e., soliton 1 after its interaction with soliton 2. As a result of this interaction, soliton 3 will be in the state $\{3, \{1, 2\}\}$. Subsequently, as indicated by the nesting of the pairs, the soliton $\{3, \{1, 2\}\}$ interacts with soliton $\{2, 1\}$ (i.e., soliton 2 after its interaction with soliton 1).

The subscripts on the right-hand side of (26) are interpreted in the same manner. That is, the matrices

$$\mathbf{c}_{\{1,2\}}(z), \quad \mathbf{c}_{\{2,1\}}(z)$$

are, respectively, the transmission coefficients relative to soliton 1 and 2 after their mutual interaction and before either has interacted with soliton 3. We note that the right-hand side of (26) is not, a priori, equivalent to the right-hand side of (25), so another step is necessary to establish the order-independence of the three-soliton interaction.

The sequence (22) of pairwise interactions is equivalent to the sequence (21), at least for soliton 3, if

$$\mathbf{S}_{\{\{3,\{1,2\}\},\{2,1\}\}} = \mathbf{S}_{\{\{3,2\},1\}},$$

which, by comparison of the right-hand sides of (25) and (26), is equivalent to the condition

$$\mathbf{c}_2(z_3)\mathbf{c}_1(z_3) = \mathbf{c}_{\{1,2\}}(z_3)\mathbf{c}_{\{2,1\}}(z_3). \tag{27}$$

In fact, a direct calculation (in which the transmission coefficients are given in terms of the phases and the eigenvalues by (15) and (16)), shows that (27) holds. We conclude that the total phase shift of soliton 3 is independent of the order in which the pairwise interactions occur.

Similar direct calculations show that, for $J = 3$, both the total phase shift of the first soliton and the total phase shift of the second soliton are equivalent to the net result of pairwise interactions with the other two solitons. Also, the phase of each of these solitons in the forward long-time limit does not depend on the order of these pairwise interactions.

By a recursive argument, one can show that the order-independence of the three-soliton interaction is sufficient to imply the order-independence of the J-soliton interaction (cf., [5]). That is, in a J-soliton interaction, the total phase shift of each soliton is the net result of the composition of pairwise interactions with the other $J-1$ solitons. Moreover, the final phase of each soliton in the forward long-time limit is independent of the order of the $J(J-1)/2$ pairwise soliton interactions.

4. Discrete vector-soliton interactions as Yang–Baxter maps

4.1. Yang-Baxter equation

The quantum, or algebraic, Yang–Baxter equation is the relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{28}$$

where R is a linear operator on the tensor product of vector spaces, $V \otimes V$, and R_{ij} is a linear operator that acts on the ith and jth components of the threefold tensor product $V \otimes V \otimes V$.

Drinfeld [7] recast the idea of the Yang–Baxter equation as a relation among maps from the Cartesian product of an arbitrary set to itself. Following Drinfeld, we consider the map

$$R(x, y) = (f(x, y), g(x, y)),$$

from the Cartesian product $X \times X$ of a *set*, X, into itself (i.e., $f, g : X \times X \to X$ and, therefore, $R : X \times X \to X \times X$). Then, naturally, we define $R_{ij} : X^n \to X^n$, by the action of functions f and g on, respectively, the ith and jth elements of the n-fold Cartesian product. That is, for $i < j$,

$$R_{ij}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, f(x_i, x_j), x_{i+1}, \ldots, x_{j-1}, g(x_i, x_j), x_{j+1}, \ldots, x_n),$$

while, for $i > j$,

$$R_{ij}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{j-1}, g(x_i, x_j), x_{j+1}, \ldots, x_{i-1}, f(x_i, x_j), x_{i+1}, \ldots, x_n).$$

Following [21], we will call a function R a *Yang–Baxter map* if it satisfies the Yang–Baxter relation (28) where the multiplication is understood as composition and the equality is in the sense of maps from $X \times X \times X$ to itself. Moreover, we call a Yang–Baxter map *reversible* if

$$R_{21}R_{12} = I, \tag{29}$$

where I is the identity map.

In the following we find it useful to consider the Yang–Baxter relation with additional parameters

$$R_{12}[\lambda_1, \lambda_2]R_{13}[\lambda_1, \lambda_3]R_{23}[\lambda_2, \lambda_3] = R_{23}[\lambda_2, \lambda_3]R_{13}[\lambda_1, \lambda_3]R_{12}[\lambda_1, \lambda_2], \tag{30}$$

where R is a family of maps parameterized by $\lambda_j \in Y$ with Y being an arbitrary set not necessarily equivalent to X. The corresponding reversibility condition is

$$R_{21}[\mu, \lambda]R_{12}[\lambda, \mu] = I, \tag{31}$$

with λ, $\mu \in Y$. We note that a parameter-dependent Yang–Baxter map can, in fact, be written in parameter-independent form. That is, if we let

$$X' = X \times Y, \quad x' = (x, \lambda), \quad y' = (y, \mu), \quad R'(x', y') = R[\lambda, \mu](x, y),$$

then (30) is equivalent to (28) and, similarly, (31) is equivalent to (29). Nonetheless, in the example of vector–soliton interaction, it is useful to consider the parameter dependence explicitly.

Recall that, in the soliton collisions of the two-component IDVNLS, the polarization vectors of the colliding solitons are shifted by the interaction. This polarization shift is a mapping from $X \times X$ to itself where, in this case, X is the space of two-component, complex unit vectors. Thus, the polarization shift caused by soliton interaction is embodied in the parameter-dependent map

$$R[z_1, z_2] : \left(\mathbf{p}_1^-, \mathbf{p}_2^-\right) \to \left(\mathbf{p}_1^+, \mathbf{p}_2^+\right), \tag{32}$$

given by (18) and (19) where the eigenvalues z_1, z_2 are the parameters.

The question of whether the polarization shift of the solitons in the interaction of three vector solitons is independent of the order of the pairwise interactions is equivalent to the question of whether the map defined by (32) is indeed a Yang–Baxter map. For the mapping defined by (32), the left-hand side of (30) corresponds to the polarization shift in the case where soliton 1 and soliton 2 are the first to interact (i.e., as in (22)) while the right-hand side corresponds to the case where soliton 2 and soliton 3 are the first to interact (i.e., as in (21)). As noted in Section 3.3, the net polarization shifts in the three-soliton interaction are, in fact, independent of the order of the soliton interactions. Therefore, (32) is indeed a Yang–Baxter map. Moreover, because soliton interactions are reversible, (32) defines a reversible Yang–Baxter Map.

The relation

$$R_{ij}R_{kl} = R_{kl}R_{ij}, \tag{33}$$

where i, j, k, l are all unequal to one another is trivially satisfied by any Yang–Baxter map. In terms of soliton collisions, Equation (33) states that, given four solitons, it is irrelevant which of two pairwise collisions takes place first whenever these two collisions involve disjoint pairs of solitons. With (33) and the Yang–Baxter relation (30), the same recursive argument cited previously (cf., [5]), shows that the Yang–Baxter relation is sufficient to imply the order-independence of the J-soliton interaction [6].

4.2. Solution of the Yang–Baxter equation by matrix factorization

In this section, we use matrix factorization as an alternative method to show that the map (32) is a Yang–Baxter map.

First, we state the method for construction of a solution of the Yang–Baxter relation via matrix factorization. Let $\mathbf{A}\,(x, \lambda)$ be a given matrix-valued function that depends on $x \in X$, and the parameter $\lambda \in Y$. We define x^+ and y^+ to be elements of X such that

$$\mathbf{A}(y^+, \mu)\mathbf{A}(x^+, \lambda) = \mathbf{A}(x^-, \lambda)\mathbf{A}(y^-, \mu), \tag{34}$$

for given x^-, y^-. If (34) uniquely defines x^+, y^+ for each x^-, y^- and λ, μ, then the map

$$R[\lambda, \mu](x^-, y^-) = (x^+, y^+),$$

defined by this relation, satisfies the parameter-dependent Yang–Baxter relation (cf., [11]).

The *transfer* (transmission) matrix for a fundamental soliton of IDVNLS (cf., (16)) is a matrix-valued function of the soliton state, $\mathbf{s}_j$, and is parameterized by the eigenvalue z_j. More precisely, the transfer matrix is a function of the polarization $\mathbf{p}_j$ and the eigenvalue, z_j. Moreover, for any generic pair of fundamental solitons, there are two pairs of transfer matrices corresponding to the polarization states in the forward and backward long-time limits (i.e., $\mathbf{c}_1^{\pm}(z)$ and $\mathbf{c}_2^{\pm}(z)$).

Now, we consider the transfer matrices $\mathbf{c}_1^{\pm}(z)$, $\mathbf{c}_2^{\pm}(z)$ under the identifications

$$x^{\pm} \leftrightarrow \mathbf{p}_1^{\pm}, \quad \lambda \leftrightarrow z_1, \quad \mathbf{A}(x^{\pm}, \lambda) \leftrightarrow \mathbf{c}_1^{\pm}(z),$$
$$y^{\pm} \leftrightarrow \mathbf{p}_2^{\pm}, \quad \mu \leftrightarrow z_2, \quad \mathbf{A}(y^{\pm}, \mu) \leftrightarrow \mathbf{c}_2^{\pm}(z).$$

With these identifications, the matrix factorization (34) is equivalent to

$$\mathbf{c}_1^{+}(z)\mathbf{c}_2^{+}(z) = \mathbf{c}_2^{-}(z)\mathbf{c}_1^{-}(z).$$

In the notation of Section 3.3, this relation is precisely (27) with $z_3 = z$, which, as previously noted, can be verified by a direct calculation. We conclude that the polarization shift induced by vector-soliton interaction is indeed a Yang–Baxter map as it can be identified with a matrix factorization of the form (34).

The matrix factorization provides an alternative proof that the net polarization shift in the three-soliton interaction is in fact order independent: in a three-soliton interaction the net shift in the polarization vectors is the result of the composition of Yang–Baxter maps and, as we have noted, the Yang–Baxter relation is equivalent to the statement that the order of the pairwise action does not affect the polarizations in the forward long-time limit. We note that the direct verification of the relation (27) is at the heart of the proof of order-independence whether one considers matrix factorization or the argument in Section 3.3. The matrix factorization argument has the slight advantage that one need not repeat the verification for all three solitons.

5. Discrete vector-soliton interactions as fractional linear transformations

For the purpose of analyzing the results of soliton interactions, a fundamental soliton of the two-component IDVNLS is characterized by three complex numbers: the primary eigenvalue, z_j, and the two components of the polarization, $p_j^{(1)}$ and $p_j^{(2)}$. In effect, the three parameters: z_j, $p_j^{(1)}$, and $p_j^{(2)}$—characterize a two-component vector soliton where the location of the envelope peak is otherwise specified. In particular, as we are interested in the interaction of a pair of solitons moving at different speeds, we know that, regardless of their separation at any fixed time, the two solitons will collide at some time between the forward and backwards long-time limits. Moreover, the results of the interaction depend neither on the location at which the interaction occurs nor on the overall phase of the solitons. Hence, for each soliton, three complex parameters suffice to describe the interaction.

Further, as we have noted, the eigenvalue of a soliton is unchanged by collisions and, therefore, the internal "state" of a given soliton is determined by just the polarization vector. If we further neglect the overall complex phase of the soliton, its state is characterized by the single complex parameter

$$\rho_j = p_j^{(1)}/p_j^{(2)},$$

where we allow ρ_1 to be defined on the extended complex plane (cf., [12]).

In a two-soliton collision, we denote the respective soliton states before the interaction as ρ_1^- and ρ_2^-, respectively. Then, the collision transforms ρ_1^- to ρ_1^+ and ρ_2^- to ρ_2^+, where we define

$$\rho_2^{\pm} = \frac{p_2^{(1)\,\pm}}{p_2^{(2)\,\pm}}, \qquad \rho_1^{\pm} = \frac{p_1^{(1)\,\pm}}{p_1^{(2)\,\pm}}.$$

It turns out that the transformations of the soliton states (from ρ_j^- to ρ_j^+) can be written as fractional linear transformations (FLTs).

Without loss of generality, we assume, as before, that the soliton corresponding to z_2 is faster than the soliton corresponding to z_1. Then, the polarization shift formulas for the two-soliton collision, (18) and (19), can be rewritten in component form as

$$p_1^{(j)\,+} = \frac{1}{\chi} C_2^{(j,1)} p_1^{(1)\,-} + \frac{1}{\chi} C_2^{(j,2)} p_1^{(2)\,-} \tag{35}$$

$$p_2^{(j)\,+} = \frac{1}{\chi} C_1^{(j,1)} p_2^{(1)\,-} + \frac{1}{\chi} C_1^{(j,2)} p_2^{(2)\,-} \tag{36}$$

where $j = 1, 2$; χ is given by (20) and

$$C_2^{(j,\ell)} = \frac{\left(z_1^2 - z_2^{*\,-2}\right)\left(z_2^2 + z_1^{-2}\right)}{\left(z_1^2 - z_2^2\right)\left(z_2^{*-2} + z_1^{-2}\right)} \left(\delta_{j,\ell} + \frac{\left(z_2^{*\,-2} - z_2^2\right)\left(z_2^2 + z_2^{*2}\right)}{\left(z_2^2 - z_1^2\right)\left(z_2^2 + z_1^{-2}\right)} p_2^{(j)\,-}\left(p_2^{(\ell)\,-}\right)^* \right)$$

$$C_1^{(j,\ell)} = \frac{\left(z_1^2 - z_2^{*-2}\right)\left(z_1^{*\,-2} + z_2^{*\,2}\right)}{\left(z_1^{*\,-2} - z_2^{*-2}\right)\left(z_1^2 + z_2^{*2}\right)} \times \left(\delta_{j,\ell} + \frac{\left(z_1^2 - z_1^{*-2}\right)\left(z_1^{*-2} + z_1^{-2}\right)}{\left(z_1^{*-2} - z_2^{*-2}\right)\left(z_1^{*-2} + z_2^{*2}\right)} p_1^{(j)\,-}\left(p_1^{(\ell)\,-}\right)^* \right)$$

with $\delta_{j,\ell}$ the Kronecker delta. Note that the coefficients $C_2^{(j,\ell)}$ are independent of $p_1^{(1)-}$ and $p_1^{(2)-}$. Moreover, leaving aside χ, the transformation (35) is linear in $p_1^{(1)-}$ and $p_1^{(2)-}$. Similarly, the transformation (36) is linear in $p_2^{(1)-}$ and $p_2^{(2)-}$, again except for the coefficient χ.

By taking the ratio of the right-hand side of (35) with $j = 1$, to the same expression with $j = 2$, one cancels the nonlinear factor χ and obtains the FLT

$$\rho_1^+ = L\left[\rho_1^-, \rho_2^-\right] = \frac{A_2 \rho_1^- + B_2}{C_2 \rho_1^- + D_2}, \tag{37}$$

where the coefficients are functions of the soliton in state ρ_2^-:

$$A_2 = 1 + [1 + f(z_1, z_2)]\left|\rho_2^-\right|^2, \quad B_2 = f(z_1, z_2)\rho_2^-,$$
$$C_2 = f(z_1, z_2)\left(\rho_2^-\right)^*, \quad D_2 = 1 + f(z_1, z_2) + \left|\rho_2^-\right|^2,$$

with the function

$$f(z_1, z_2) = \frac{\left(z_2^{*\,-2} - z_2^2\right)\left(z_2^2 + z_2^{*\,2}\right)}{\left(z_2^2 - z_1^2\right)\left(z_2^2 + z_1^{-2}\right)},$$

depending only on the eigenvalues. The notation L in (37) indicates that soliton 1 is on the *left* after the soliton collision as it has a lower velocity (as determined by the relation between the eigenvalues z_1 and z_2). We remark that coefficients of (37) are not uniquely defined because an FLT is invariant if one multiplies the numerator and denominator by the same, nonzero, complex number. However, we have chosen a normalization that leaves the coefficients in a simple form.

Similarly, by taking the ratio of the left- and right-hand sides of (36) with $j = 1$ to the same expression with $j = 2$, one obtains the FLT

$$\rho_2^+ = R\left[\rho_1^-, \rho_2^-\right] = \frac{A_1\rho_2^- + B_1}{C_1\rho_2^- + D_1}, \tag{38}$$

where

$$A_1 = 1 + [1 + g(z_1, z_2)]\left|\rho_1^-\right|^2, \quad B_1 = g(z_1, z_2)\rho_1^-,$$
$$C_1 = g(z_1, z_2)\left(\rho_1^-\right)^*, \quad D_1 = 1 + g(z_1, z_2) + \left|\rho_1^-\right|^2,$$

with

$$g(z_1, z_2) = \frac{\left(z_1^2 - z_1^{*\,-2}\right)\left(z_1^{*\,-2} + z_1^{-2}\right)}{\left(z_1^{*\,-2} - z_2^{*\,-2}\right)\left(z_1^{*-2} + z_2^{*\,2}\right)}.$$

The notation R in (38) indicates that soliton 2 is on the *right* after the soliton collision.

We note that, just as for formulas (18) and (19), which describe the polarization shift, the FLTs (37) and (38) are not symmetric with respect to the interchange of ρ_1^- with ρ_2^- and z_1 with z_2. That is,

$$R\left[\rho_2^-, \rho_1^-\right] \neq L\left[\rho_1^-, \rho_2^-\right],$$

where z_1 and z_2 are also interchanged in (37) and (38). However, the symmetry

$$g(z_1, z_2) = f^*(z_2, z_1),$$

holds.

In the continuum limit, the FLTs (37) and (38) converge to the FLTs that describe the polarization shift for the solitons of VNLS (cf., [6, 12]). To obtain the FLT that gives the transformation of ρ_1 for the solitons of VNLS, we make the familiar change of variables $z_j = e^{-ik_j h}$ and let $h \to 0$. That is, in (37), we replace $f(z_1, z_2)$ with $\hat{f}(k_1, k_2)$ where

$$\hat{f}(k_1, k_2) = \lim_{h \to 0} f(e^{-ik_1 h}, e^{-ik_2 h}),$$

and similarly for $g(z_1, z_2)$.

A direct calculation shows that the FLTs must have nonzero determinants (i.e., $A_j D_j - B_j C_j \neq 0$ for $j = 1, 2$). Therefore, the transformations are invertible. Moreover, the unique inverse transformation is obtained by the substitution $\rho_j \to -1/\rho_j^*$ in (37) and (38). That is,

$$\rho_1^- = L\big[\rho_1^+, -\big(1/\rho_2^-\big)^*\big],$$

and, similarly,

$$\rho_2^- = R\big[-\big(1/\rho_1^-\big)^*, \rho_2^+\big].$$

The uniqueness of the inverses follows from the fact that the set of all invertible FLTs form a group [1].

Following [12], we refer to a pair of solitons with polarization states ρ_1 and $\rho_2 = -1/\rho_1^*$, respectively, as an *inverse pair*. If a third soliton, with state ρ_3, collides, sequentially, with the two solitons of an inverse pair, the state is unchanged by the net effect of the two collisions. That is, $\rho_3^+ = \rho_3^-$. As we discuss in Section 6, this property is especially useful in designing logical operators because any information encoded in ρ_3 is preserved even after it "operates" on the polarization states of the inverse pair.

Finally, we remark that one can use the FLT (37) and (38) to provide an alternative proof of the order-invariance of the polarization shift in the three-soliton interaction and, consequently, the order-invariance of the J-soliton interaction. Essentially, the proof is a direct calculation by iteration of the FLTs. In fact, such a proof is a direct adaptation of a proof of the order-invariance of the soliton interactions of VNLS (cf., [6] for details).

6. Computation with discrete vector-solitons

6.1. Computation and logic gates

Computation is generally conceptualized as a series of logical operations executed by a sequence of logic "gates." Each gate produces logical output values that depend on the values of its inputs. In turn, the outputs become the inputs of subsequent gates. In standard binary logic, the individual inputs and outputs of the gates are binary integers,"bits," in one of two states, typically identified as TRUE and FALSE or 1 and 0. To implement logical operations by

Table 1
One-input, One-output Binary Logic Gates. The Value of the Input Bit Is Denoted ρ_{in} and the Value of the Output Bit Is Denoted ρ_{out}

	$\rho_{\text{in}} = 1$	$\rho_{\text{in}} = 0$
COPY	$\rho_{\text{out}} = 1$	$\rho_{\text{out}} = 0$
NOT	$\rho_{\text{out}} = 0$	$\rho_{\text{out}} = 1$
ONE	$\rho_{\text{out}} = 1$	$\rho_{\text{out}} = 1$
ZERO	$\rho_{\text{out}} = 0$	$\rho_{\text{out}} = 0$

means of soliton collisions, we identify the complex polarization state, ρ, of each soliton as a bit where $\rho = 1$ encodes the value TRUE and $\rho = 0$ encodes the value FALSE.

A one-input, one-output logical gate computes the value of a single output bit (here encoded in a complex polarization state ρ_{out}) based on the value of a single input bit (encoded in a complex polarization state ρ_{in}). There are four possible one-input, one-output gates which embody, respectively the logical operations COPY, NOT, ONE, and ZERO. The actions of these gates are given in Table 1. Note that the ZERO gate can be obtained by the sequence of a ONE gate and a NOT gate (i.e., the output of a ONE gate becomes the input of a NOT gate), and need not be considered separately.

The one-input, one-output gates listed in Table 1 are, in themselves, insufficient for general logical computations. Instead, one needs (i) an appropriate two-input, one-output gate; (ii) the one-input, two output FANOUT gate that creates two copies of a single input; and (iii) a method of "wiring" to direct the outputs of "upstream" gates to the inputs of "downstream" gates. Indeed, these three elements are sufficient for a Turing-equivalent, general-purpose computer (cf. [16]).

There are a number of two-input, one-output gates that can be used to construct a general-purpose computer, one of which is the "not and" or NAND gate. Below, we describe how to construct the NAND gate with discrete vector-soliton interactions. The starting point in the construction of both the NAND gate and the FANOUT is the one-input, one-output gate.

The construction of the FANOUT and NAND gates described in the following sections follows the scheme of Steiglitz [16] that constructs these gates by the interaction of Manakov solitons (i.e., solutions of VNLS). We conclude that, like Manakov solitons, the *discrete* vector solitons of IDVNLS can, in principle, be configured as a general-purpose computer. As in the case of VNLS, this result is mathematical and, at this time, the feasibility of the implementation of such a scheme in a physical device remains an open problem.

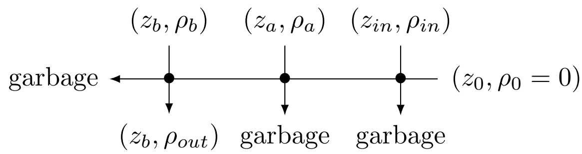

Figure 2. One-input, one-output, and three-soliton gate.

6.2. Implementation of one-input, one-output gates

The one-input, one-output COPY, NOT, and ONE operations can be implemented by a four-soliton, three-collision gate introduced in [16] for VNLS (illustrated in Figure 2). The soliton with eigenvalue and polarization state labeled *in* is the input of the gate and, before the interaction, has a logical value, either $\rho_{\text{in}} = 0$ or $\rho_{\text{in}} = 1$. The actuator, labeled with the subscript 0, is a soliton in polarization state $\rho = 0$ that collides, sequentially, with both the input soliton and the remaining two solitons of the gate. To implement the gate, the input, the actuator and the remaining two solitons (whose eigenvalues and polarization states before the interactions are labeled with subscripts a and b respectively) are arranged spatially so that that the collisions take place as illustrated in Figure 2. That is, only the actuator soliton collides with each of the remaining solitons, and does so in the order indicated by the figure. In particular, we choose the eigenvalues such that the actuator soliton is the "left" soliton after each of its three collisions. Finally, we must specify ρ_a and ρ_b so that the relation between the input state, ρ_{in} and the output state, ρ_{out}, corresponds to one of the logic operations.

To illustrate the working of the three-collision gate, we consider the logical operation COPY. To construct a COPY gate, we must determine the polarization states ρ_a and ρ_b that satisfy two equations: one that fixes $\rho_{\text{out}} = 1$ when $\rho_{\text{in}} = 1$ and a second that $\rho_{\text{out}} = 0$ when $\rho_{\text{in}} = 0$. Recall that the actuator soliton interacts first with the input, then with soliton a and finally with soliton b. Moreover, as noted, the actuator is always the "left" soliton after each of these collisions. Therefore, using the notation of (37) and (38), we can write the equations for the COPY gate as:

$$R[L[L[0, 0], \rho_a], \rho_b] = 0, \tag{39}$$

$$R[L[L[0, 1], \rho_a], \rho_b] = 1. \tag{40}$$

These rational, complex equations are the same as for the COPY gate of continuous vector solitons [6] except that the expressions for the coefficients f and g are different.

Although the system (39) and (40) is a system of two equations with two unknowns, it is not known whether a solution exists in general. We note that, in continuous case, there are parameter regimes for which a unique local solution exists (cf., [6]). Moreover, for a fixed choice of eigenvalues, the equations can be solved numerically. To simplify the numerical calculations, one can use (39) to solve for ρ_b in terms of ρ_a and then substitute this expression into (40). The resulting rational complex equation has only one unknown, ρ_a, which, for fixed eigenvalues, can be computed with a basic root-finding routine. Finally, ρ_a determines ρ_b by back-substitution. (See Table 2 for example values.)

It follows immediately from the definition of the NOT gate that the corresponding equations are

$$R[L[L[0,0],\rho_a],\rho_b] = 1,$$

$$R[L[L[0,1],\rho_a],\rho_b] = 0.$$

Similarly, the equations for the ONE gate are

$$R[L[L[0,0],\rho_a],\rho_b] = 1,$$

$$R[L[L[0,1],\rho_a],\rho_b] = 1.$$

Again, for fixed eigenvalues, both sets of rational complex equations can be solved numerically (see Table 2).

We note that, unlike the NOT and COPY gates, the ONE gate is not invertible because information is lost. Soliton interactions are, however, invertible. Therefore, the ONE operation could not be implemented by any gate in which the input and output are the polarization states of the same soliton. In the four-soliton gate described here, the input and output are, in fact, the polarization states of distinct solitons. The "garbage" outputs of the three-soliton gate embody the loss of information that makes the gate potentially noninvertible. For this reason, the four-soliton gate described here can be configured as a ONE gate as well as a COPY and NOT gate.

6.3. FANOUT gate and wiring

The FANOUT can be implemented by a modification of the COPY gate that restores the original polarization/logical state of the input soliton (see Figure 3). Specifically, a fifth soliton with the same speed and the inverse polarization of the actuator (i.e., a soliton corresponding to an octet with primary eigenvalue z_0 and polarization state $\rho = \infty$) will, upon collision with the soliton that serves as the input, precisely reverse the polarization shift of the input that was effected by the actuator soliton (cf., [16]). Thus, the input soliton emerges from the gate not as "garbage," but rather unchanged. Such a construction of a FANOUT gate for discrete, vector solitons is precisely the same as the construction in the continuous (i.e., Manakov) vector case.

Table 2
Polarization States of Solitons in One-input, One-output Gates with $k_0 = -1 + 7i$, $k_a = 1 + 7i$, $k_b = 1 + 7i$, $k_{\text{in}} = 1 + 7i$ in VNLS, and $z_j = e^{-ihk_j}$ in IDVNLS

Gate	IDVNLS, $h = .1$		VNLS	
	ρ_a	ρ_b	ρ_a	ρ_b
COPY	$-0.453021 + 0.120541i$	$0.137265 - 1.30982i$	$-0.420097 + 0.0896611i$	$0.731203 - 1.55794i$
NOT	$-0.485589 - 1.04269i$	$0.930712 + 0.658363i$	$-0.237235 - 1.03717i$	$0.596457 + 0.990249i$
ONE	$1.40827 - 0.671727i$	$-0.368282 + 1.53988i$	$1.46562 - 0.556233i$	$-0.595714 + 1.07481i$
a-CONV	$0.565104 + 1.01487i$	$-0.252996 - 0.50047i$	$0.421026 + 0.705305i$	$-0.48767 - 0.799749i$
b-CONV	$2.20435 + 0.200956i$	$-0.676611 + 0.573384i$	$1.81413 + 1.04661i$	$-0.722064 + 0.0452077i$

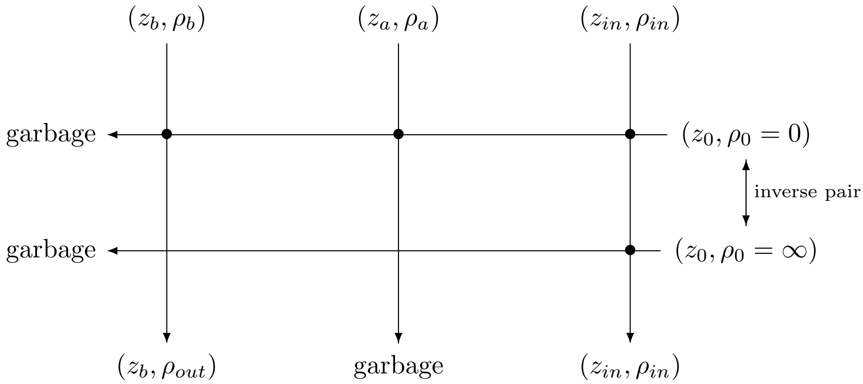

Figure 3. FANOUT gate: Five-soliton four-collision gate. Collisions are indicated by a bullet.

We note that the above-described implementation of FANOUT requires "wire-crossing" in the sense that the fifth soliton must not interact with the soliton "*b*" even as it travels at the same speed as the actuator that collides all the three data solitons ("*a*", "*b*", and "*in*"). However, such control of the soliton intersections is possible if the solitons can be "time-gated," i.e. localized in both time and space (see [16]).

The connection between gates ("wiring") can be accomplished by the implementation of gates with a second speed [16]. Parameters for the one-input, one-output gates constructed with solitons moving at a second speed are given in Table 3. The Galilean symmetry of IDVNLS is more complex than that of VNLS. Nonetheless, the implementation of gates with a second set of speeds suggests that the method used for wiring with Manakov solitons also applies to discrete vector solitons.

6.4. NAND gate

The NAND gate operates on two input solitons and encodes a single output soliton (cf. Table 4).

This gate can be built from a cascaded combination of a FANOUT gate and one-input, one-output gates of the configuration in Figure 2 [16]. In the NAND gate, the first of the two input solitons is directed to a FANOUT, which creates two copies of this first input. The two copies of the input soliton are inputs, respectively, to a pair of parallel one-input, one-output "converter" gates (described below). The outputs of these intermediate gates cascade become the "*a*" and "*b*" solitons of a final one-input, one-output gate whose input soliton (i.e., the "*in*" soliton) is the second input of the NAND gate. The single output soliton of this final gate is the output of the NAND gate (see Figure 4).

One of the two intermediate converter gates converts a logical input (i.e., $\rho_{\rm in} = 0$ or $\rho_{\rm in} = 1$) to the polarization state corresponding to the *a* soliton of a

Table 3
Polarization States of Solitons in One-input, One-output Gates with $k_0 = -1 + 8i$, $k_a = 1 + 8i$, $k_b = 1 + 8i$, $k_{\text{in}} = 1 + 8i$ in VNLS, and $z_j = e^{-ihk_j}$ in IDVNLS

Gate	IDVNLS, $h = .1$		VNLS	
	ρ_a	ρ_b	ρ_a	ρ_b
COPY	$-0.446926 + 0.118085i$	$0.172705 - 1.33931i$	$-0.417811 + 0.0806951i$	$0.940322 - 1.56709i$
NOT	$-0.455302 - 1.04467i$	$0.903275 + 0.703102i$	$-0.196312 - 1.03168i$	$0.514188 + 1.02898i$
ONE	$1.41916 - 0.662805i$	$-0.396825 + 1.4881i$	$1.46043 - 0.522696i$	$-0.635376 + 0.984983i$
a-CONV	$0.544087 + 0.965607i$	$-0.280154 - 0.530536i$	$0.394356 + 0.66162i$	$-0.536982 - 0.878172i$
b-CONV	$2.17428 + 0.302488i$	$-0.699395 + 0.516938i$	$1.7103 + 1.18825i$	$-0.687345 - 0.0512793i$

Table 4
Two-input ($\rho_{\text{in}}^{(1)}$, $\rho_{\text{in}}^{(2)}$) - One-output (ρ_{out}) NAND Gate. The Output Is 1, Unless both Inputs Are 1

NAND Gate		
$\rho_{\text{in}}^{(1)} = 0$	$\rho_{\text{in}}^{(2)} = 0$	$\rho_{\text{out}} = 1$
$\rho_{\text{in}}^{(1)} = 0$	$\rho_{\text{in}}^{(2)} = 1$	$\rho_{\text{out}} = 1$
$\rho_{\text{in}}^{(1)} = 1$	$\rho_{\text{in}}^{(2)} = 0$	$\rho_{\text{out}} = 1$
$\rho_{\text{in}}^{(1)} = 1$	$\rho_{\text{in}}^{(2)} = 1$	$\rho_{\text{out}} = 0$

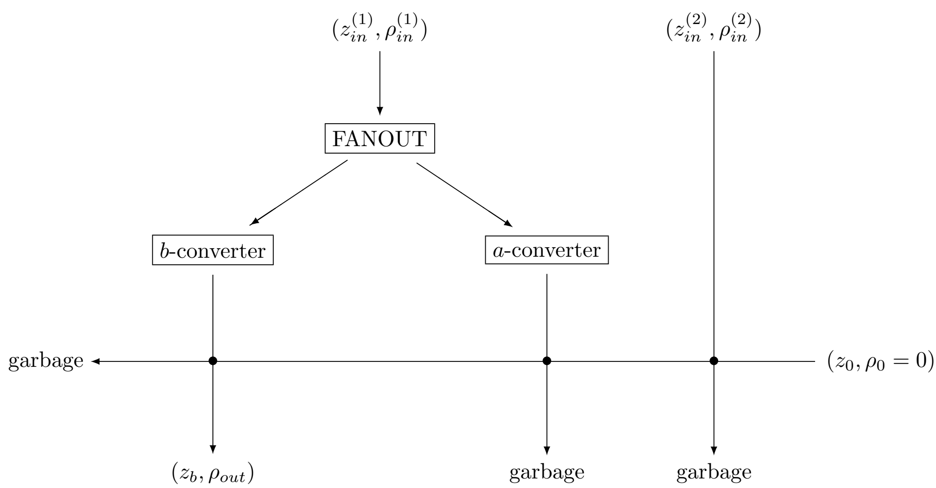

Figure 4. NAND gate. The inputs are encoded in $\rho_{\text{in}}^{(1)}$ and $\rho_{\text{in}}^{(2)}$. The output is encoded in ρ_{out}.

ONE or NOT gate (according to the logical value of the input state). Adapting the terminology of [16], we refer to such a gate, that takes $\rho_{\text{in}} = 0$ to ρ_a of a ONE gate $\rho_{\text{in}} = 1$ to $\rho_{\text{out}} = \rho_a$ of a NOT gate, as an a-converter. Similarly, the second (parallel) intermediate gate is the b-converter that maps $\rho_{\text{in}} = 0$ to $\rho_{\text{out}} = \rho_b$ of a ONE gate and $\rho_{\text{in}} = 1$ to ρ_b of a NOT gate. Just as for the other four, one-input, one-output gates, the polarization states of the "a" and "b" solitons of the converter gates are determined by the solution of a system of two complex rational equations. For example, for an a-converter, the necessary polarization states, $\tilde{\rho}_a$ and $\tilde{\rho}_b$, are solutions of the equations

$$R[L[L[0, 0], \tilde{\rho}_a], \tilde{\rho}_b] = \rho_a^{\text{ONE}},$$

$$R[L[L[0,1],\tilde{\rho}_a],\tilde{\rho}_b]=\rho_a^{\mathrm{NOT}}$$

where, for a specified eigenvalue, ρ_a^{ONE} is the polarization state that characterizes soliton "a" of a ONE gate and ρ_a^{NOT} is the polarization state that characterizes soliton "a" of a NOT gate. As for the other one-input, one-output gates, the polarization states ρ_a and ρ_b can be computed numerically for fixed eigenvalues (see Tables 2 and 3).

Acknowledgment

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Appendix: Moving breathers in IDNLS

The parallel composite soliton described in Section 2.3.3 is of the form

$$\mathbf{Q}_n=\mathbf{p}Q_n,$$

where $\mathbf{p}$ is a constant unit vector (the polarization) and Q_n is a scalar function. Therefore, if we drop the polarization vector, the remaining scalar function is a moving, breathing solution of the scalar IDNLS (5). This moving-breather solution of the *scalar* IDNLS, has not been described previously in the literature. Like the composite solitons of IDVNLS, this moving and breathing solution has no counterpart in the PDE continuum limit, i.e., scalar NLS. In fact, in the scalar case (IDNLS), there is a more general class of moving, breathing solutions that includes the scalar reduction of the parallel composite soliton as a special case.

Whether considered as a solution of IDNLS (scalar) or IDVNLS (vector), a parallel composite soliton of IDVNLS is associated with the same eigenvalue octet. However, in the scattering problem associated with IDNLS, the symmetries dictate only that the eigenvalues appear in quartets (cf., e.g., [5]). (There is one less symmetry in the scattering problem associated with IDNLS.) Therefore, considered as a solution of IDNLS, the parallel composite-soliton solution of Section 2.3.3 does *not* have a minimal spectrum. Instead, because it is associated with two quartets, the solution constitutes a moving and breathing two-soliton solution of IDNLS in which the solitons do not separate spatially in the long-time limits. These two constituent solitons have the same envelope amplitude and velocity. We note that, by contrast, in NLS it is not possible to have distinct solitons that have both the same envelope velocity and the same envelope amplitude.

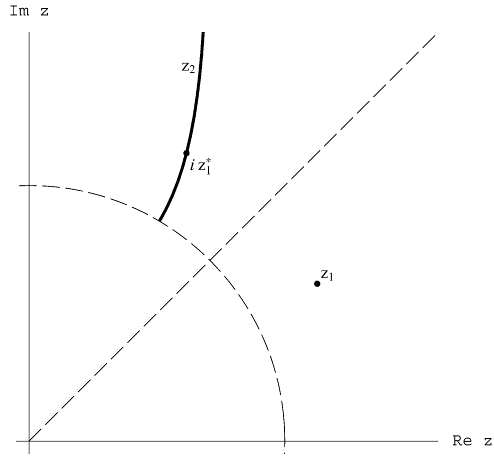

Figure A.1. Relation between the primary eigenvalues in a moving, breathing solution of IDNLS that has no counterpart in the continuum limit. The dashed lines are $|z| = 1$ and $\mathrm{Arg}\, z = \frac{\pi}{4}$ in the complex z-plane. The relation (A.1) holds if z_2 is anywhere on the thick line. The case $z_2 = iz_1^*$ corresponds to the spectrum associated with the parallel-composite-soliton solution of IDVNLS.

The more general moving, breathing states of IDNLS are also associated with a spectrum containing two quartets. The quartets are each defined by a primary eigenvalue, i.e.,

$$z_1 = e^{a_1+ib_1}, \qquad z_2 = e^{a_2+ib_2},$$

respectively, such that

$$\frac{\sinh 2a_1 \sin 2b_1}{a_1} = \frac{\sinh 2a_2 \sin 2b_2}{a_2}. \tag{A.1}$$

The freedom of z_2 for a fixed z_1 is illustrated in Figure A.1. The relation (A.1) ensures that the two solitons (each associated with one of the quartets) have the same envelope velocity and, therefore, do not separate spatially in the long-time limits. The parallel composite solitons are the special case in which $a_2 = a_1$ and $b_2 = \frac{\pi}{2} - b_1$ and, consequently, the two quartets are aligned as an octet of the type associated with the solitons of IDVNLS.

To explicitly compute the formula for a moving, breathing state whose eigenvalues satisfy (A.1), one can, for example, explicitly solve the inverse problem in the IST (cf., e.g., [5]). The resulting solution formula is itself long and complex, but can be obtained with the help of a computer algebra system. Moreover, the analytic solution can be plotted at fixed times to illustrate the dynamics (see Figure A.2). The terms of the formula, which are unbounded as $n, \tau \to \infty$ are exponentials of linear phases of the form (12) with $a = a_1$ and $b = b_1$. The periodic τ- and n-dependence of the solution is contained in

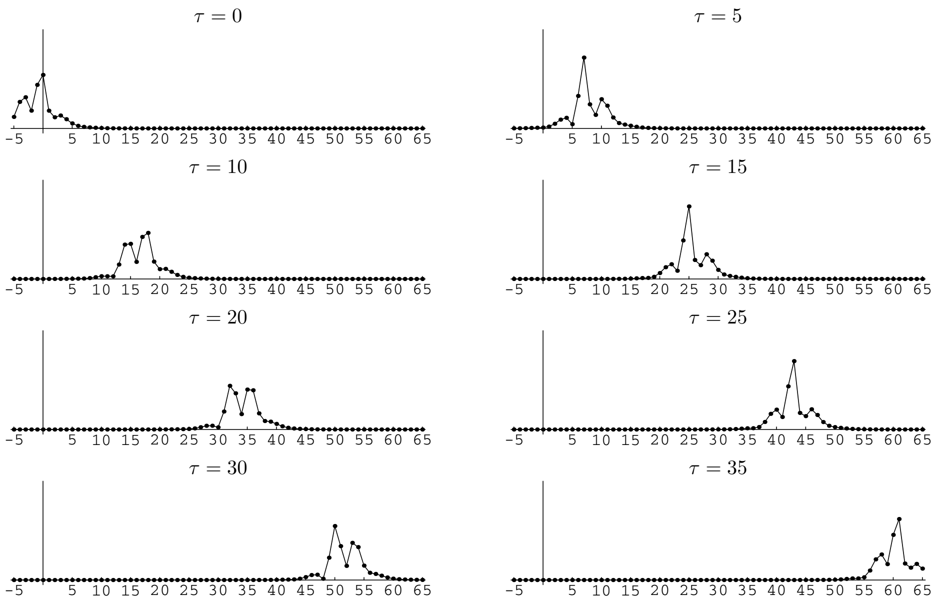

Figure A.2. Moving and breathing two-octet solution of IDNLS with $a_1 = 0.25$, $b_1 = 0.5$, $C_1(0) = 1$, $a_2 = 0.4$, $b_2 = 1.115$, $C_2(0) = 1$. This solution does not have a counterpart in the continuum limit.

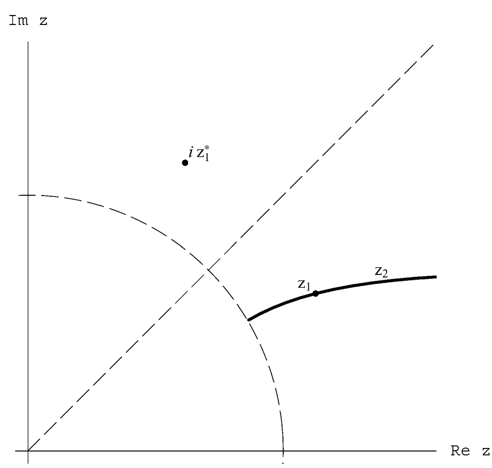

Figure A.3. Relation between the primary eigenvalues in a moving, breathing solution of IDNLS that *has* a counter part in the continuum limit. The dashed lines are $|z| = 1$ and $\operatorname{Arg} z = \frac{\pi}{4}$ in the complex z-plane.

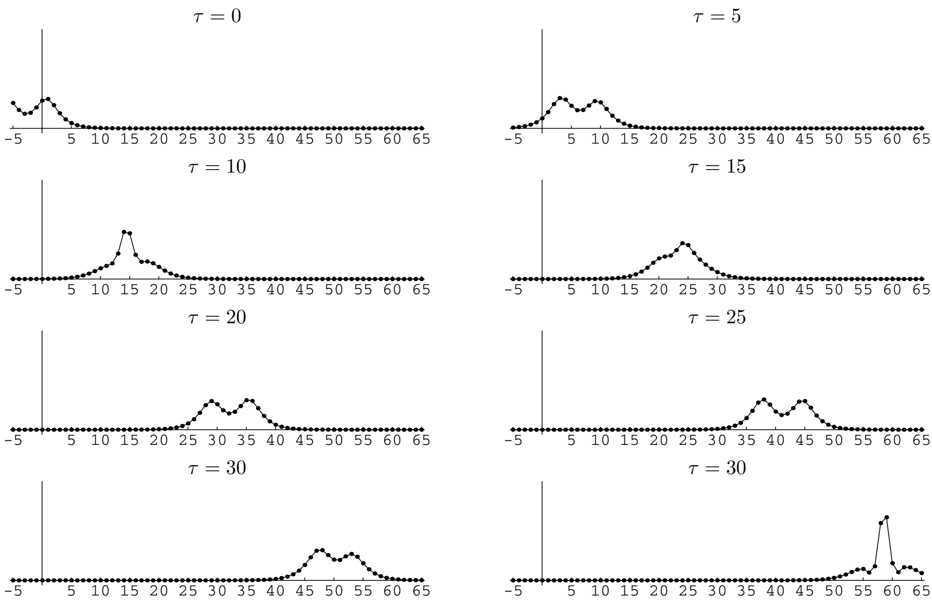

Figure A.4. Moving and breathing two-octet solution of IDNLS with $a_1 = 0.25$, $b_1 = 0.5$, $C_1(0) = 1$, $a_2 = 0.4$, $b_2 = 0.45531$, $C_2(0) = 1$. This solution has a counterpart in the continuum limit.

complex exponentials of phases of the form (13), with $a = a_1$ and $b = b_1$ or $a = a_2$ and $b = b_2$. As is typical for such solutions, the real exponentials yield the spatially localized traveling envelope and the complex exponentials with distinct phases generate the "breathing" of the envelope as it travels.

In fact, for a given z_1, there are two, one-parameter families of solutions of (A.1). The solution curve that comprises the second set of possible primary eigenvalues passes through z_1, as illustrated in Figure A.3. If z_2 is on this second curve, the corresponding solution is also a breathing two-soliton state. The second solution (cf. Figure A.4), however, is analogous to the two-soliton bound state in NLS in which there are two solitons whose respective associated eigenvalues have the same real part (and hence the same envelope speed), but different imaginary parts (and hence different envelope amplitudes). Therefore, unlike the solution whose spectrum falls on the curve in Figure A.1, this second solution has a (known) counterpart in the continuum limit (i.e., NLS).

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