

# Four-wave mixing in wavelength-division-multiplexed soliton systems: damping and amplification

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Four-wave mixing in wavelength-division-multiplexed soliton systems with damping and amplification is studied. An analytical model is introduced that explains the dramatic growth of the four-wave terms. The model yields a resonance condition relating the soliton frequency and the amplifier distance. It correctly predicts all essential features regarding the resonant growth of the four-wave contributions. © 1996 Optical Society of America

In realistic long-distance fiber-optic communication systems damping effects are significant. Use of erbium-doped fiber amplifiers to overcome damping has made soliton communication systems feasible. The problem that we consider here is wavelength-division-multiplexed soliton systems in the presence of both damping and amplification. This case differs from that of the ideal fiber, in which it can be demonstrated analytically<sup>1</sup> that four-wave interaction terms grow from a vanishing background, become nonnegligible for a period of time, and then decay.

The practical relevance of the growth of four-wave mixing (FWM) in fiber-optic soliton systems was discussed in Ref. 2 mainly through numerical simulations. In this Letter we develop a suitable analytical model to explain this growth. The model predicts resonance at frequency locations that are commensurate with a given amplifier spacing. We discuss a number of different situations: (i) In the general case we solve the model both analytically and numerically and obtain the growth and saturation of the FWM terms. (ii) As a special case we discuss the pure resonance situation, in which we assume that the solitons provide a slowly varying background. This yields resonant amplifier distances consistent with maximum values of the amplitude of the FWM terms, in agreement with the values discussed in Ref. 2. (iii) We show that the analytical results of the model are in complete agreement with direct numerical simulations of the full system. It is clear that FWM phenomena in soliton systems (even beyond those considered in fiber optics) are extremely important. In fiber-optic systems the resonant growth of FWM terms can cause interference jitter, thereby increasing bit-error rates. This Letter clarifies the interplay between two building blocks associated with nonlinear

systems: solitons and FWM<sup>3,4</sup> in the presence of amplification and damping.

The relevant wave equation is the nonlinear Schrödinger (NLS) equation with damping and periodic amplification:

$$iq_z + \frac{1}{2}q_{tt} + |q|^2q = -i\Gamma q + i[\exp(\Gamma z_a) - 1] \sum_{n=1}^N \delta(z - nz_a)q, \quad (1)$$

where  $\Gamma$  is the normalized loss coefficient,  $z_a$  is the normalized characteristic amplifier spacing, and  $z$  and  $t$  are the normalized propagation distance and the normalized retarded time, respectively, expressed in the usual nondimensional units. We take into account<sup>5,6</sup> the loss and amplification cycles by looking for a solution of the form  $q(z, t) = A(z)u(z, t)$ , with  $A$  real. Taking  $A$  to satisfy

$$A_z + \Gamma A - [\exp(\Gamma z_a) - 1] \sum_{n=1}^N \delta(z - nz_a)A = 0, \quad (2)$$

we show that Eq. (1) becomes

$$iu_z + \frac{1}{2}u_{tt} + g(z)|u|^2u = 0, \quad (3)$$

where  $g(z) = A^2(z) = a_0^2 \exp[-2\Gamma(z - nz_a)]$  for  $nz_a \leq z < (n+1)z_a$ ,  $n \geq 0$  and  $a_0 = \{2\Gamma z_a / [1 - \exp(-2\Gamma z_a)]\}^{1/2}$ , so that  $\langle g(z) \rangle = 1$  over each amplification period.

In the ideal case [ $g(z) \equiv 1$ ]<sup>1</sup> it is possible to write  $u = u_s + u_{\text{FWM}} + O(\epsilon^3)$ , where  $\epsilon$ , defined below, is a small parameter corresponding to widely spaced wavelength-division-multiplexed channel frequencies;  $u_s$  represents the  $O(1)$  soliton contributions [possibly

corrected by the  $O(\epsilon)$  frequency shift during the collision], and  $u_{\text{FWM}} \sim O(\epsilon^2)$  represents the FWM terms. When  $N = 2$  we have simply  $u_s = u_1 + u_2$ , where  $u_1$  and  $u_2$  are localized in frequency around  $\Omega_1$  and  $\Omega_2$ . In this case the four-wave contribution  $u_{\text{FWM}}$  consists of two separate terms localized around the FWM frequencies  $\Omega_{112} = 2\Omega_1 - \Omega_2$  and  $\Omega_{221} = 2\Omega_2 - \Omega_1$ . That is, we decompose  $u_{\text{FWM}}$  as  $u_{\text{FWM}} = u_{112} + u_{221}$ .

Figure 1 shows a numerical simulation of a typical two-soliton collision with  $\Gamma = 10$ ,  $z_a = 0.2$ , and  $\Omega_1 = -\Omega_2 = -3$ , and Fig. 2 displays the corresponding frequency spectrum. It is evident that the amplification process induces instability and growth of the FWM terms whose amplitude is an order of magnitude larger than that in the unperturbed case (see also Ref. 2), and the process saturates after the collision is completed. Also, the interplay between the four-wave components and the amplification process produces a rich structure of secondary maxima in the frequency spectrum.

The resonant growth of the four-wave products can be analytically explained by the same basic decomposition as that of the ideal case. That is, we substitute  $u \sim u_s + u_{\text{FWM}}$  into Eq. (3) and look only for terms that are located in frequency near, say,  $\Omega_{221}$  (that is, we discard terms located near the frequency channels of the solitons and the remaining FWM term). In this way, neglecting small terms, we obtain a model equation for the growth of FWM terms (or, more explicitly,  $u_{221}$ ) in the presence of damping and amplification:

$$iu_z + \frac{1}{2}u_{tt} + g(z)u_2^2u_1^* = 0, \quad (4)$$

where for simplicity  $u_{221}$  is now called  $u$ . To the leading order we replace  $u_1$  and  $u_2$  with the one-soliton solution of the NLS equation:  $u_j(z, t) = A_j \exp[i\chi_j(z, t)] \text{sech } S_j(z, t)$ , where  $S_j(z, t) = A(t - T_j - \Omega_j z)$  and  $\chi_j(z, t) = \Omega_j t - (\Omega_j^2 - A_j^2)z/2$ . We consider the physically relevant case  $A_1 = A_2 = 1$  and  $\Omega_2 = -\Omega_1 \equiv \Omega/2 > 0$ , so that  $\Omega$  is the frequency-channel separation and  $\epsilon = 2/\Omega$ . Also, we set  $T_1 = T_2 = 0$  so that the collision between the two solitons is located at  $z = 0$  [in an improved model we also use the  $O(\epsilon)$  corrections to the soliton channels; cf. Eq. (4) of Ref. 7].

It is convenient to write the 221 FWM term as  $u(z, t) = F(z, t) \exp[-i(k_4 z - \omega_4 t)]$ , where  $k_4$  and  $\omega_4$  are the characteristic parameters of the FWM contribution:  $\omega_4 \equiv \Omega_{221} = 2\Omega_2 - \Omega_1 = 3\Omega/2$  and  $k_4 \equiv (2\Omega_2^2 - \Omega_1^2)/2 = 1/8\Omega^2$ . Introducing the above ansatz into Eq. (4) yields

$$iF_z + \frac{1}{2}F_{tt} + i\omega_4 F_t - \left(\frac{1}{2}\omega_4^2 - k_4\right)F + g(z) \times u_2^2 u_1^* \exp[i(k_4 z - \omega_4 t)] = 0. \quad (5)$$

As a first approximation we neglect all derivatives in Eq. (5), which corresponds to assuming that  $F(z, t)$  is a slowly varying envelope of  $u(z, t)$ . In the ideal case [if  $g(z) \equiv 1$ ] this allows us to obtain immediately the result that  $F(z, t) = u_2^2(z, t)u_1^*(z, t)/\Omega^2$ , which reproduces the four-wave contributions coming from the asymptotic expansion of the two-soliton solution of the NLS equation.<sup>1</sup>

If  $g(z)$  is not constant  $F$  does indeed have significant spatial modulations, and, in the slowly varying envelope approximation, we have

$$iF_z - \left(\frac{1}{2}\omega_4^2 - k_4\right)F = -g(z)u_2^2 u_1^* \exp[i(k_4 z - \omega_4 t)]. \quad (6)$$

The essence of the ordinary differential equation (6) is represented by the relation between the frequency of free oscillations,  $\omega_4^2/2$ , and the frequency of the forced oscillations that are due to the presence of  $g(z)u_2^2 u_1^* \exp[i(k_4 z - \omega_4 t)]$ . In particular, a resonance is reached whenever the frequency of the forcing terms coincides with  $\Omega^2$  (i.e., when there is phase matching). More precisely, we expand  $g(z)$  in Fourier series:  $g(z) = \sum_{n=-\infty}^{+\infty} g_n \exp(-ink_a z)$ , where  $k_a = 2\pi/z_a$ . The oscillations of  $u_2^2 u_1^*$  are almost exactly canceled by  $\exp[i(k_4 z - \omega_4 t)]$ , since  $\arg[u_2^2 u_1^*] = 2\chi_2 - \chi_1 = \omega_4 t - (k_4 - 1/2)z$ . Hence the resonance condition is verified when  $nk_a - 1/2 = \Omega^2$ . For a fixed value of  $\Omega$  the previous condition determines the values of the amplifier distance that produce resonance:

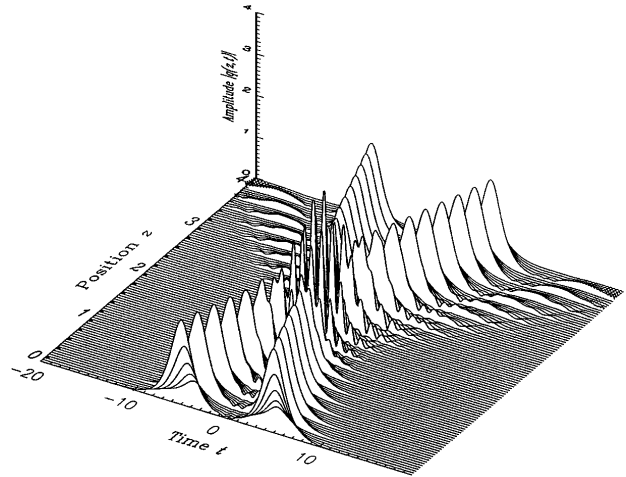


Fig. 1. Two-soliton collision in the presence of damping and periodic amplification.  $\Gamma = 10$ ,  $z_a = 0.2$ ,  $\Omega_1 = -\Omega_2 = -3$ ,  $A_1 = A_2 = 1$ , and  $T_1 = -T_2 = 5$ .

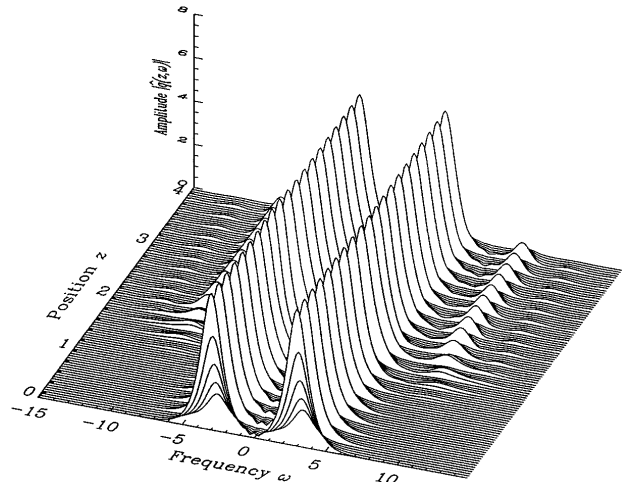


Fig. 2. Fourier spectrum relative to the collision shown in Fig. 1.

$$z_a = 4n\pi/(2\Omega^2 + 1). \quad (7)$$

Equation (7) is essentially the resonance condition that appears in Ref. 2 and agrees with the results obtained by numerical simulations, which are discussed below.

Now we turn our attention to the full model. Equation (5) can be solved exactly by use of Fourier transforms. We define  $\hat{F}(z, \omega) \equiv \mathcal{F}_\omega[F(z, t)] = \int_{-\infty}^{+\infty} dt \exp(-i\omega t)F(z, t)$ , so that

$$i\hat{F}_z - \left[ \frac{1}{2}(\omega + \omega_4)^2 - k_4 \right] \hat{F} + g(z)\exp(ik_4 z) \mathcal{F}_\omega[u_2^2(z, t)u_1^*(z, t)\exp(-i\omega_4 t)] = 0. \quad (8)$$

The Fourier coefficients of  $g(z)$  are given by  $g_n = a_0^2 z_a^{-1} \int_0^{z_a} dz \exp(ink_a z - 2\Gamma z) = \Gamma z_a / (\Gamma z_a - n\pi i)$ . Also, we use  $\mathcal{F}_\omega[\text{sech}^2 t \text{sech}(t + \Delta)] = \pi \text{sech}(\pi\omega/2) \times I(\Delta, \omega)$ , where  $I(\Delta, \omega) = [\cosh \Delta + i\omega \sinh \Delta - \exp i\omega\Delta]/\sinh^2 \Delta$ , and solve Eq. (8) to obtain (after the change of variable  $\zeta = \Omega z$ )

$$\hat{F}(z, \omega) = \frac{i\pi}{\Omega} \text{sech}(\pi\omega/2) \exp\left[-\frac{1}{2}(\omega^2 + 3\Omega + 2\Omega^2)z\right] \times \sum_{n=-\infty}^{+\infty} g_n \int_{-\infty}^{\Omega z} d\zeta \exp[i\gamma_n(\omega)\zeta] I(\zeta, \omega), \quad (9)$$

where  $\gamma_n(\omega) = (\omega^2 + 2\Omega\omega + 2\Omega^2 - 2nk_a + 1)/2\Omega$ .

As  $z \rightarrow \infty$ , for large  $\Omega$  we expect the major contribution to  $\hat{F}$  to come from the vicinity of  $\omega_n$  where  $\omega_n$  denotes the zeros of  $\gamma_n(\omega)$ :  $\omega_n^\pm = -\Omega \pm (2nk_a - \Omega^2 - 1)^{1/2}$ . In fact this condition (which is the general phase-matching condition) gives a good approximation of the location of the maxima and their dependence on the amplifier distance.

Because of the  $\text{sech}(\pi\omega/2)$  in front of the integral, we expect the value of  $|\hat{F}|$  to be exponentially small unless  $\omega_n^\pm \approx 0$ , which implies that the only relevant root is  $\omega_n^+$ . Conversely, if  $k_a$  and  $\Omega$  are such that  $\omega_n^+ = 0$  this approach reproduces the resonance condition previously obtained in the slowly varying amplitude approximation [Eq. (7)]. The resonance is reached when the maximum of  $|\hat{F}|$  is located at  $\omega \approx 0$ , that is, when the carrier frequency of  $u_{221}$  coincides asymptotically with the frequency of the unperturbed case (i.e.,  $\Omega_{221}$ ).

When  $\omega_n^\pm \neq 0$  the actual maximum of  $\hat{F}$  is slightly displaced from  $\omega_n^\pm$ , as a result of the role played by the  $\text{sech}(\pi\omega/2)$ . To compute this deviation we must analyze the structure of  $\hat{F}$ . When  $z \rightarrow \infty$  the integral on the right-hand side of Eq. (9) is equivalent to a Fourier transform. In fact, such a transform can also be calculated exactly:  $\mathcal{F}_\alpha[I(\zeta, \omega)] = \pi(\alpha + \omega)\cosh(\tau\omega/2)/\sinh[\pi(\alpha + \omega)/2]\cosh(\pi\alpha/2)$ . Equation (9) then yields

$$\hat{F}(z, \omega) \underset{z \rightarrow \infty}{\sim} \frac{i\pi^2}{\Omega} \exp[-1/2(\omega^2 + 3\Omega\omega + 2\Omega^2)z] \times \sum_{n=-\infty}^{+\infty} n \frac{g_n[\gamma_n(\omega) + \omega]}{\sinh\{1/2\pi[\gamma_n(\omega) + \omega]\}\cosh[1/2\pi\gamma_n(\omega)]}. \quad (10)$$

Given  $\Omega$  and  $k_a$ , we use expression (10) to look for the maxima of  $|\hat{F}|$  and their frequency location. The corresponding predictions for  $\Omega = 12$  are compared in Fig. 3 with the results coming from numerical

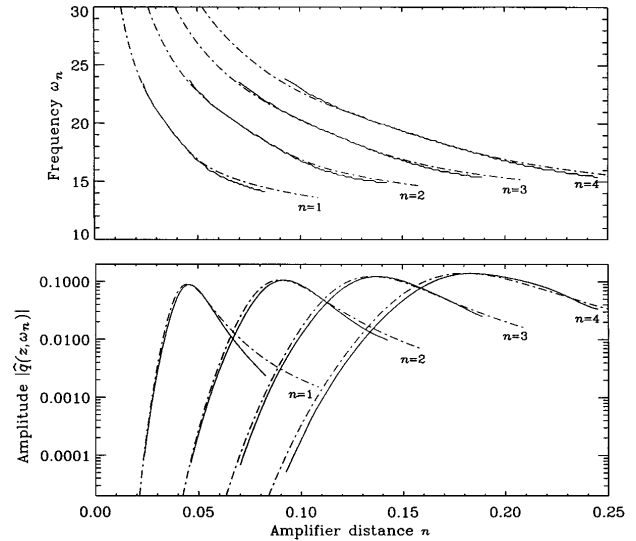


Fig. 3. Amplitude and frequency of the secondary maxima of  $|\hat{q}|$  for  $\Omega = 12$  as a function of the amplifier distance. Solid curves, numerical results from Eq. (1); dashed curves, theoretical predictions from expression (10).

integration of the full NLS equation with damping and amplification [Eq. (1)]. The agreement between the predicted amplitudes and frequencies of the maxima and the values computed numerically is remarkable.

The phenomenon of resonant growth of FWM terms in the presence of periodic amplification raises a number of important questions. For example, how are multisoliton interactions affected by FWM? Preliminary numerical results of three-soliton collisions indicate that the presence of nonnegligible FWM in the same frequency channel as a soliton can cause significant interference problems. Also, investigations of collisions between solitons and the resonant FWM products must be undertaken, and in particular whether these collisions can significantly modify relevant soliton parameters needs to be studied. Finally, it is not clear *a priori* what the resulting effect of introducing filters in the system is and how the presence of filters modifies the production and the growth of FWM terms. We plan to investigate these problems.

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