

**Magnetic field, Lorentz forces and stored energy of multipole  $\cos(n\theta)$   
and sector magnets with and without iron yoke**

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**Abstract**

When approaching the design of a multipole magnet, such as a dipole, quadrupole, sextupole, and so on, it is highly advantageous to initiate the process by establishing the fundamental parameters. These parameters include conductor size, current density, inner and outer radius of the iron yoke, and more. This preliminary dimensioning enables the acquisition of the necessary specifications for the design. Within this report, analytical expressions for the magnetic field, Lorentz forces, and stored energy of multipole magnets with the  $\cos(n\theta)$  and sector coil configurations, both with and without the presence of an iron yoke, are derived. These derivations are based on the vector potential of a current line.

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## 1 Multipole $\cos(n\theta)$ without iron yoke

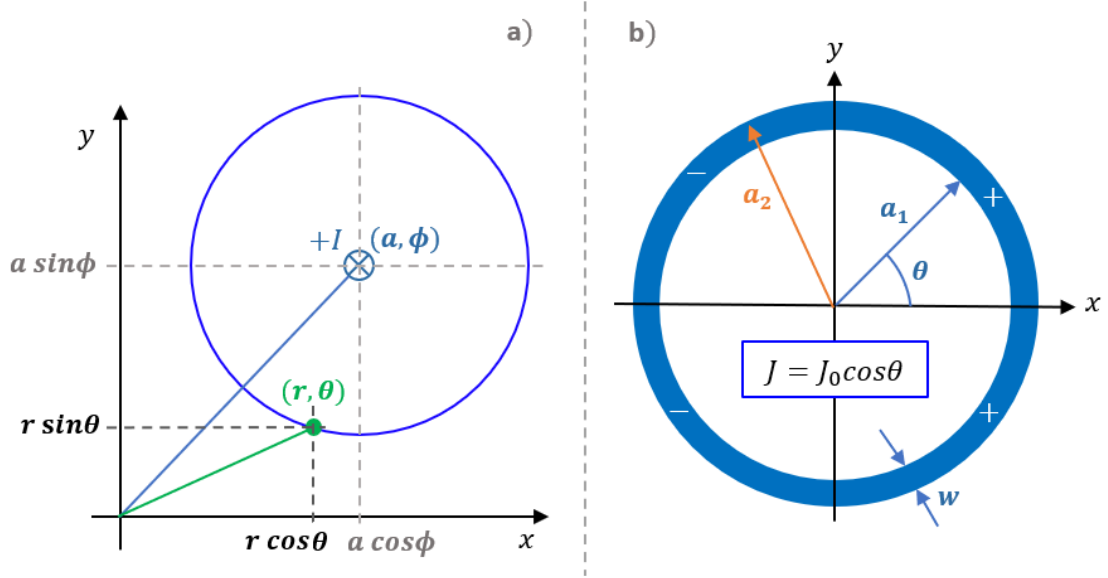


Figure 1: a) Representation of a current line, b) Dipole representation in  $\cos\theta$  configuration.

The vector potential in cylindrical coordinates  $(r, \theta, z)$  of a current line  $I$  directed along the  $z$  axis and located at  $(a, \phi)$  (see Figure 1a) is the initial point for the calculation of the magnetic field in the aperture and conductors, Lorentz forces, and stored energy. The only non-zero component of the vector potential,  $A_z$ , is expressed as:

$$A_z(r, \theta) = \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \cos(n(\phi - \theta)) \quad \text{for } r < a \quad (1)$$

$$A_z(r, \theta) = \frac{\mu_0 I}{2\pi} \left[ \ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{a}{r}\right)^n \cos(n(\phi - \theta)) \right] \quad \text{for } r > a \quad (2)$$

where  $n$  represents the index of the series expansion. The magnetic field can be computed using the following two relationships:

$$B_r = \frac{1}{r} \frac{\partial A_z}{\partial \theta} \quad ; \quad B_\theta = -\frac{\partial A_z}{\partial r} \quad (3)$$

For  $r < a$ :

$$B_r(r, \theta) = \frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{n-1} \sin(n(\phi - \theta)) \quad (4)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{n-1} \cos(n(\phi - \theta)) \quad (5)$$

For  $r > a$ :

$$B_r(r, \theta) = \frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sin(n(\phi - \theta)) \quad (6)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n+1} \cos(n(\phi - \theta)) \quad (7)$$

As shown in Figure 1b) for the dipole case, a perfect  $\cos(m\theta)$  multipole is visualized as a cylindrical shell with an inner radius of  $a_1$  and an outer radius of  $a_2$ . This shell features a current density distribution denoted as  $J = J_0 \cos(m\phi)$ , where  $m$  represents the multipole index ( $m = 1$  for the dipole). To compute the resulting magnetic field, it is necessary to integrate the field generated by a current line over the entire cylindrical shell, with the substitution  $I \rightarrow J_0 \cos(m\phi) a da d\phi$ .

For an accurate representation, it's crucial to distinguish between the magnetic field inside the aperture and inside the conductor.

### 1.1 Magnetic field in the aperture

The magnetic field in the bore is given by:

$$\begin{aligned} B_r(r, \theta) &= \frac{\mu_0 J_0}{2\pi} \sum_{n=1}^{\infty} \int_{a_1}^{a_2} \left(\frac{r}{a}\right)^{n-1} da \int_0^{2\pi} \sin(n(\phi - \theta)) \cos(m\phi) d\phi \\ &= -\frac{\mu_0 J_0}{2} r^{n-1} \left(\frac{a_2^{2-n} - a_1^{2-n}}{2-n}\right) \sin(n\theta) \end{aligned} \quad (8)$$

$$\begin{aligned} B_\theta(r, \theta) &= -\frac{\mu_0 J_0}{2\pi} \sum_{n=1}^{\infty} \int_{a_1}^{a_2} \left(\frac{r}{a}\right)^{n-1} da \int_0^{2\pi} \cos(n(\phi - \theta)) \cos(m\phi) d\phi \\ &= -\frac{\mu_0 J_0}{2} r^{n-1} \left(\frac{a_2^{2-n} - a_1^{2-n}}{2-n}\right) \cos(n\theta) \end{aligned} \quad (9)$$

where the integral over the angle is nonzero only if  $m = n$ .

To convert from cylindrical coordinates to Cartesian coordinates, one can proceed as follows:

$$B_x = B_r \cos\theta - B_\theta \sin\theta \quad (10)$$

$$B_y = B_r \sin\theta + B_\theta \cos\theta \quad (11)$$

For  $n = 1$ :

$$B_r = -\frac{\mu_0 J_0}{2} (a_2 - a_1) \sin\theta \quad (12)$$

$$B_\theta = -\frac{\mu_0 J_0}{2} (a_2 - a_1) \cos\theta \quad (13)$$

$$|B| = \frac{\mu_0 J_0 (a_2 - a_1)}{2} \quad (14)$$

For  $n = 2$ :

$$B_r = -\frac{\mu_0 J_0}{2} r \ln\left(\frac{a_2}{a_1}\right) \sin(2\theta) \quad (15)$$

$$B_\theta = -\frac{\mu_0 J_0}{2} r \ln\left(\frac{a_2}{a_1}\right) \cos(2\theta) \quad (16)$$

$$|B| = \frac{\mu_0 J_0}{2} r \ln\left(\frac{a_2}{a_1}\right) \quad (17)$$

To derive these equations with  $n = 2$ , the following limit was solved:

$$\lim_{n \rightarrow 2} \frac{a_2^{2-n} - a_1^{2-n}}{2-n} = \ln\left(\frac{a_2}{a_1}\right) \quad (18)$$

and the gradient can be calculated as follows:

$$G = \frac{\partial B}{\partial r} = \frac{\mu_0 J_0}{2} \ln\left(\frac{a_2}{a_1}\right) \quad (19)$$

## 1.2 Magnetic field in the conductor

In the conductor, if the point is located at  $(r, \theta)$ , the integral must be split into two parts:

$$B(r, \theta) = \int_0^{2\pi} \int_{a_1}^r B_{r>a} a d\phi da + \int_0^{2\pi} \int_r^{a_2} B_{r<a} a d\phi da \quad (20)$$

Following a similar approach as in the previous paragraph, the magnetic field can be calculated as:

$$B_r(r, \theta) = -\frac{\mu_0 J_0}{2} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} r^{n-1} + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right) \sin(n\theta) \quad (21)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 J_0}{2} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} r^{n-1} - \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right) \cos(n\theta) \quad (22)$$

For  $n = 1$ :

$$B_r(r, \theta) = -\frac{\mu_0 J_0}{2} \left( a_2 - \frac{2}{3}r - \frac{1}{3} \frac{a_1^3}{r^2} \right) \sin\theta \quad (23)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 J_0}{2} \left( a_2 - \frac{4}{3}r + \frac{1}{3} \frac{a_1^3}{r^2} \right) \cos\theta \quad (24)$$

By definition, the peak field inside the conductor refers to the maximum value of the

magnetic field modulus. For symmetry reasons, it is situated at  $\theta = \pi/2n$ , and for the dipole configuration, this corresponds to  $\theta = \pi/2$ , where:

$$|B(r)| = \frac{\mu_0 J_0}{2} \left( a_2 - \frac{2}{3}r - \frac{1}{3} \frac{a_1^3}{r^2} \right) \quad (25)$$

The maximum value can be determined by setting the first derivative of  $|B(r)|$  to zero, resulting in the equation in terms of  $r$ :

$$-\frac{2}{3} + \frac{2}{3} \frac{a_1^3}{r^3} = 0 \quad (26)$$

The solution of this equation is  $r = a_1$  for a dipole magnet in  $\cos\theta$  approximation, as shown in the Ansys simulation (Fig. 2).

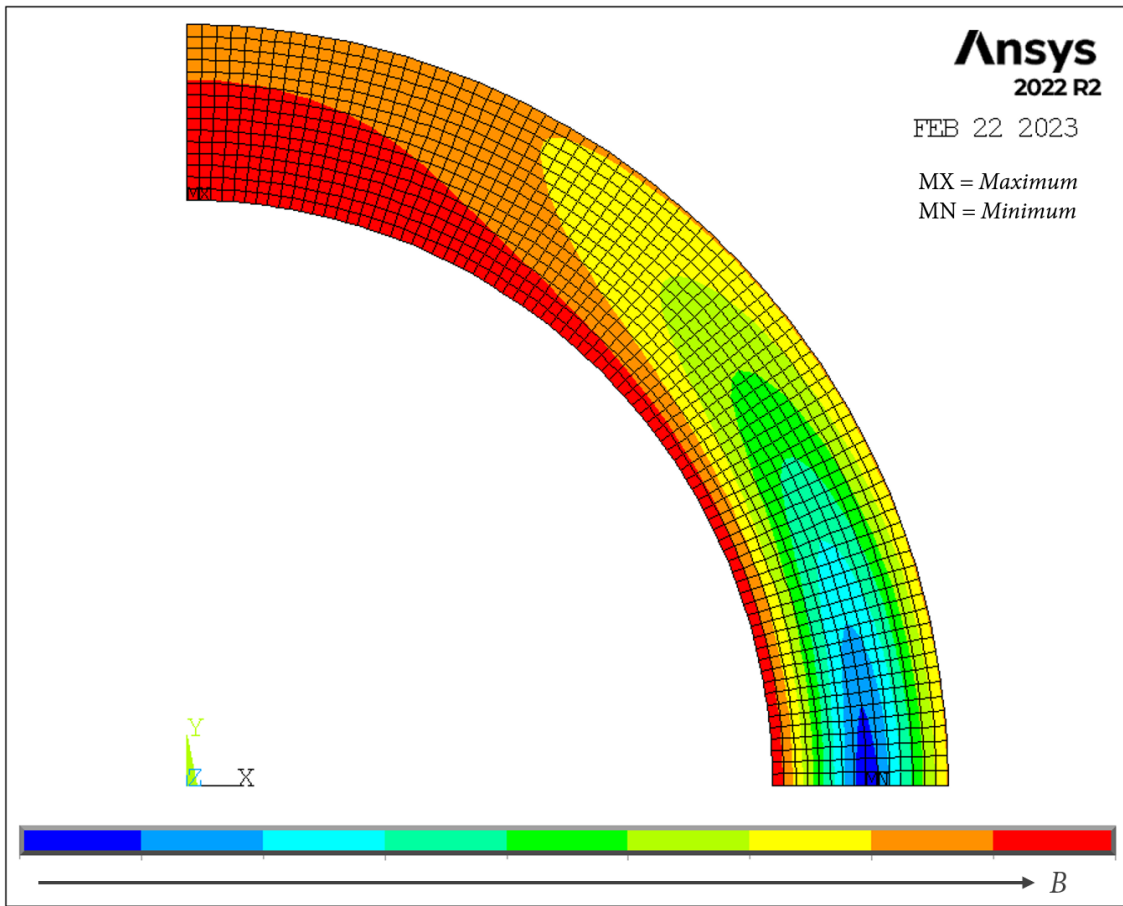


Figure 2: Simulation with Ansys of the magnetic field produced by a dipole in  $\cos\theta$  configuration.

Using this value for  $(r, \theta)$ , one can find:

$$B_{peak} = -\frac{\mu_0 J_0 (a_2 - a_1)}{2} \quad (27)$$

This value is equal to the field in the aperture for continuity.

For  $n = 2$ :

$$B_r = -\frac{\mu_0 J_0}{2} \left[ r \left( \ln \left( \frac{a_2}{r} \right) + \frac{1}{4} \right) - \frac{1}{4} \frac{a_1^4}{r^3} \right] \sin(2\theta) \quad (28)$$

$$B_\theta = -\frac{\mu_0 J_0}{2} \left[ r \left( \ln \left( \frac{a_2}{r} \right) - \frac{1}{4} \right) + \frac{1}{4} \frac{a_1^4}{r^3} \right] \cos(2\theta) \quad (29)$$

For symmetry reasons, the peak field inside the conductor is situated at  $\theta = 45^\circ$  and  $r = r(a_1, a_2)$  for a quadrupole magnet in  $\cos(\theta)$  approximation, as shown in the Ansys simulation (Fig. 3).

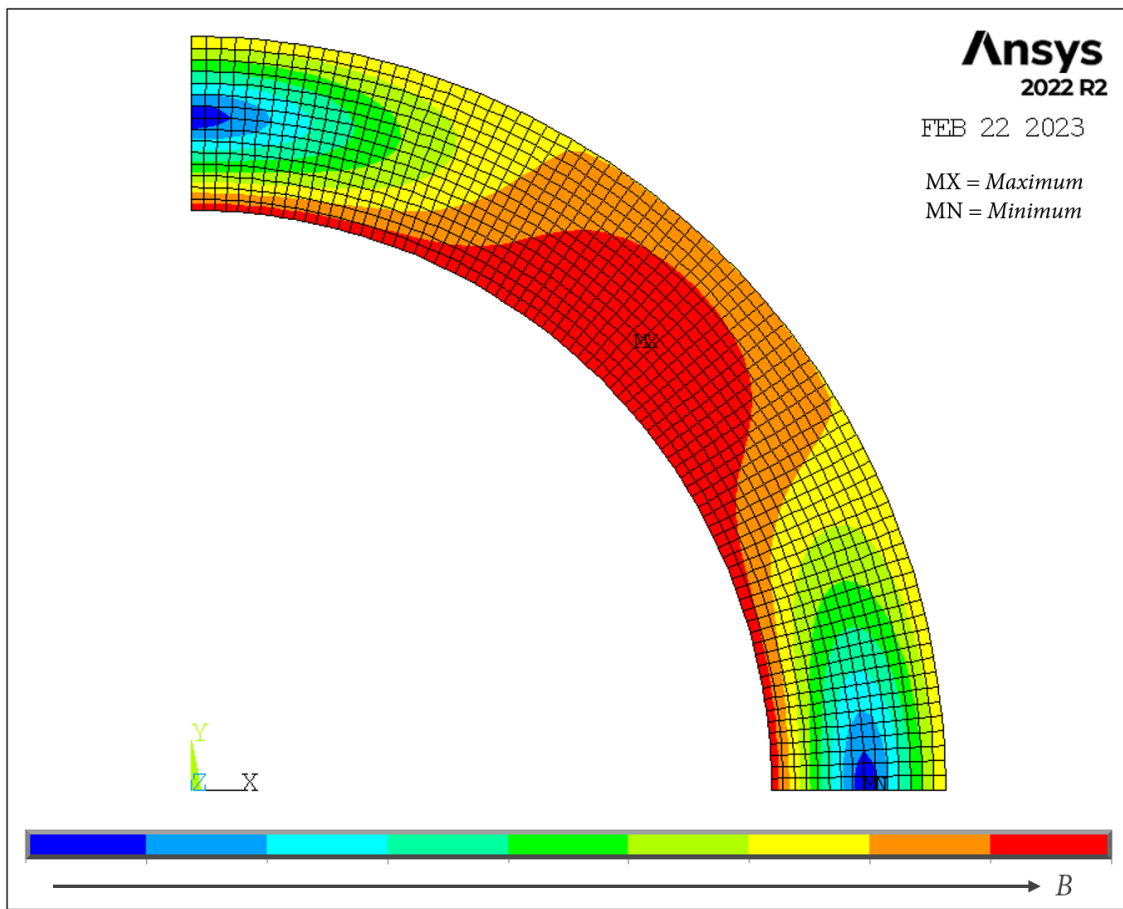


Figure 3: Simulation with Ansys of the magnetic field produced by a quadrupole in  $\cos 2\theta$  configuration.

It can be demonstrated that the  $r$  peak is a function of  $a_1$  (internal radius) and  $w$  (coil



width) as follows:

$$r(a_1, w) = (a_1 + w) c_1 \exp \left[ \frac{1}{4} W \left( \frac{\epsilon_1(a_1)}{(\epsilon_2(a_1) w + \epsilon_3(a_1))^4} \right) \right] \quad (30)$$

Where  $W$  is the *Lambert W function*, and the parameter  $c_1 = 0.47237$  is a constant, while  $\epsilon_{1,2,3}$  are functions of  $a_1$ . For an internal radius of  $a_1 = 75 \text{ mm}$ :  $\epsilon_1 = 60.25661$ ,  $\epsilon_2 = 0.01333 \frac{1}{\text{mm}}$ ,  $\epsilon_3 = 1$ . For instance, with a fixed coil width of  $30 \text{ mm}$ , a value of  $r$  is obtained, which is equal to  $82.59279 \text{ mm}$ . The peak magnetic field is given by:

$$B_{peak}(w) = -\frac{\mu_0 J_0}{2} \left[ r(w) \left( \ln \left( \frac{a_2}{r(w)} \right) + \frac{1}{4} \right) - \frac{1}{4} \frac{a_1^4}{r(w)^3} \right] \quad (31)$$

In the thin coil limit ( $w \rightarrow 0$ ), it is located at  $\theta = 45^\circ$  with  $r = a_1$ , and for continuity, it coincides with the expression in equation (17).

### 1.3 Stored energy

The expression for the vector potential inside the conductor is as follows:

$$A_z(r, \theta) = \int_0^{2\pi} \int_{a_1}^r A_{r>a} a d\phi da + \int_0^{2\pi} \int_r^{a_2} A_{r<a} a d\phi da \quad (32)$$

where  $A_{r>a}$  is defined in equation (2) and  $A_{r<a}$  is defined in equation (1). Integrating in cylindrical coordinates:

$$A_z(r, \theta) = \frac{\mu_0 J_0}{2n} \left[ \frac{a_2^{2-n} - r^{2-n}}{2-n} r^n + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^n} \right] \cos(n\theta) \quad (33)$$

The stored energy per unit length [ $J/m$ ] is calculated as:

$$\frac{E}{l} = \frac{1}{2} \int_{conductor} \vec{A} \cdot \vec{J} dS \quad (34)$$

with  $J = J_0 \cos(m\phi)$ . Then, integrating over the surface of the conductor:

$$\frac{E}{l} = \frac{\pi \mu_0 J_0^2}{4n(4-n^2)} \left[ a_1^4 + a_2^4 - \frac{n}{2} (a_2^4 - a_1^4) - 2 a_1^{2+n} a_2^{2-n} \right] \quad (35)$$

For  $n = 1$ :

$$\frac{E}{l} = \frac{\pi \mu_0 J_0^2}{24} (a_2^4 + 3 a_1^4 - 4 a_1^3 a_2) \quad (36)$$

For  $n = 2$ :

$$\frac{E}{l} = \frac{\pi\mu_0 J_0^2}{64} \left[ a_2^4 - a_1^4 + 4 a_1^4 \ln \left( \frac{a_1}{a_2} \right) \right] \quad (37)$$

Here, the following limit was used:

$$\lim_{n \rightarrow 2} \frac{a_1^4 + a_2^4 - \frac{n}{2}(a_2^4 - a_1^4) - 2 a_1^{2+n} a_2^{2-n}}{4 - n^2} = \frac{1}{8} \left[ a_2^4 - a_1^4 + 4 a_1^4 \ln \left( \frac{a_1}{a_2} \right) \right] \quad (38)$$

#### 1.4 Lorentz forces

A conducting element carrying a current density  $J$  [ $A/mm^2$ ] is subject to a force density  $f_L$  [ $N/m^3$ ] given by:

$$\vec{f}_L = \vec{J} \times \vec{B} \quad (39)$$

If the current density is  $\vec{J} = (0, 0, J_0 \cos m\theta)$  and the magnetic field is  $\vec{B} = (B_r, B_\theta, 0)$ , the components of the force density are:

$$f_r = -J_0 \cos(m\theta) B_\theta$$

$$f_\theta = +J_0 \cos(m\theta) B_r$$

$$f_z = 0$$

Utilizing the previously calculated  $B_\theta$  and  $B_r$  values from section 1.2 and taking into account that  $n$  becomes equal to  $m$  after the integrations, the complete forms can be derived as follows:

$$f_r(r, \theta) = -\frac{\mu_0 J_0^2}{2} \cos^2(n\theta) \left( \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} - \frac{a_2^{2-n} - r^{2-n}}{2-n} r^{n-1} \right) \quad (40)$$

$$f_\theta(r, \theta) = -\frac{\mu_0 J_0^2}{2} \cos(n\theta) \sin(n\theta) \left( \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} + \frac{a_2^{2-n} - r^{2-n}}{2-n} r^{n-1} \right) \quad (41)$$

By integrating over the cross-section, one can obtain the total forces per unit of longitudinal length  $F$  [ $N/m$ ] in  $1/2n$  symmetry as follows:

$$F_r = \int_0^{\pi/2n} \int_{a_1}^{a_2} f_r r dr d\theta \quad ; \quad F_\theta = \int_0^{\pi/2n} \int_{a_1}^{a_2} f_\theta r dr d\theta \quad (42)$$

The radial force per unit length is:

$$F_r = -\frac{\pi\mu_0 J_0^2}{8n} \left( \frac{4 a_2^3 - a_1^3}{3} \frac{1}{4-n^2} - \frac{a_1^{2+n} a_2^{1-n} - a_1^3}{(2+n)(1-n)} - \frac{a_2^3 - a_1^{1+n} a_2^{2-n}}{(1+n)(2-n)} \right) \quad (43)$$

and the angular one is:

$$F_\theta = -\frac{\mu_0 J_0^2}{4n} \left( -\frac{2n a_2^3 - a_1^3}{3} \frac{1}{4-n^2} - \frac{a_1^{2+n} a_2^{1-n} - a_1^3}{(2+n)(1-n)} + \frac{a_2^3 - a_1^{1+n} a_2^{2-n}}{(1+n)(2-n)} \right) \quad (44)$$

For  $n = 1$ :

$$F_r = -\frac{\pi\mu_0 J_0^2}{8} \left( \frac{4}{3} \frac{a_2^3 - a_1^3}{3} - \frac{a_1^3}{3} \ln \left( \frac{a_2}{a_1} \right) - \frac{a_2^3 - a_1^2 a_2}{2} \right) \quad (45)$$

$$F_\theta = -\frac{\mu_0 J_0^2}{4} \left( -\frac{2}{3} \frac{a_2^3 - a_1^3}{3} - \frac{a_1^3}{3} \ln \left( \frac{a_2}{a_1} \right) + \frac{a_2^3 - a_1^2 a_2}{2} \right) \quad (46)$$

where the following limit was used:

$$\lim_{n \rightarrow 1} \frac{a_1^{2+n} a_2^{1-n} - a_1^3}{1-n} = a_1^3 \ln \left( \frac{a_2}{a_1} \right) \quad (47)$$

For  $n = 2$ :

$$F_r = -\frac{\pi\mu_0 J_0^2}{16} \left( -\frac{a_2^3 - a_1^3}{36} + \frac{a_1^3}{3} \ln \left( \frac{a_2}{a_1} \right) + \frac{a_1^4 - a_1^3 a_2}{4 a_2} \right) \quad (48)$$

$$F_\theta = -\frac{\mu_0 J_0^2}{8} \left( 7 \frac{a_2^3 - a_1^3}{36} - \frac{a_1^3}{3} \ln \left( \frac{a_2}{a_1} \right) + \frac{a_1^4 - a_1^3 a_2}{4 a_2} \right) \quad (49)$$

where the following limits were used:

$$\lim_{n \rightarrow 2} \left( \frac{4}{3} \frac{a_2^3 - a_1^3}{4 - n^2} - \frac{a_2^3 - a_1^{1+n} a_2^{2-n}}{(1+n)(2-n)} \right) = -\frac{1}{36} (a_2^3 - a_1^3) + \frac{1}{3} a_1^3 \ln \left( \frac{a_2}{a_1} \right) \quad (50)$$

and

$$\lim_{n \rightarrow 2} \left( -\frac{2n}{3} \frac{a_2^3 - a_1^3}{4 - n^2} + \frac{a_2^3 - a_1^{1+n} a_2^{2-n}}{(1+n)(2-n)} \right) = \frac{7}{36} (a_2^3 - a_1^3) - \frac{1}{3} a_1^3 \ln \left( \frac{a_2}{a_1} \right) \quad (51)$$

## 2 Multipole $\cos(n\theta)$ with iron yoke

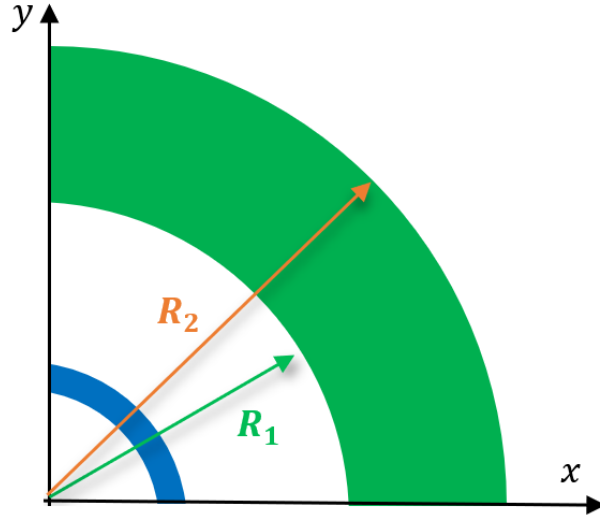


Figure 4: Schematic representation of the iron yoke (in green).

In this section, we follow the same procedures as in the previous section, but this time, we take into account the effects introduced by an iron yoke with linear permeability. It's important to note that this is applicable when magnetic fields are below 1.5 – 2 T, remaining within the range of iron's magnetic saturation. If we consider an iron yoke with inner radius  $R_1$ , outer radius  $R_2$ , and a relative permeability of  $\mu_r$ , as illustrated in Figure 4, the only non-zero component of the vector potential, denoted as  $A_z$ , corresponding to a current line  $I$  oriented along the  $z$ -axis and positioned at  $(a, \phi)$ , is as follows:

$$A_z(r, \theta) = \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] \cos(n(\phi - \theta)) \quad \text{for } r < a \quad (52)$$

$$A_z(r, \theta) = \frac{\mu_0 I}{2\pi} \left[ \ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty} \frac{1}{n} \left[ \left(\frac{a}{r}\right)^n + k_n \left(\frac{r}{a}\right)^n \left(\frac{a}{R_1}\right)^{2n} \right] \cos(n(\phi - \theta)) \right] \quad \text{for } r > a \quad (53)$$

where

$$k_n = \frac{\mu_r - 1}{\mu_r + 1} \frac{1 - \left(\frac{R_1}{R_2}\right)^{2n}}{1 - \left(\frac{\mu_r - 1}{\mu_r + 1}\right)^2 \left(\frac{R_1}{R_2}\right)^{2n}} \quad (54)$$

Utilizing equation (3), the magnetic field generated by a current line is expressed as follows:

for  $r < a$ :

$$B_r(r, \theta) = \frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{n-1} \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] \sin(n(\phi - \theta)) \quad (55)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{n-1} \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] \cos(n(\phi - \theta)) \quad (56)$$

for  $r > a$ :

$$B_r(r, \theta) = \frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left[ \left(\frac{a}{r}\right)^{n+1} + \left(\frac{r}{a}\right)^{n-1} k_n \left(\frac{a}{R_1}\right)^{2n} \right] \sin(n(\phi - \theta)) \quad (57)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2\pi a} \sum_{n=1}^{\infty} \left[ \left(\frac{a}{r}\right)^{n+1} - \left(\frac{r}{a}\right)^{n-1} k_n \left(\frac{a}{R_1}\right)^{2n} \right] \cos(n(\phi - \theta)) \quad (58)$$

Therefore, by applying the superposition principle, the magnetic field remains identical to the field without iron, except for the additional term introduced by the iron's permeability. Now, in the case of a perfect  $\cos\theta$  multipole featuring a current density distribution expressed as  $J = J_0 \cos(m\phi)$ , with  $m$  representing the multipole index, the magnetic field within both the aperture and the conductor can be calculated.

## 2.1 Magnetic field in the aperture, including iron yoke

The integration over both the angle and the radius results in the magnetic field within the aperture, where  $r < a$ :

$$B_r(r, \theta) = -\frac{\mu_0 J_0}{2} r^{n-1} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \sin(n\theta) \quad (59)$$

$$B_\theta(r, \theta) = -\frac{\mu_0 J_0}{2} r^{n-1} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \cos(n\theta) \quad (60)$$

For  $n = 1$ :

$$B_r = -\frac{\mu_0 J_0}{2} \left( a_2 - a_1 + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \right) \sin\theta \quad (61)$$

$$B_\theta = -\frac{\mu_0 J_0}{2} \left( a_2 - a_1 + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \right) \cos\theta \quad (62)$$

$$|B| = \frac{\mu_0 J_0}{2} \left( a_2 - a_1 + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \right) \quad (63)$$

For  $n = 2$ :

$$B_r = -\frac{\mu_0 J_0}{2} r \left[ \ln\left(\frac{a_2}{a_1}\right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right] \sin(2\theta) \quad (64)$$

$$B_\theta = -\frac{\mu_0 J_0}{2} r \left[ \ln \left( \frac{a_2}{a_1} \right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right] \cos(2\theta) \quad (65)$$

$$|B| = \frac{\mu_0 J_0}{2} r \left[ \ln \left( \frac{a_2}{a_1} \right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right] \quad (66)$$

$$G = \frac{\partial B}{\partial r} \Rightarrow G = \frac{\mu_0 J_0}{2} \left[ \ln \left( \frac{a_2}{a_1} \right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right] \quad (67)$$

## 2.2 Magnetic field in the conductor, including iron yoke

When  $(r, \theta)$  is located inside the conductor, the magnetic field within the conductor includes both the original field without the presence of iron, denoted as  $B_{r, no\ iron}$ , and an additional contribution due to the permeability  $\mu_r$ :

$$B_r = B_{r, no\ iron} + B_{r, iron} \quad (68)$$

where  $B_{r, no\ iron}$  is defined in eq.(21), and the extra term is given by:

$$B_{r, iron} = \frac{\mu_0 J_0}{2\pi} \sum_{n=1}^{\infty} \int_{a_1}^{a_2} \left( \frac{r}{a} \right)^{n-1} k_n \left( \frac{a}{R_1} \right)^{n-1} da \int_0^{2\pi} \sin(n(\phi - \theta)) \cos(m\phi) d\phi \quad (69)$$

Then, following the necessary integrations and calculations, one can derive:

$$B_r = -\frac{\mu_0 J_0}{2} \left[ \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) r^{n-1} + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right] \sin(n\theta) \quad (70)$$

and likewise:

$$B_\theta = -\frac{\mu_0 J_0}{2} \left[ \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) r^{n-1} - \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right] \cos(n\theta) \quad (71)$$

For  $n = 1$ :

The peak of the dipole field occurs at an angle  $\theta = \pi/2$  and a radius  $r = a_1$ , which is the solution for  $\partial B_r / \partial r = 0$  with an index set to 1.

$$B_\theta = 0 \quad ; \quad B_r = -\frac{\mu_0 J_0}{2} \left( a_2 - r + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} + \frac{r^3 - a_1^3}{3r^2} \right) \quad (72)$$

Replacing  $r$  with  $a_1$ , the peak field is:

$$|B_{max}| = \frac{\mu_0 J_0}{2} \left( a_2 - a_1 + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \right) \quad (73)$$

For  $n = 2$ :

The peak of the quadrupole field occurs at an angle  $\theta = \pi/4$  and a radius  $r(a_1, a_2)$ , which is the solution for  $\partial B_r / \partial r = 0$  with an index set to 2.

$$B_\theta = 0 \quad ; \quad B_r = -\frac{\mu_0 J_0}{2} \left[ r \left( \ln \left( \frac{a_2}{r} \right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right) + \frac{r^4 - a_1^4}{4r^3} \right] \quad (74)$$

The radius that corresponds to the maximum field can be obtained by solving this following equation:

$$\ln \left( \frac{a_2}{r} \right) + \frac{3}{4} \left( \frac{a_1^4}{r^4} - 1 \right) + \frac{k_2}{R_1^4} \left( \frac{a_2^4 - a_1^4}{4} \right) = 0 \quad (75)$$

A numerical estimate, with  $a_1 = 75 \text{ mm}$ ,  $a_2 = a_1 + w$  (where  $w = 30 \text{ mm}$  is the coil width),  $R_1 = a_2 + 20 \text{ mm}$ ,  $R_2 = R_1 + W$  (where  $W = 30 \text{ mm}$  is the iron yoke width) and  $\mu_r = 10$ , results in a radius of  $r = 84.30302 \text{ mm}$ .

### 2.3 Stored energy, including iron yoke

Using the superposition principle, the vector potential inside the conductor is found as a sum of two terms:

$$A_z = A_{z, \text{no iron}} + A_{z, \text{iron}} \quad (76)$$

where  $A_{z, \text{no iron}}$  is defined in eq.(33), and the extra term due to iron is given by:

$$A_{z, \text{iron}} = \frac{\mu_0 J_0}{2n} \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} r^n \cos(n\theta) \quad (77)$$

Then, combining the two terms:

$$A_z(r, \theta) = \frac{\mu_0 J_0}{2n} \left[ \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) r^n + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^n} \right] \cos(n\theta) \quad (78)$$

The stored energy per unit length  $[J/mm]$  is:

$$\begin{aligned} \frac{E}{l} &= \left( \frac{E}{l} \right)_{\text{no iron}} + \frac{\pi \mu_0 J_0^2}{4n} \int_{a_1}^{a_2} \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} r^{n+1} dr \\ &= \left( \frac{E}{l} \right)_{\text{no iron}} + \frac{\pi \mu_0 J_0^2}{4n} \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \end{aligned} \quad (79)$$

where  $(E/l)_{\text{no iron}}$  is defined in eq.(35) .

For  $n = 1$ :

$$\frac{E}{l} = \frac{\pi\mu_0 J_0^2}{4} \left[ \frac{a_2^4 + 3a_1^4 - 4a_1^3 a_2}{6} + \frac{k_1}{R_1^2} \left( \frac{a_2^3 - a_1^3}{3} \right)^2 \right] \quad (80)$$

For  $n = 2$ :

$$\frac{E}{l} = \frac{\pi\mu_0 J_0^2}{8} \left[ \frac{a_2^4 - a_1^4 + 4a_1^4 \ln\left(\frac{a_1}{a_2}\right)}{8} + \frac{k_2}{R_1^4} \left( \frac{a_2^4 - a_1^4}{4} \right)^2 \right] \quad (81)$$

## 2.4 Lorentz forces, including iron yoke

The force densities  $f_r$  and  $f_\theta$  in  $[N/m^3]$  are given by:

$$\begin{aligned} f_r(r, \theta) &= -J_0 \cos(n\theta) B_\theta = \\ &= -\frac{\mu_0 J_0^2}{2} \cos^2(n\theta) \left[ \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} - \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) r^{n-1} \right] \end{aligned} \quad (82)$$

$$\begin{aligned} f_\theta(r, \theta) &= +J_0 \cos(n\theta) B_r = \\ &= -\frac{\mu_0 J_0^2}{2} \cos(n\theta) \sin(n\theta) \left[ \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} + \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) r^{n-1} \right] \end{aligned} \quad (83)$$

Following the same path of the previous paragraphs, the total radial force per unit length,  $F_r$   $[N/m]$ , is:

$$F_r = F_{r, no\ iron} + \frac{\pi\mu_0 J_0^2}{8n} \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \frac{a_2^{1+n} - a_1^{1+n}}{1+n} \quad (84)$$

and the total angular force per unit length,  $F_\theta$   $[N/m]$ , is:

$$F_\theta = F_{\theta, no\ iron} - \frac{\mu_0 J_0^2}{4n} \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \frac{a_2^{1+n} - a_1^{1+n}}{1+n} \quad (85)$$

where  $F_{r, no\ iron}$  and  $F_{\theta, no\ iron}$  are defined respectively in eq.(43) and in eq.(44) .

For  $n = 1$ :

$$F_r = -\frac{\pi\mu_0 J_0^2}{8} \left( \frac{4}{3} \frac{a_2^3 - a_1^3}{3} - \frac{a_1^3}{3} \ln\left(\frac{a_2}{a_1}\right) - \frac{a_2^3 - a_1^2 a_2}{2} - \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \frac{a_2^2 - a_1^2}{2} \right) \quad (86)$$

$$F_\theta = -\frac{\mu_0 J_0^2}{4} \left( -\frac{2}{3} \frac{a_2^3 - a_1^3}{3} - \frac{a_1^3}{3} \ln\left(\frac{a_2}{a_1}\right) + \frac{a_2^3 - a_1^2 a_2}{2} + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \frac{a_2^2 - a_1^2}{2} \right) \quad (87)$$



For  $n = 2$ :

$$F_r = -\frac{\pi\mu_0 J_0^2}{16} \left( -\frac{a_2^3 - a_1^3}{36} + \frac{a_1^3}{3} \ln\left(\frac{a_2}{a_1}\right) + \frac{a_1^4 - a_1^3 a_2}{4 a_2} - \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \frac{a_2^3 - a_1^3}{3} \right) \quad (88)$$

$$F_\theta = -\frac{\mu_0 J_0^2}{8} \left( 7 \frac{a_2^3 - a_1^3}{36} - \frac{a_1^3}{3} \ln\left(\frac{a_2}{a_1}\right) + \frac{a_1^4 - a_1^3 a_2}{4 a_2} + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \frac{a_2^3 - a_1^3}{3} \right) \quad (89)$$

It's important to specify that if the magnetic field is high, above  $1.5 - 2 \text{ T}$ , the iron becomes saturated and non-linear, and these formulas are no longer valid.

### 3 Sector coil

To describe the current distributions of the sector coil type, a similar approach can be used as previously discussed for the  $\cos\theta$  configurations. The key difference lies in the integrals over the angle  $d\phi$ , given that the current distribution changes from  $J = J_0 \cos(m\phi)$  to  $J = J_0$ .

The focus will be on the cases of dipole and quadrupole configurations, with the iron yoke term being incorporated from the beginning.

#### 3.1 Sector dipole

For a sector coil dipole, let's start by considering 4 current lines placed as in Figure 5. In this context, the magnetic field is composed of four separate contributions, each corresponding to one of the current lines and distinguished by their angular components.

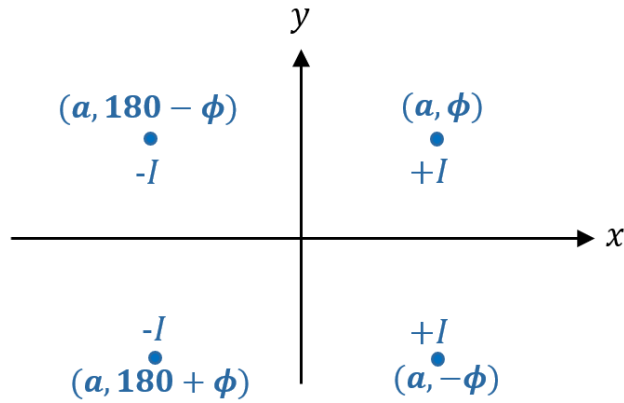


Figure 5: Schematic representation of a dipole with 4 current lines, two positive and two negative, all with their pair of angular coordinates.

For  $B_r$ , the four angular contributions are:

$$\begin{aligned} & \sin(n(\phi - \theta)) + \sin(n(-\phi - \theta)) - \sin(n(\pi - \phi - \theta)) - \sin(n(\pi + \phi - \theta)) \\ & = -4 \cos(n\phi) \sin(n\theta) \quad \text{with } n \text{ odd} \end{aligned}$$

while for  $B_\theta$ , they are:

$$\begin{aligned} & \cos(n(\phi - \theta)) + \cos(n(-\phi - \theta)) - \cos(n(\pi - \phi - \theta)) - \cos(n(\pi + \phi - \theta)) \\ & = +4 \cos(n\phi) \cos(n\theta) \quad \text{with } n \text{ odd} \end{aligned}$$

In the case of a sector dipole with an angle  $\alpha$ , as shown in Figure 6, **the magnetic field**

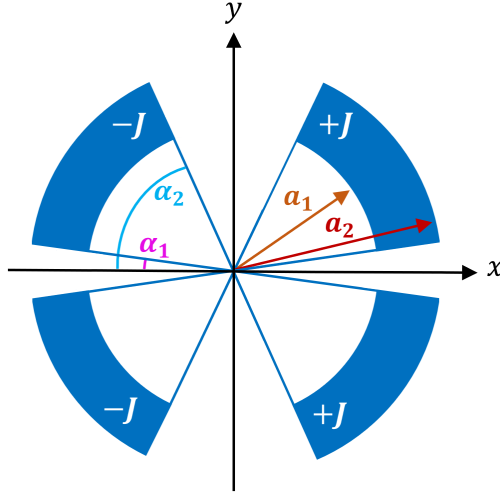


Figure 6: Schematic representation of a sector dipole.

within the aperture is given by:

$$B_r(r, \theta) = \frac{\mu_0 J_0}{2\pi a} \sum_{n=1}^{\infty} \int_{a_1}^{a_2} \left(\frac{r}{a}\right)^{n-1} \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] a da \int_{\alpha_1}^{\alpha_2} (-4) \cos(n\phi) \sin(n\theta) d\phi \quad (90)$$

$$B_\theta(r, \theta) = \frac{\mu_0 J_0}{2\pi a} \sum_{n=1}^{\infty} \int_{a_1}^{a_2} \left(\frac{r}{a}\right)^{n-1} \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] a da \int_{\alpha_1}^{\alpha_2} (+4) \cos(n\phi) \cos(n\theta) d\phi \quad (91)$$

The component proportional to  $k_n$ , which is defined in equation (54), is attributed to the presence of the iron yoke. After computation, the magnetic field can be expressed as an infinite series of terms as follows:

$$B_r = -\frac{2\mu_0 J_0}{\pi} \sum_{n \text{ odd}} \frac{r^{n-1}}{n} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \times \sin(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (92)$$

$$B_\theta = -\frac{2\mu_0 J_0}{\pi} \sum_{n \text{ odd}} \frac{r^{n-1}}{n} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \times \cos(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (93)$$

In order to determine the contribution of each harmonic component, it is possible to evaluate the following integral at a fixed radius, denoted as  $r = R_{ref}$  and called reference radius, which is typically defined as  $\frac{2}{3}a_1$ .

$$B_n = \frac{1}{\pi} \int_0^{2\pi} B_\theta(R_{ref}, \theta) \cos(n\theta) d\theta \quad (94)$$

The first harmonic, without iron contribution, is:

$$B_1 = -\frac{2\mu_0 J_0 w}{\pi} (\sin\alpha_2 - \sin\alpha_1) \quad (95)$$

while, with the contribution of iron, it is:

$$B_1 = -\frac{2\mu_0 J_0}{\pi} \left[ w + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \right] (\sin\alpha_2 - \sin\alpha_1) \quad (96)$$

The high-order harmonics, without iron contribution, are:

$$B_{n>1} = -\frac{2\mu_0 J_0 R_{ref}^{n-1}}{\pi n(n-2)} \left( \frac{1}{a_1^{n-2}} - \frac{1}{a_2^{n-2}} \right) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (97)$$

where  $n$  must be odd, while, with the contribution of iron, they are:

$$B_{n>1} = -\frac{2\mu_0 J_0 R_{ref}^{n-1}}{\pi n} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (98)$$

In the case of **the magnetic field inside the conductor**, the integration over the angle remains the same, while the results of the integration over the radius have been previously discussed in the context of the  $\cos\theta$  configuration.

$$B_r(r, \theta) = -\sum_{n \text{ odd}}^{\infty} \frac{2\mu_0 J_0}{n\pi} \left[ r^{n-1} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right] \sin(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (99)$$

$$B_\theta(r, \theta) = -\sum_{n \text{ odd}}^{\infty} \frac{2\mu_0 J_0}{n\pi} \left[ r^{n-1} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right] \cos(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (100)$$

In order to compute **the stored energy**, the potential vector within the conductor must be determined, namely:

$$A_z(r, \theta) = \int_{\alpha_1}^{\alpha_2} \int_{a_1}^r A_{r>a} a d\phi da + \int_{\alpha_1}^{\alpha_2} \int_r^{a_2} A_{r<a} a d\phi da \quad (101)$$

where  $A_{r>a}$  is defined in equation (53) and  $A_{r<a}$  is defined in equation (52). After performing the integration:

$$A_z(r, \theta) = \sum_{n \text{ odd}} \frac{2\mu_0 J_0}{n^2 \pi} \left[ r^n \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) + \right.$$

$$+ \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^n} \Big] \cos(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (102)$$

Now, the stored energy per unit length [ $J/mm$ ] can be expressed as:

$$\begin{aligned} \frac{E}{l} &= \frac{1}{2} \int_{\text{conductor}} \vec{A} \cdot \vec{J} dS = \frac{1}{2} \cdot 4 \int_{\alpha_1}^{\alpha_2} \int_{a_1}^{a_2} A_z J_0 r dr d\theta = \quad (103) \\ &= \sum_{n \text{ odd}} \frac{4\mu_0 J_0^2}{n^3 \pi} \left[ \frac{a_2^4(2-n) + a_1^4(2+n) - 4a_1^{2+n} a_2^{2-n}}{2(4-n^2)} + \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \right] \times \\ &\quad \times [\sin(n\alpha_2) - \sin(n\alpha_1)]^2 \quad (104) \end{aligned}$$

Finally, **the Lorentz forces** acting on the coil can be expressed starting from the force density  $\vec{f}_L = \vec{J} \times \vec{B}$ , in [ $N/m^3$ ], with  $\vec{J} = (0, 0, J_0)$  and  $\vec{B} = (B_r, B_\theta, 0)$ .

$$\begin{aligned} f_r(r, \theta) &= - \sum_{n \text{ odd}} \frac{2\mu_0 J_0^2}{n\pi} \left[ \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} - r^{n-1} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \right. \right. \\ &\quad \left. \left. + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \right] \cos(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (105) \end{aligned}$$

$$\begin{aligned} f_\theta(r, \theta) &= - \sum_{n \text{ odd}} \frac{2\mu_0 J_0^2}{n\pi} \left[ \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} + r^{n-1} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \right. \right. \\ &\quad \left. \left. + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \right] \sin(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (106) \end{aligned}$$

$$f_z = 0 \quad (107)$$

The total forces per unit of longitudinal length  $F$  [ $N/m$ ] can be obtained as:

$$F_r = \int_{\alpha_1}^{\alpha_2} \int_{a_1}^{a_2} f_r r dr d\theta \quad ; \quad F_\theta = \int_{\alpha_1}^{\alpha_2} \int_{a_1}^{a_2} f_\theta r dr d\theta \quad (108)$$

then:

$$\begin{aligned} F_r &= - \sum_{n \text{ odd}} \frac{2\mu_0 J_0^2}{n^2 \pi} [\sin(n\alpha_2) - \sin(n\alpha_1)]^2 \left[ \frac{4}{3} \frac{a_2^3 - a_1^3}{4-n^2} - \frac{a_1^{2+n} a_2^{1-n} - a_1^3}{(2+n)(1-n)} + \right. \\ &\quad \left. - \frac{a_2^3 - a_1^{1+n} a_2^{2-n}}{(1+n)(2-n)} - \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \frac{a_2^{1+n} - a_1^{1+n}}{1+n} \right] \quad (109) \end{aligned}$$

$$F_{\theta} = + \sum_{n \text{ odd}}^{\infty} \frac{2\mu_0 J_0^2}{n^2 \pi} [\sin(n\alpha_2) - \sin(n\alpha_1)] [\cos(n\alpha_2) - \cos(n\alpha_1)] \left[ -\frac{2n}{3} \frac{a_2^3 - a_1^3}{4 - n^2} + \right. \\ \left. - \frac{a_1^{2+n} a_2^{1-n} - a_1^3}{(2+n)(1-n)} + \frac{a_2^3 - a_1^{1+n} a_2^{2-n}}{(1+n)(2-n)} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \frac{a_2^{1+n} - a_1^{1+n}}{1+n} \right] \quad (110)$$

The first term of the series ( $n = 1$ ) can be calculated using the limit already solved in eq. (47) and is equal to:

$$F_{r, n=1} = -\frac{2\mu_0 J_0^2}{\pi} [\sin(\alpha_2) - \sin(\alpha_1)]^2 \left[ \frac{4}{3} \frac{a_2^3 - a_1^3}{3} - \frac{a_1^3}{3} \ln\left(\frac{a_2}{a_1}\right) + \right. \\ \left. - \frac{a_2^3 - a_1^2 a_2}{2} - \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \frac{a_2^2 - a_1^2}{2} \right] \quad (111)$$

$$F_{\theta, n=1} = \frac{2\mu_0 J_0^2}{\pi} [\sin(\alpha_2) - \sin(\alpha_1)] [\cos(\alpha_2) - \cos(\alpha_1)] \left[ -\frac{2}{3} \frac{a_2^3 - a_1^3}{3} - \frac{a_1^3}{3} \ln\left(\frac{a_2}{a_1}\right) + \right. \\ \left. + \frac{a_2^3 - a_1^2 a_2}{2} + \frac{k_1}{R_1^2} \frac{a_2^3 - a_1^3}{3} \frac{a_2^2 - a_1^2}{2} \right] \quad (112)$$

All these formulas are applicable as long as the iron is not saturated, meaning the magnetic field remains below approximately  $1.5 - 2 \text{ T}$ .

Before moving on to the quadrupole, it is important to specify that the series index  $n$  indicates the order of the harmonic, but since the sector dipole is not a perfect dipole, it has all the terms of the series, not just the first one.

### 3.2 Sector quadrupole

In the case of the sector coil quadrupole, the analysis is initiated with 8 current lines arranged as illustrated in Figure 7. This configuration results in 8 distinct contributions to the magnetic field, each associated with one of the current lines, which vary in terms of their angular components.

For  $B_r$ , the eight angular contributions, through trigonometric operations, give:

$$- 8 \cos(n\phi) \sin(n\theta) \quad \text{with } n = 2(2\tilde{n} - 1), \text{ where } \tilde{n} \text{ is a positive integer.} \quad (113)$$

while for  $B_{\theta}$ , they give:

$$+ 8 \cos(n\phi) \cos(n\theta) \quad \text{with } n = 2(2\tilde{n} - 1), \text{ where } \tilde{n} \text{ is a positive integer.} \quad (114)$$

**The magnetic field in the aperture** for a sector quadrupole with angle  $\alpha$ , illustrated in

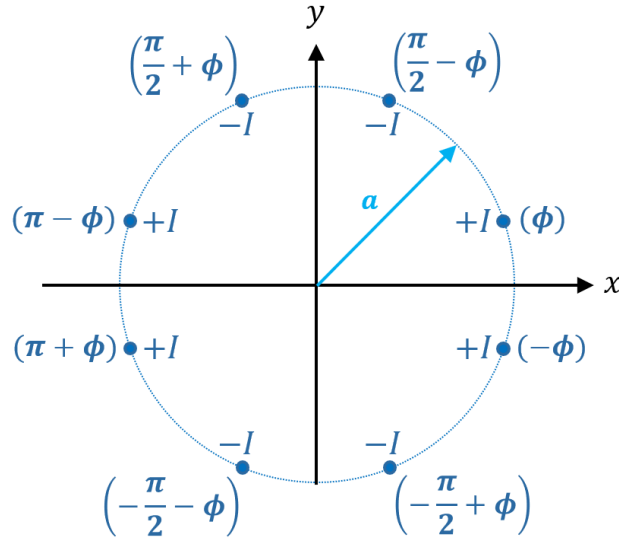


Figure 7: Schematic representation of a quadrupole with 8 current lines, four positive and four negative, all with their angular coordinate.

$\tilde{n}$	1	2	3	4	5
$n = 2(2\tilde{n} - 1)$	2	6	10	14	18

Table 1: The first five values of this infinite sequence.

Figure 8, is expressed as follows:

$$B_r(r, \theta) = \frac{\mu_0 J_0}{2\pi a} \sum_{n=2,6,\dots}^{\infty} \int_{a_1}^{a_2} \left(\frac{r}{a}\right)^{n-1} \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] a da \int_{\alpha_1}^{\alpha_2} (-8) \cos(n\phi) \sin(n\theta) d\phi \quad (115)$$

$$B_\theta(r, \theta) = \frac{\mu_0 J_0}{2\pi a} \sum_{n=2,6,\dots}^{\infty} \int_{a_1}^{a_2} \left(\frac{r}{a}\right)^{n-1} \left[1 + k_n \left(\frac{a}{R_1}\right)^{2n}\right] a da \int_{\alpha_1}^{\alpha_2} (+8) \cos(n\phi) \cos(n\theta) d\phi \quad (116)$$

where  $n = 2(2\tilde{n} - 1)$  with  $\tilde{n} = 1, 2, 3, \dots$ . Similarly to the dipole, the term proportional to  $k_n$ , which is defined in eq. (54), is due to the iron yoke. After performing the calculations, an infinite series of terms is obtained. The first term in the series expansion corresponds to  $n = 2$  (obtained by  $\tilde{n} = 1$ ), the second term corresponds to  $n = 6$  (obtained by  $\tilde{n} = 2$ ), and so on (refer to Table 1). Therefore, the magnetic field in the aperture can be expressed as:

$$B_r = -\frac{4\mu_0 J_0}{\pi} \sum_{n=2,6,\dots}^{\infty} \frac{r^{n-1}}{n} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \times \sin(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (117)$$

$$B_\theta = -\frac{4\mu_0 J_0}{\pi} \sum_{n=2,6,\dots}^{\infty} \frac{r^{n-1}}{n} \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \times$$

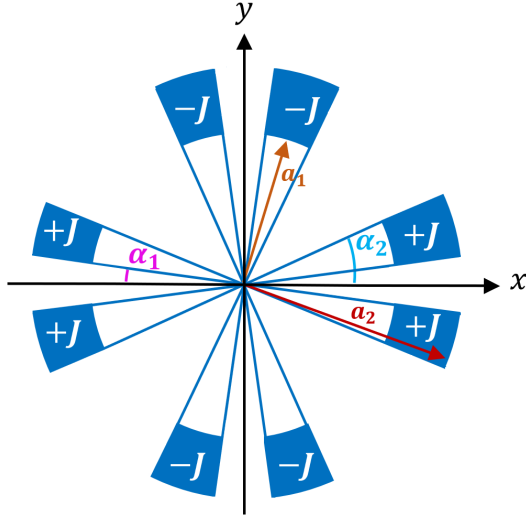


Figure 8: Schematic representation of a sector quadrupole.

$$\times \cos(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (118)$$

specifying that for the first term  $n = 2$  the limit already performed in eq.(18) must be applied. To compute the harmonics of the bore magnetic field, it's possible to use the equation (94), which, in this case, gives as the first harmonic ( $n = 2$ )  $B_2$ :

$$B_2 = -\frac{2\mu_0 J_0 R_{ref}}{\pi} [\sin(2\alpha_2) - \sin(2\alpha_1)] \left[ \ln\left(\frac{a_2}{a_1}\right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right] \quad (119)$$

where  $R_{ref}$  is the reference radius, often defined as  $\frac{2}{3}a_1$ , and  $R_1$  is the inner radius of the ferromagnetic yoke. The resulting gradient is:

$$G = \frac{B_2}{R_{ref}} = -\frac{2\mu_0 J_0}{\pi} [\sin(2\alpha_2) - \sin(2\alpha_1)] \left[ \ln\left(\frac{a_2}{a_1}\right) + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \right] \quad (120)$$

The higher-order harmonics ( $n = 6, 10, 14, \dots$ ) are:

$$B_n = -\frac{4\mu_0 J_0 R_{ref}^{n-1}}{\pi n} [\sin(n\alpha_2) - \sin(n\alpha_1)] \left( \frac{a_2^{2-n} - a_1^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) \quad (121)$$

**The magnetic field inside the conductor** can be calculated as already seen in the sector dipole, considering the variations just illustrated between dipole and quadrupole:

$$B_r(r, \theta) = -\sum_{n=2,6,\dots}^{\infty} \frac{4\mu_0 J_0}{n\pi} \left[ r^{n-1} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) + \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right] \sin(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (122)$$



$$B_{\theta}(r, \theta) = - \sum_{n=2,6,\dots}^{\infty} \frac{4\mu_0 J_0}{n\pi} \left[ r^{n-1} \left( \frac{a_2^{2-n} - r^{2-n}}{2-n} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right) + \right. \\ \left. - \frac{r^{2+n} - a_1^{2+n}}{2+n} \frac{1}{r^{n+1}} \right] \cos(n\theta) [\sin(n\alpha_2) - \sin(n\alpha_1)] \quad (123)$$

Similarly, **the stored energy per unit length**  $[J/mm]$  can be expressed as:

$$\frac{E}{l} = \sum_{n=2,6,\dots}^{\infty} \frac{16\mu_0 J_0^2}{n^3\pi} \left[ \frac{a_2^4(2-n) + a_1^4(2+n) - 4a_1^{2+n}a_2^{2-n}}{2(4-n^2)} + \right. \\ \left. + \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \right] [\sin(n\alpha_2) - \sin(n\alpha_1)]^2 \quad (124)$$

where for the first term ( $n = 2$ ) the limit is already been made in eq.(38).

**The total forces** per unit of longitudinal length  $[N/m]$  which act on the coil is given by:

$$F_r = - \sum_{n=2,6,\dots}^{\infty} \frac{4\mu_0 J_0^2}{n^2\pi} [\sin(n\alpha_2) - \sin(n\alpha_1)]^2 \left[ \frac{4}{3} \frac{a_2^3 - a_1^3}{4-n^2} - \frac{a_1^{2+n}a_2^{1-n} - a_1^3}{(2+n)(1-n)} + \right. \\ \left. - \frac{a_2^3 - a_1^{1+n}a_2^{2-n}}{(1+n)(2-n)} - \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \frac{a_2^{1+n} - a_1^{1+n}}{1+n} \right] \quad (125)$$

$$F_{\theta} = \sum_{n=2,6,\dots}^{\infty} \frac{4\mu_0 J_0^2}{n^2\pi} [\sin(n\alpha_2) - \sin(n\alpha_1)] [\cos(n\alpha_2) - \cos(n\alpha_1)] \left[ - \frac{2n}{3} \frac{a_2^3 - a_1^3}{4-n^2} + \right. \\ \left. - \frac{a_1^{2+n}a_2^{1-n} - a_1^3}{(2+n)(1-n)} + \frac{a_2^3 - a_1^{1+n}a_2^{2-n}}{(1+n)(2-n)} + \frac{k_n}{R_1^{2n}} \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \frac{a_2^{1+n} - a_1^{1+n}}{1+n} \right] \quad (126)$$

remembering that  $n = 2(2\tilde{n} - 1)$  with  $\tilde{n} = 1, 2, 3, 4, 5, \dots$ . The first term of the series ( $n = 2$ ) can be calculated using the limits already solved in eq. (50)-(51), and is equal to:

$$F_{r, n=2} = - \frac{\mu_0 J_0^2}{\pi} [\sin(2\alpha_2) - \sin(2\alpha_1)]^2 \left[ - \frac{1}{36} (a_2^3 - a_1^3) + \right. \\ \left. + \frac{a_1^3}{3} \ln \left( \frac{a_2}{a_1} \right) + \frac{a_1^4 - a_1^3 a_2}{4a_2} - \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \frac{a_2^3 - a_1^3}{3} \right] \quad (127)$$

$$F_{\theta, n=2} = \frac{\mu_0 J_0^2}{\pi} [\sin(2\alpha_2) - \sin(2\alpha_1)] [\cos(2\alpha_2) - \cos(2\alpha_1)] \left[ \frac{7}{36} (a_2^3 - a_1^3) + \right. \\ \left. - \frac{a_1^3}{3} \ln \left( \frac{a_2}{a_1} \right) + \frac{a_1^4 - a_1^3 a_2}{4a_2} + \frac{k_2}{R_1^4} \frac{a_2^4 - a_1^4}{4} \frac{a_2^3 - a_1^3}{3} \right] \quad (128)$$

In conclusion, it should be noted that the angles  $\alpha_1 < \alpha_2$ , which delimit the angular size of the sector, can take values between  $0^\circ$  and  $45^\circ$  in the sector quadrupole, while they can assume values between  $0^\circ$  and  $90^\circ$  in the sector dipole.

### 3.3 Multiple sector dipole

In the previous section, the single-sector dipole was studied: a configuration that approaches the cos-theta approximation of a perfect dipole. This simplified model can be implemented to get closer and closer to a real-world case, for instance, by increasing the number of sectors, transitioning from a single-sector to a multiple-sector configuration. In particular, if there are two sectors involved, it is called a double-sector dipole.

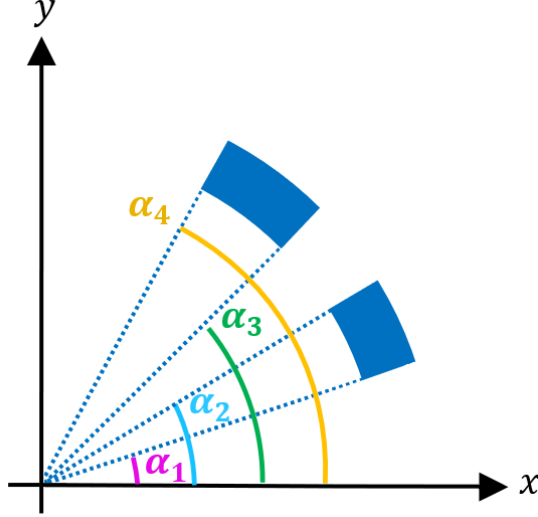


Figure 9: Schematic representation of a double-sector dipole.

For a double-sector dipole, the situation is analogous to what has already been seen for the single sector. This is because the contributions of the individual sectors overlap linearly for the magnetic field and forces. An exception is the energy stored per unit length, as in addition to the contributions of the two sectors, mutual inductances between the two sectors must be taken into account. Defining the total stored energy as  $E_T$ :

$$E_T = E_1 + E_2 + E_{12} + E_{21} \quad (129)$$

where  $E_1$  is the stored energy of the single sector between  $\alpha_1$  and  $\alpha_2$ ,  $E_2$  is the stored energy of the single sector between  $\alpha_3$  and  $\alpha_4$ , while  $E_{12}$  and  $E_{21}$  are the stored energies due to the mutual inductances. Defining all terms:

$$E_1 = \sum_{n \text{ odd}} \frac{4\mu_0 J_0^2}{n^3 \pi} \left[ \frac{a_2^4(2-n) + a_1^4(2+n) - 4a_1^{2+n}a_2^{2-n}}{2(4-n^2)} + \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \right] \times \\ \times [\sin(n\alpha_2) - \sin(n\alpha_1)]^2 \quad (130)$$

$$E_2 = \sum_{n \text{ odd}} \frac{4\mu_0 J_0^2}{n^3 \pi} \left[ \frac{a_2^4(2-n) + a_1^4(2+n) - 4a_1^{2+n}a_2^{2-n}}{2(4-n^2)} + \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \right] \times \\ \times [\sin(n\alpha_4) - \sin(n\alpha_3)]^2 \quad (131)$$

The stored energies by the mutual inductances are equal and given by:

$$E_{21} = E_{12} \quad (132)$$

$$E_{12} = \sum_{n \text{ odd}} \frac{4\mu_0 J_0^2}{n^3 \pi} \left[ \frac{a_2^4(2-n) + a_1^4(2+n) - 4a_1^{2+n}a_2^{2-n}}{2(4-n^2)} + \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \right] \times \\ \times [\sin(n\alpha_2) - \sin(n\alpha_1)] [\sin(n\alpha_4) - \sin(n\alpha_3)] \quad (133)$$

it's possible to calculate the total stored energy in a double-sector dipole as follows:

$$E_T = \sum_{n \text{ odd}} \frac{4\mu_0 J_0^2}{n^3 \pi} \left[ \frac{a_2^4(2-n) + a_1^4(2+n) - 4a_1^{2+n}a_2^{2-n}}{2(4-n^2)} + \frac{k_n}{R_1^{2n}} \left( \frac{a_2^{2+n} - a_1^{2+n}}{2+n} \right)^2 \right] \times \\ \times [\sin(n\alpha_2) - \sin(n\alpha_1) + \sin(n\alpha_4) - \sin(n\alpha_3)]^2 \quad (134)$$

In the case of a dipole with more than two sectors, it is possible to easily calculate all the various contributions by exploiting the linear superposition and, as just demonstrated, taking into account the influence of each sector on the others when calculating the energy stored in the coils. From section 2 onwards, the contribution due to the ferromagnetic yoke that typically surrounds the coils of a magnet has been taken into account. The iron yoke is essential for the correct calculation of various parameters involved. Iron was assumed linear to facilitate analytical processing. However, when the iron saturates for magnetic fields above 1.5 - 2 T, this assumption is no longer valid. Studies without iron yoke can be carried out by setting  $k_n = 0$ .