A. Turrin: QUANTUM-MECHANICAL FORM OF THE DAMPED BLOCH EQUATIONS.
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ABSTRACT.

The damped optical Bloch equations have been used to derive the form of the "corresponding" two-level atomic dynamics. It turns out that the resulting equations involve non-linear damping terms.
Formally, collisional effects can be introduced into the two-level atomic dynamics by combining Eqs. 1)

\begin{align*}
1a) \quad & \dot{X} = -\Delta Y \\
1b) \quad & \dot{Y} = \Delta X + \omega Z \\
1c) \quad & \dot{Z} = -\omega Y
\end{align*}

for the Bloch vector\(^{(1)}\) (in the rotating wave approximation and in a reference frame rotating with the rotating wave) and Eqs. 2)

\begin{align*}
2a) \quad & \dot{X} = -\Gamma_2 X \\
2b) \quad & \dot{Y} = -\Gamma_2 Y \\
2c) \quad & \dot{Z} = -\Gamma_1 (Z+1)
\end{align*}

for the relaxation. Thus, the damped Bloch equations are

\begin{align*}
3a) \quad & \dot{X} = -\Delta Y - \Gamma_2 X \\
3b) \quad & \dot{Y} = \Delta X - \Gamma_2 Y + \omega Z \\
3c) \quad & \dot{Z} = -\omega Y - \Gamma_1 (Z+1)
\end{align*}

where \(\omega = p\hat{e}(t)/\hbar\); \(p\) is the dipole matrix element between the upper state and the lower state, and \(\hat{e}(t)\) is the envelope of the optical pulse. \(\Delta = \Delta(t)\) is the detuning.

The phenomenological constants \(\Gamma_1\) and \(\Gamma_2\) denote the population rate and the phase relaxation rate respectively.

In this letter we give a quantum-mechanical representation of Eqs. 3) by use of suitable transformations.

We will restrict our analysis to the case where \(\Gamma_1 = 0\), since if \(\Gamma_1 \neq 0\) the two-level system decays back to its ground state \(X=0, Y=0, Z=-1\) as the pulse goes out (i.e. the solution at \(t=+\infty\) is known in the case \(\Gamma_1 \neq 0\)).

From Eqs. 3) it follows that the absolute value of the Bloch vector \(R = \sqrt{X^2 + Y^2 + Z^2}\) decays in length during the pulse. This decay law is governed by the equation

\begin{align*}
4) \quad & \dot{R}/R = -\Gamma_2 \left\{1 - (Z/R)^2\right\}
\end{align*}
Introduce in Eqs. 3) and 4) the two complex functions $\sigma$ and $\varepsilon$ defined by the relationships

\begin{align*}
5a) \quad X + iY &= -(R+Z)/(\varepsilon E) \\
5b) \quad X - iY &= (R+Z) \sigma^* E,
\end{align*}

where $E = \exp(\Gamma_2 t)$, and the star denotes complex conjugation. Transformations 5a) and 5b) are a generalization of the ones given by Darboux (2) for the undamped ($\dot{R}=0$, $R=1$) case.

It follows for $Z/R$ the expression

\begin{align*}
6a) \quad Z/R &= (\varepsilon^* \varepsilon - E^{-2})/(\varepsilon \varepsilon^* + E^{-2})
\end{align*}

or, alternatively,

\begin{align*}
6b) \quad Z/R &= (E^{-2} - \sigma \sigma^*)/(\sigma \sigma^* + E^{-2})
\end{align*}

and the relationship

\begin{align*}
7) \quad \sigma \varepsilon &= -E^{-2}
\end{align*}

holds.

A straightforward (although rather tedious) calculation leads to the following differential equations for $\sigma$ and $\varepsilon$:

\begin{align*}
8a) \quad \dot{\varepsilon} &= (i/2)\omega E \varepsilon^2 - i A \varepsilon - (i/2)\omega E - [2 \Gamma_2 E^{-2}/(\varepsilon \varepsilon^* + E^{-2})] \varepsilon \\
8b) \quad \dot{\sigma} &= - (i/2)\omega E \sigma^2 + i A \sigma + (i/2)\omega E - [2 \Gamma_2 E^{-2}/(\sigma \sigma^* + E^{-2})] \sigma
\end{align*}

On introduction of two new functions $f$ and $g$ by the substitutions

\begin{align*}
9a) \quad \varepsilon &= i(2/\omega)(\dot{f}/f) E^{-1} \\
9b) \quad \sigma &= -i(2/\omega)(\dot{g}/g) E^{-1}
\end{align*}

one obtains the equations for $f$ and $g$:

\begin{align*}
10a) \quad \ddot{f} + (iA - \dot{\omega}/\omega - \Gamma_2 Z/R) \dot{f} + (\omega/2)^2 f = 0 \\
10b) \quad \ddot{g} - (iA + \dot{\omega}/\omega - \Gamma_2 Z/R) \dot{g} + (\omega/2)^2 g = 0
\end{align*}

where

\begin{align*}
Z/R = \{(2/\omega)^2 |\dot{f}/f|^2 - 1\}/\{(2/\omega)^2 |\dot{f}/f|^2 + 1\} \\
\text{or} \quad Z/R = -\{(2/\omega)^2 |\dot{g}/g|^2 - 1\}/\{(2/\omega)^2 |\dot{g}/g|^2 + 1\}.
\end{align*}
As a consequence of the relationship 7), Eqs. 10a) and 10b) are coupled equations, i.e.

\[ (\dot{f}/f)(\dot{g}/g) = -(\omega/2)^2 \]

In the case \( T_2 = 0 \), Eqs. 10) reduce just to the equations encountered by Zener(3), Froissart and Stora(4) and Horwitz(5) as a consequence of their quantum-mechanical formulation of the problem.

Now, if the two Eqs. 10a) and 10b) are multiplied by \( \dot{g}/(g f) \) and \( \dot{f}/(g f) \), respectively, taking into account that the relationship 11) holds, one obtains the equations

12a) \[ -\ddot{g}/g + \dot{f}/f = -(\lambda \dot{\omega}/\omega - T_2 Z/R) \]

12b) \[ -\ddot{f}/f + \dot{g}/g = (\lambda \dot{\omega}/\omega - T_2 Z/R) \]

Integrating once, one gets

13a) \[ \dot{f} = (\dot{f}(0)/g(0)) \{\omega/\omega(0)\} g \exp \left[ - \int_0^t i\Delta dt + \int_0^t (T_2 Z/R) dt \right] \]

13b) \[ \dot{g} = (\dot{g}(0)/f(0)) \{\omega/\omega(0)\} f \exp \left[ - \int_0^t i\Delta dt - \int_0^t (T_2 Z/R) dt \right] \]

According to Eq. 11) and in order to maintain symmetry between the functions \( f \) and \( g \) we write

\[ \dot{f}(0)/g(0) = \dot{g}(0)/f(0) = -i\omega(0)/2 \]

and get

14a) \[ \dot{f} = -i(\omega/2) g \exp \left[ - \int_0^t i\Delta dt + \int_0^t (T_2 Z/R) dt \right] \]

14b) \[ \dot{g} = -i(\omega/2) f \exp \left[ - \int_0^t i\Delta dt - \int_0^t (T_2 Z/R) dt \right] \]

in the form obtained at once by the quoted Authors(3), (4), (5) in their quantum-mechanical treatment (with \( T_2 = 0 \)).

Now, to facilitate comparison with other work(6), (7), (8), we convert Eqs. 14a) and 14b), using
\[ f = a \exp \left[ -\frac{i}{2} \int_0^t \Delta dt \right] \]
\[ g = b \exp \left[ \frac{i}{2} \int_0^t \Delta dt \right] \]

16a) \[ \dot{a} - \frac{i}{2} \Delta a = -\frac{i}{2} \omega b \exp \left[ \int_0^t (\Gamma_0 Z/R) dt \right] \]
16b) \[ \dot{b} + \frac{i}{2} \Delta b = -\frac{i}{2} \omega a \exp \left[ -\int_0^t (\Gamma_0 Z/R) dt \right] \]

where
\[ Z/R = \frac{(2/\omega)^2 |a/a-(i/2)\Delta|^2 - 1}{(2/\omega)^2 |a/a-(i/2)\Delta|^2 + 1} \]
or
\[ Z/R = -\frac{(2/\omega)^2 |b/b+(i/2)\Delta|^2 - 1}{(2/\omega)^2 |b/b+(i/2)\Delta|^2 + 1} \]

which agree, for \( \Gamma_0 = 0 \), with those given by Kroll and Watson \(^7\) and Lau \(^8\), \(^8\).

It is worth drawing attention to a point about the occupation numbers \( a \) and \( b \), when \( \Gamma_0 = 0 \): with the normalization condition \( a^*a + b^*b = 1 \), \( X, Y \) and \( Z \) assume the expressions \( X = ab^* + ba^* \), \( Y = i(ab^* - ba^*) \), \( Z = aa^* - bb^* \). These are, in fact, the three components of the Bloch vector defined by Feynman, Vernon and Hellwarth \(^1\). This can be very quickly derived by Eqs. 5), 9), 14) and 15), with \( E = 1 \).

The conclusion can be drawn that the quantum-mechanical version of the damped Bloch equations puts us in a certain difficulty, because one cannot obtain insight into the meaning of the phase relaxation terms that come in. This is a consequence of the phenomenological nature of the damped Bloch equations and of the fact that the microscopic interpretation of dephasing processes is still a contumacious problem.
REFERENCES.


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(4) M. Froissart and R. Stora, Nucl. Instr. and Methods, 7, 297 (1960)


