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Research Article

Liouville Theorems for a Class of Linear Second-Order Operators with Nonnegative Characteristic Form

Alessia Elisabetta Kogoj and Ermanno Lanconelli

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We report on some Liouville-type theorems for a class of linear second-order partial differential equation with nonnegative characteristic form. The theorems we show improve our previous results.

1. Introduction

In this paper, we survey and improve some Liouville-type theorems for a class of hypoelliptic second-order operators, appeared in the series of papers [1–4].

The operators considered in these papers can be written as follows:

$$\mathscr{L} := \sum_{i,j=1}^{N} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}\right) + \sum_{i=1}^{N} b_i(x)\partial_{x_i} - \partial_t, \tag{1.1}$$

where the coefficients a_{ij}, b_i are t-independent and smooth in $\mathbb{R}^N$. The matrix $A = (a_{ij})_{i,j=1,\dots,N}$ is supposed to be symmetric and nonnegative definite at any point of $\mathbb{R}^N$.

We will denote by $z = (x,t)$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, the point of $\mathbb{R}^{N+1}$, by Y the first-order differential operator

$$Y := \sum_{i=1}^{N} b_i(x)\partial_{x_i} - \partial_t, \tag{1.2}$$

and by $\mathscr{L}_0$ the *stationary* counterpart of $\mathscr{L}$, that is,

$$\mathscr{L}_0 := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}) + \sum_{i=1}^{N} b_i(x)\partial_{x_i}. \tag{1.3}$$

We always assume the operator Y to be divergence free, that is, $\sum_{i=1}^{N} \partial_{x_i} b_i(x) = 0$ at any point $x \in \mathbb{R}^N$. Moreover, as in [2], we assume the following hypotheses.

(H1) $\mathscr{L}$ is homogeneous of degree two with respect to the group of dilations $(d_\lambda)_{\lambda>0}$ given by

$$\begin{gathered} d_\lambda(x,t) = (D_\lambda(x), \lambda^2 t), \\ D_\lambda(x) = D_\lambda(x_1,\dots,x_N) = (\lambda^{\sigma_1}x_1,\dots,\lambda^{\sigma_N}x_N), \end{gathered} \tag{1.4}$$

where $\sigma = (\sigma_1,\dots,\sigma_N)$ is an N-tuple of natural numbers satisfying $1 = \sigma_1 \le \sigma_2 \le \cdots \le \sigma_N$. When we say that $\mathscr{L}$ is d_λ-homogeneous of degree two, we mean that

$$\mathscr{L}(u(d_\lambda(x,t))) = \lambda^2 (\mathscr{L}u)(d_\lambda(x,t)) \quad \forall u \in C^\infty(\mathbb{R}^{N+1}). \tag{1.5}$$

(H2) For every $(x,t),(y,\tau) \in \mathbb{R}^{N+1}$, $t > \tau$, there exists an $\mathscr{L}$-admissible path $\eta : [0,T] \to \mathbb{R}^{N+1}$ such that $\eta(0) = (x,t)$, $\eta(T) = (y,\tau)$.

An $\mathscr{L}$-*admissible* path is any continuous path η which is the sum of a finite number of diffusion and drift trajectories.

A *diffusion trajectory* is a curve η satisfying, at any points of its domain, the inequality

$$(\langle \eta'(s), \xi\rangle)^2 \le \langle \hat{A}(\eta(s))\xi, \xi\rangle \quad \forall \xi \in \mathbb{R}^N. \tag{1.6}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^{N+1}$ and $\hat{A}(z) = \hat{A}(x,t) = \hat{A}(x)$ stands for the $(N+1)\times(N+1)$ matrix

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.7}$$

A *drift trajectory* is a positively oriented integral curve of Y.

Throughout the paper, we will denote by Q the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to the dilations (1.4), that is,

$$Q = \sigma_1 + \cdots + \sigma_N + 2 \tag{1.8}$$

and assume

$$Q \ge 5. \tag{1.9}$$

Then, the D_λ-homogeneous dimension of $\mathbb{R}^N$ is $Q-2 \ge 3$.

We explicitly remark that the smoothness of the coefficients of $\mathscr{L}$ and the homogeneity assumption in (H1) imply that the a_{ij}'s and the b_i's are polynomial functions (see [5, Lemma 2]). Moreover, the "oriented" connectivity condition in (H1) implies the hypoellipticity of $\mathscr{L}$ and of $\mathscr{L}_0$ (see [1, Proposition 10.1]). For any $z = (x,t) \in \mathbb{R}^{N+1}$, we define the d_λ-homogeneous norm $|z|$ by

$$|z| = \big|(x,t)\big| := \big(|x|^4 + t^2\big)^{1/4}, \tag{1.10}$$

where

$$|x| = \big|(x_1,\dots,x_N)\big| = \left(\sum_{j=1}^{N} (x_j^2)^{\sigma/\sigma_j}\right)^{1/2\sigma}, \quad \sigma = \prod_{j=1}^{N} \sigma_j. \tag{1.11}$$

Hypotheses (H1) and (H2) imply the existence of a fundamental solution $\Gamma(z,\zeta)$ of $\mathscr{L}$ with the following properties (see [2, page 308]):

(i) Γ is smooth in $\{(z,\zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \mid z \neq \zeta\}$,
(ii) $\Gamma(\cdot,\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\mathscr{L}\Gamma(\cdot,\zeta) = -\delta_\zeta$ for every $\zeta \in \mathbb{R}^{N+1}$,
(iii) $\Gamma(z,\cdot) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\mathscr{L}^*\Gamma(z,\cdot) = -\delta_z$ for every $z \in \mathbb{R}^{N+1}$,
(iv) $\limsup_{\zeta\to z}\Gamma(z,\zeta) = \infty$ for every $z \in \mathbb{R}^{N+1}$,
(v) $\Gamma(0,\zeta) \to 0$ as $\zeta \to \infty$, $\Gamma(0,d_\lambda(\zeta)) = \lambda^{-Q+2}\Gamma(0,\zeta)$,
(vi) $\Gamma((x,t),(\xi,\tau)) \geq 0$, > 0 if and only if $t > \tau$,
(vii) $\Gamma((x,t),(\xi,\tau)) = \Gamma((x,0),(\xi,\tau - t))$.

In (iii) $\mathscr{L}^*$ denotes the formal adjoint of $\mathscr{L}$.

These properties of Γ allow to obtain a mean value formula at $z = 0$ for the entire solutions to $\mathscr{L}u = 0$. We then use this formula to prove a *scaling* invariant *Harnack inequality* for the *nonnegative* solutions $\mathscr{L}u = f$ in $\mathbb{R}^{N+1}$. Our first Liouville-type theorems will follow from this Harnack inequality. All these results will be showed in Section 2.

In Section 3, we show some *asymptotic* Liouville theorem for nonnegative solution to $\mathscr{L}u = 0$ in the halfspace $\mathbb{R}^N\times]-\infty,0[$ assuming that $\mathscr{L}$, together with (H1) and (H2), is left invariant with respect to some Lie groups in $\mathbb{R}^{N+1}$.

Finally, in Section 4 some examples of operators to which our results apply are showed.

2. Polynomial Liouville theorems

Throughout this section, we will assume that $\mathscr{L}$ in (1.1) satisfies hypotheses (H1) and (H2). Let Γ be the fundamental solution of $\mathscr{L}$ with pole at the origin. With a standard procedure based on the Green identity for $\mathscr{L}$ and by using the properties of Γ recalled in the introduction, one obtains a mean value formula at $z = 0$ for the solution to $\mathscr{L}u = 0$. Precisely, for every point $(0,T) \in \mathbb{R}^{N+1}$ and $r > 0$, define the $\mathscr{L}$-ball centered at $(0,T)$ and with radius r, as follows:

$$\Omega_r(0,T) := \left\{\zeta \in \mathbb{R}^{N+1} : \Gamma\big((0,T),\zeta\big) > \left(\frac{1}{r}\right)^{Q-2}\right\}. \tag{2.1}$$

Then, if $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$, one has

$$u(0,T) = \left(\frac{1}{r}\right)^{Q-2} \int_{\Omega_r(0,T)} K(T,\zeta)u(\zeta)\,d\zeta, \tag{2.2}$$

where

$$K(T,\zeta) = \frac{\langle A(\xi)\nabla_\xi \Gamma, \nabla_\xi \Gamma\rangle}{\Gamma^2}, \quad \zeta = (\xi,\tau), \tag{2.3}$$

and Γ stands for $\Gamma((0,T),(\xi,\tau))$. Moreover, $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^N$ and ∇_ξ is the gradient operator $(\partial_{\xi_1},\dots,\partial_{\xi_N})$.

Formula (2.2) is just one of the numerous extensions of the classical Gauss mean value theorem for harmonic functions. For a proof of it, we directly refer to [6, Theorem 1.5]. We would like to stress that in this proof one uses our assumption $\operatorname{div} Y = 0$.

The kernel $K(T,\cdot)$ is strictly positive in a dense open subset of $\Omega_r(0,T)$ for every $T, r > 0$ (see [2, Lemma 2.3]). This property of $K(T,\cdot)$, together with the d_λ-homogeneity of $\mathscr{L}$, leads to the following Harnack-type inequality for entire solutions to $\mathscr{L}u = 0$.

Theorem 2.1. *Let $u : \mathbb{R}^{N+1} \to \mathbb{R}$ be a nonnegative solution to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$. Then, there exist two positive constants $C = C(\mathscr{L})$ and $\theta = \theta(\mathscr{L})$ such that*

$$\sup_{C_{\theta r}} u \le C u(0,r^2) \quad \forall r > 0, \tag{2.4}$$

where, for $\rho > 0$, C_ρ denotes the d_λ-symmetric ball

$$C_\rho := \{z \in \mathbb{R}^{N+1} \mid |z| < \rho\}. \tag{2.5}$$

The proof of this theorem is contained in [2, page 310].

By using inequality (2.4) together with some basic properties of the fundamental solution Γ, one easily gets the following a priori estimates for the positive solution to $\mathscr{L}u = f$ in $\mathbb{R}^{N+1}$.

Corollary 2.2. *Let f be a smooth function in $\mathbb{R}^{N+1}$ and let u be a nonnegative solution to*

$$\mathscr{L}u = f \quad \text{in } \mathbb{R}^{N+1}. \tag{2.6}$$

Then there exists a positive constant C independent of u and f such that

$$u(z) \le C u\left(0, \left(\frac{|z|}{\theta}\right)^2\right) + |z|^2 \sup_{|\zeta| \le |z|/\theta^2} |f(\zeta)|, \tag{2.7}$$

where θ is the constant in Theorem 2.1.

This result allows to use the Liouville-type theorem of Luo [5] to obtain our main result in this section.

Theorem 2.3. *Let $u : \mathbb{R}^{N+1} \to \mathbb{R}$ be a smooth function such that*

$$\begin{aligned} \mathscr{L}u &= p \quad \text{in } \mathbb{R}^{N+1}, \\ u &\ge q \quad \text{in } \mathbb{R}^{N+1}, \end{aligned} \tag{2.8}$$

where p and q are polynomial function. Assume

$$u(0,t) = O(t^m) \quad \text{as } t \longrightarrow \infty. \tag{2.9}$$

Then, u is a polynomial function.

Proof. We split the proof into two steps.
Step 1. There exists $n > 0$ such that

$$u(z) = O(|z|^n) \quad \text{as } z \longrightarrow \infty. \tag{2.10}$$

Indeed, letting $v := u - q$, we have

$$\begin{gathered} \mathcal{L}v = p - \mathcal{L}q \quad \text{in } \mathbb{R}^{N+1}, \\ v \geq 0 \quad \text{in } \mathbb{R}^{N+1}, \end{gathered} \tag{2.11}$$

and $v(0,t) = u(0,t) - q(0,t) = O(t^{n_1})$ as $t \to \infty$, for a suitable $n_1 > 0$. Moreover, since p and $\mathcal{L}q$ are polynomial functions, $(p - \mathcal{L}q)(z) = O(|z|^{m_1})$ as $z \to \infty$ for a suitable $m_1 > 0$. Then, by the previous corollary, there exists $m_2 > 0$ such that

$$v(z) = O(|z|^{m_2}) \quad \text{as } z \longrightarrow \infty. \tag{2.12}$$

From this estimate, since $v = u + q$, and q is a polynomial function, the assertion (2.10) follows.
Step 2. Since p is a polynomial function and $\mathcal{L}$ is d_λ-homogeneous, there exists $m \in \mathbb{N}$ such that

$$\mathcal{L}^{(m)} p \equiv 0, \tag{2.13}$$

where $\mathcal{L}^{(m)} = \mathcal{L} \circ \cdots \circ \mathcal{L}$ is the mth iterated of $\mathcal{L}$. It follows that

$$\mathcal{L}^{(m+1)} u = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{2.14}$$

Moreover, since $\mathcal{L}$ is *d_λ-homogeneous* and *hypoelliptic*, the same properties hold for $\mathcal{L}^{(m+1)}$. On the other hand, by Step 1, $u(z) = O(z^m)$ as $z \to \infty$, so that u is a *tempered distribution*. Then, by Luo's paper [5, Theorem 1], u is a *polynomial function*. □

Remark 2.4. It is well known that hypothesis (2.9) in the previous theorem cannot be removed. Indeed, if $\mathcal{L} = \Delta - \partial_t$ is the classical heat operator and $u(x,t) = \exp(x_1 + \cdots + x_N + Nt)$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $t \in \mathbb{R}$, we have

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^{N+1},\ u \geq 0, \tag{2.15}$$

and u is *not* a polynomial function.

In the previous theorem, the degree of the polynomial function u can be estimated in terms of the ones of p and q. For this, we need some more notation. If $\alpha = (\alpha_1, \dots, \alpha_N, \alpha_{N+1})$ is a multi-index with $(N+1)$ nonnegative integer components, we let

$$|\alpha|_{d_\lambda} := \sigma_1\alpha_1 + \cdots + \sigma_N\alpha_N + 2\alpha_{N+1}, \tag{2.16}$$

and, if $z=(x,t)=(x_1,\dots,x_N,t)\in\mathbb{R}^{N+1}$,

$$z^\alpha := x_1^{\alpha_1}\cdots x_N^{\alpha_N}t^{\alpha_{N+1}}. \tag{2.17}$$

As a consequence, we can write every polynomial function p in $\mathbb{R}^{N+1}$, as follows:

$$p(z)=\sum_{|\alpha|_{d_\lambda}\le m} c_\alpha z^\alpha \tag{2.18}$$

with $m\in\mathbb{Z}$, $m\ge 0$, and $c_\alpha\in\mathbb{R}$ for every multi-index α. If

$$\sum_{|\alpha|_{d_\lambda}=m} c_\alpha z^\alpha \not\equiv 0 \quad \text{in } \mathbb{R}^{N+1}, \tag{2.19}$$

then we set

$$m=\deg_{d_\lambda} p. \tag{2.20}$$

If p is independent of t, that is, if p is a polynomial function in $\mathbb{R}^N$, we denote by

$$\deg_{D_\lambda} p \tag{2.21}$$

the degree of p with respect to the dilations $(D_\lambda)_{\lambda>0}$. Obviously, in this case, $\deg_{D_\lambda} p = \deg_{d_\lambda} p$.

Proposition 2.5. *Let $u,p:\mathbb{R}^{N+1}\to\mathbb{R}$ be polynomial functions such that*

$$\mathcal{L}u=p \quad \textit{in } \mathbb{R}^{N+1}. \tag{2.22}$$

Assume $u\ge 0$. Thus, the following statements hold.

(i) *If $p\equiv 0$, then $u=$* constant.

(ii) *If $p\not\equiv 0$, then*

$$\deg_{d_\lambda} u = 2+\deg_{d_\lambda} p. \tag{2.23}$$

This proposition is a consequence of the following lemma.

Lemma 2.6. *Let $u:\mathbb{R}^{N+1}\to\mathbb{R}$ be a nonnegative polynomial function d_λ-homogeneous of degree $m>0$. Then $\mathcal{L}u\not\equiv 0$ in $\mathbb{R}^{N+1}$.*

Proof. We argue by contradiction and assume $\mathcal{L}u=0$. Since u is nonnegative and d_λ-homogeneous of strictly positive degree, we have

$$u(0,0)=0=\min_{\mathbb{R}^{N+1}} u. \tag{2.24}$$

Let us now denote by $\mathcal{P}$ the $\mathcal{L}$-propagation set of $(0,0)$ in $\mathbb{R}^{N+1}$, that is, the set

$$\begin{aligned}\mathcal{P} := \{z\in\mathbb{R}^{N+1} &: \text{there exists an } \mathcal{L}\text{-admissible path } \eta:[0,T]\longrightarrow\mathbb{R}^{N+1},\\ &\text{s.t. } \eta(0)=(0,0),\ \eta(T)=z\}.\end{aligned} \tag{2.25}$$

From hypotheses (H2), we obtain $\mathcal{P} = \mathbb{R}^N \times]-\infty, 0]$ so that, since $(0,0)$ is a minimum point of u and the minimum spread all over $\mathcal{P}$ (see [7]), we have

$$u(z) = u(0,0) = 0 \quad \forall z \in \mathbb{R}^N \times]-\infty, 0]. \tag{2.26}$$

Then, being u a polynomial function, $u \equiv 0$ in $\mathbb{R}^{N+1}$. This contradicts the assumption $\deg_{d_\lambda} u > 0$, and completes the proof. □

Proof of Proposition 2.5. Obviously, if $u =$ constant, we have nothing to prove. If we assume $m := \deg_{d_\lambda} u > 0$ and prove that

$$m \geq 2, \; p \not\equiv 0, \quad \deg_{d_\lambda} p = m - 2, \tag{2.27}$$

then it would complete the proof. Let us write u as follows:

$$u = u_0 + u_1 + \cdots + u_m, \tag{2.28}$$

where u_j is a polynomial function d_λ-homogeneous of degree j, $j = 0, \dots, m$, and $u_m \not\equiv 0$ in $\mathbb{R}^{N+1}$.

Then

$$p = \mathcal{L}u = \mathcal{L}u_0 + \mathcal{L}u_1 + \cdots + \mathcal{L}u_m, \tag{2.29}$$

and, since $\mathcal{L}$ is d_λ-homogeneous of degree two,

$$(\mathcal{L}u_j)(d_\lambda(x)) = \lambda^{j-2}\mathcal{L}u_j(x) \tag{2.30}$$

so that $\mathcal{L}u_0 = \mathcal{L}u_1 \equiv 0$ and $\deg_{d_\lambda} \mathcal{L}u_j = j - 2$ if and only if $\mathcal{L}u_j \not\equiv 0$.

On the other hand, the hypothesis $u \geq 0$ implies $u_m \geq 0$ so that, being $u_m \not\equiv 0$ and d_λ-homogeneous of degree $m > 0$, by Lemma 2.6, we get $\mathcal{L}u_m \not\equiv 0$. Hence $m \geq 2$. Moreover, by (2.29), $p = \mathcal{L}u \not\equiv 0$ and

$$\deg_{d_\lambda} p = \deg_{d_\lambda} \mathcal{L}u_m = m - 2. \tag{2.31}$$

□

This proposition allows us to make more precise the conclusion of Theorem 2.3. Indeed, we have the following.

Proposition 2.7. *Let $u, p, q : \mathbb{R}^{N+1} \to \mathbb{R}$ be polynomial functions such that*

$$\begin{aligned} \mathcal{L}u = p \quad & \text{in } \mathbb{R}^{N+1}, \\ u \geq q \quad & \text{in } \mathbb{R}^{N+1}. \end{aligned} \tag{2.32}$$

Then

$$\deg_{d_\lambda} u \leq \max\{2 + \deg_{d_\lambda} p, \deg_{d_\lambda} q\}. \tag{2.33}$$

In particular, and more precisely, if $q = 0$, that is, if $u \geq 0$, then

$$\begin{aligned} \deg_{d_\lambda} u = 2 + \deg_{d_\lambda} p \quad & \text{if } p \not\equiv 0, \\ u = \text{constant} \quad & \text{if } p \equiv 0. \end{aligned} \tag{2.34}$$

Proof. If $q \equiv 0$, the assertion is the one of Proposition 2.5. Suppose $q \not\equiv 0$. By letting $v := u - q$, we have

$$\mathcal{L}v = p - \mathcal{L}q, \quad v \geq 0. \tag{2.35}$$

By Proposition 2.5, we have

$$\deg_{d_\lambda} v \leq 2 + \deg_{d_\lambda}(p - \mathcal{L}q) \leq 2 + \max\{\deg_{d_\lambda} p, \deg_{d_\lambda} q - 2\} = \max\{2 + \deg_{d_\lambda} p, \deg_{d_\lambda} q\} \tag{2.36}$$

and (2.33) follows. □

Proposition 2.7, together with Theorem 2.3, extends and improves the Liouville-type theorems contained in [2, 4] (precisely [2, Theorem 1.1] and [4, Theorem 1.2]).

From Theorem 2.3 and Proposition 2.7, we straightforwardly get the following polynomial Liouville theorem for the stationary operator $\mathcal{L}_0$ in (1.3).

Theorem 2.8. *Let $P, Q : \mathbb{R}^N \to \mathbb{R}$ be polynomial functions and let $U : \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that*

$$\mathcal{L}_0 U = P, \quad U \geq Q, \text{ in } \mathbb{R}^N. \tag{2.37}$$

Then, U is a polynomial function and

$$\deg_{D_\lambda} U \leq \max\{2 + \deg_{D_\lambda} P, \deg_{D_\lambda} Q\}. \tag{2.38}$$

In particular, and more precisely, if $Q \equiv 0$, that is, if $U \geq 0$, then

$$\begin{gathered} \deg_{D_\lambda} U = 2 + \deg_{D_\lambda} P \quad \text{if } P \not\equiv 0, \\ U = \text{constant} \quad \text{if } P \equiv 0. \end{gathered} \tag{2.39}$$

Proof. Let us define

$$u(x,t) = U(x), \qquad p(x,t) = P(x), \qquad q(x,t) = Q(x). \tag{2.40}$$

Then p, q are polynomial functions in $\mathbb{R}^{N+1}$ and u is a smooth solution to the equation

$$\mathcal{L}u = p \quad \text{in } \mathbb{R}^{N+1}, \tag{2.41}$$

such that $u \geq q$. Moreover,

$$u(0,t) = U(0) = O(1) \quad \text{as } t \longrightarrow \infty. \tag{2.42}$$

Then, by Theorem 2.3, u is a polynomial function in $\mathbb{R}^{N+1}$. This obviously implies that U is a polynomial in $\mathbb{R}^N$. The second part of the theorem immediately follows from Proposition 2.5. □

Remark 2.9. The class of our stationary operators $\mathcal{L}_0$ also contains "parabolic"-type operators like, for example, the following "forward-backward" heat operator

$$\mathcal{L}_0 := \partial_{x_1}^2 + x_1\partial_{x_2} \quad \text{in } \mathbb{R}^2. \tag{2.43}$$

Nevertheless, in Theorem 2.8, we do not require any a priori behavior at infinity, like condition (2.9) in Theorem 2.3.

3. Asymptotic Liouville theorems in halfspaces

The operator $\mathcal{L}$ in our class do not satisfy the *usual* Liouville property. Precisely, if u is a nonnegative solution to

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^{N+1}, \tag{3.1}$$

then we cannot conclude that $u \equiv$ constant without asking an extra condition on the solution u (see Theorem 2.3 and Remark 2.4).

However, if we also assume that $\mathcal{L}$ is left translation invariant with respect to the composition law of some Lie group in $\mathbb{R}^{N+1}$, then we can show that *every nonnegative solution of (3.1) is constant at $t = -\infty$*.

To be precise, let us fix the new hypothesis on $\mathcal{L}$ and give the definition of *$\mathcal{L}$-parabolic trajectory*.

Suppose $\mathcal{L}$ satisfies (H2) of the introduction and, instead of (H1), the following condition

(H1)* There exists a homogeneous Lie group in $\mathbb{R}^{N+1}$,

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda) \tag{3.2}$$

such that $\mathcal{L}$ is left translation invariant on $\mathbb{L}$ and d_λ-homogeneous of degree two.

We assume the composition law $\circ$ is Euclidean in the time variable, that is,

$$(x,t) \circ (x',t') = \big(c(x,t,x',t'), t+t'\big), \tag{3.3}$$

where $c(x,t,x',t')$ denotes a suitable function of (x,t) and (x',t').

It is a standard matter to prove the existence of a positive constant C such that

$$|z \circ \zeta| \le C\big(|z| + |\zeta|\big) \quad \forall z, \zeta \in \mathbb{R}^{N+1}. \tag{3.4}$$

Let $\gamma : [0,\infty[\to \mathbb{R}^N$ be a continuous function such that

$$\limsup_{s\to\infty} \frac{|\gamma(s)|^2}{s} < \infty \tag{3.5}$$

(here $|\cdot|$ denotes the D_λ-homogeneous norm (1.11)).

Then, the path

$$s \longmapsto \eta(s) = (\gamma(s), T - s), \quad T \in \mathbb{R}, \tag{3.6}$$

will be called an $\mathcal{L}$-*parabolic* trajectory.

Obviously, the curve

$$s \longmapsto \eta(s) = (\alpha, T - s), \quad \alpha \in \mathbb{R}^N, \ T \in \mathbb{R} \tag{3.7}$$

is an $\mathcal{L}$-parabolic trajectory. It can be proved that every integral curve of the vector fields Y in (1.2) also is an $\mathcal{L}$-parabolic trajectory (see [3, Lemma 3]).

Our first asymptotic Liouville theorem is the following one.

Theorem 3.1. *Let $\mathcal{L}$ satisfy hypotheses (H1)* and (H2), and let u be a nonnegative solution to the equation*

$$\mathcal{L}u = 0 \tag{3.8}$$

in the halfspace

$$S = \mathbb{R}^N \times] - \infty, 0[. \tag{3.9}$$

Then, for every $\mathcal{L}$-parabolic trajectory η,

$$\lim_{s \to \infty} u(\eta(s)) = \inf_S u. \tag{3.10}$$

In particular

$$\lim_{t \to -\infty} u(x,t) = \inf_S u \quad \forall x \in \mathbb{R}^N. \tag{3.11}$$

The proof of this theorem relies on a *left translation* and *scaling invariant* Harnack inequality for nonnegative solutions to $\mathcal{L}u = 0$.

For every $z_0 \in \mathbb{R}^{N+1}$ and $M > 0$, let us put

$$P_{z_0}(M) := z_0 \circ P(M), \tag{3.12}$$

where

$$P(M) := \{(x,t) \in \mathbb{R}^{N+1} : |x|^2 \leq -Mt\}. \tag{3.13}$$

Then, the following theorem holds.

Theorem 3.2 (left and scaling invariant Harnack inequality). *Let u be a nonnegative solution to*

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^N \times] - \infty, 0[. \tag{3.14}$$

Then, for every $z_0 \in \mathbb{R}^N \times]-\infty, 0[$ and $M > 0$, there exists a positive constant $C = C(M)$, independent of z_0 and u, such that

$$\sup_{P_{z_0}(M)} u \le Cu(z_0). \tag{3.15}$$

Proof. It follows from Theorem 2.1 and the left translation invariance of $\mathscr{L}$. The details are contained in [3, Proof of Theorem 3]. □

From this theorem we obtain the proof of Theorem 3.1.

Proof of Theorem 3.1. We may assume $\inf_S u = 0$. Let $\eta(s) = (\gamma(s), s_0 - s)$, $s_0 \le 0$, $s \ge s_0$ be an $\mathscr{L}$-parabolic trajectory. Then, there exists $M_0 > 0$ such that

$$|\gamma(s)|^2 \le M_0 s \quad \forall s \ge s^*, \tag{3.16}$$

where $s^* > 0$ is big enough. Let us put $M = 2C(M_0^2 + 1)^{1/4}$ where C is the positive constant in the triangular inequality (3.4). Let $\varepsilon > 0$ be arbitrarily fixed and choose $z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in S$ such that

$$u(z_\varepsilon) < \varepsilon. \tag{3.17}$$

Now, for every $s \ge s^*$, we have

$$\begin{aligned} |z_\varepsilon^{-1} \circ \eta(s)| &\le C(|z_\varepsilon^{-1}| + |\eta(s)|) \\ &\le C(|z_\varepsilon^{-1}| + (M_0^2 + 1)^{1/4}\sqrt{s}) \\ &= C\sqrt{s - s_0 + t_\varepsilon}\left(\frac{|z_\varepsilon^{-1}|}{\sqrt{s - s_0 + t_\varepsilon}} + (M_0^2 + 1)^{1/4}\sqrt{\frac{s}{s - s_0 + t_\varepsilon}}\right). \end{aligned} \tag{3.18}$$

Then, there exists $T = T(\varepsilon) > 0$ such that

$$|z_\varepsilon^{-1} \circ \eta(s)| \le M\sqrt{s - s_0 + t_\varepsilon} \quad \forall s > T. \tag{3.19}$$

This implies that

$$\eta(s) \in z_\varepsilon \circ P(M) \equiv P_{z_\varepsilon}(M) \quad \forall s > T. \tag{3.20}$$

On the other hand, by the Harnack inequality of Theorem 3.2, there exists $C^* = C^*(M) > 0$ independent of z_ε and ε such that

$$\sup_{P_{z_\varepsilon}(M)} u \le C^* u(z_\varepsilon). \tag{3.21}$$

Therefore,

$$u(\eta(s)) \le C^* \varepsilon \quad \forall s > T. \tag{3.22}$$

Since C^* is independent of ε, this proves the theorem. □

Theorem 3.1 is contained in [3, Theorem 1]. The idea of our proof is taken from Glagoleva's paper [8], in which classical parabolic operators of Cordes-type are considered. For the heat equation, a stronger version of Theorem 3.1 was proved by Bear [9].

The following theorem improves Theorem 3.1.

Theorem 3.3. *Let $\mathcal{L}$ and u as in Theorem 3.1. For every $M > 0$ and $t < 0$, define*

$$M(u,t) = \sup\{u(x,t) : |x|^2 \leq -Mt\}. \tag{3.23}$$

Then

$$\lim_{t\to-\infty} M(u,t) = \inf_S u. \tag{3.24}$$

Proof. Let ε be arbitrarily fixed and let $z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in S$ be such that

$$u(z_\varepsilon) < m + \varepsilon, \quad m := \inf_S u. \tag{3.25}$$

Let M_0 be a positive constant that will be chosen later independently of ε. Since $u - m$ is a nonnegative solution to $\mathcal{L}v = 0$ in S, the Harnack inequality of Theorem 3.2 implies

$$u(z) - m \leq C_0(u(z_\varepsilon) - m) \quad \forall z \in P_{z_\varepsilon}(M_0), \tag{3.26}$$

where $C_0 = C_0(M_0)$ is independent of ε (and u).

Let C be the constant in the triangularity inequality (3.4) and choose $T = T(u,\varepsilon) > 0$ such that

$$T > 2|z_\varepsilon - 1|^2 + 2|t_\varepsilon|. \tag{3.27}$$

Then, if $z = (x,t) \in S$ with $t < -T$ and $|x|^2 < -Mt$, we have

$$\begin{aligned} |z_\varepsilon^{-1} \circ z| &\leq C(|z_\varepsilon|^{-1} + |z|) \leq C(|z_\varepsilon|^{-1} + (\sqrt{M}+1)\sqrt{-t}) \\ &= C\sqrt{t_\varepsilon - t}\left(\frac{|z_\varepsilon^{-1}|}{\sqrt{t_\varepsilon - t}} + (\sqrt{M}+1)\sqrt{\frac{1}{1-|t_\varepsilon/t|}}\right) \\ &\leq C\sqrt{t_\varepsilon - t}(1 + \sqrt{2}(\sqrt{M}+1)) =: M_0. \end{aligned} \tag{3.28}$$

Then, by (3.25) and (3.26),

$$m \leq u(z) \leq m + C_0\varepsilon \tag{3.29}$$

for every $z = (x,t) \in S$ with $t < -T$ and $|x|^2 < -Mt$. Thus

$$m \leq M(u,t) \leq m + C_0\varepsilon \quad \forall t < -T. \tag{3.30}$$

Since C_0 does not depend on ε, this completes the proof. □

4. Some examples

In this section, we show some explicit examples of operators to which our results apply.

Example 4.1 (heat operators on Carnot groups). Let $(\mathbb{R}^N,\circ)$ be a Lie group in $\mathbb{R}^N$. Assume that $\mathbb{R}^N$ can be split as follows:

$$\mathbb{R}^N = \mathbb{R}^{N_1}\times\cdots\times\mathbb{R}^{N_m} \tag{4.1}$$

and that the dilations

$$\begin{gathered} D_\lambda:\mathbb{R}^N\longrightarrow\mathbb{R}^N,\qquad D_\lambda\left(x^{(N_1)},\dots,x^{(N_m)}\right)=\left(\lambda x^{(N_1)},\dots,\lambda^m x^{(N_m)}\right)\\ x^{(N_i)}\in\mathbb{R}^{N_i},\quad i=1,\dots,m,\ \lambda>0, \end{gathered} \tag{4.2}$$

are automorphisms of $(\mathbb{R}^N,\circ)$.

We also assume

$$\operatorname{rank}\operatorname{Lie}\{X_1,\dots,X_{N_1}\}(x)=N\quad\forall x\in\mathbb{R}^N, \tag{4.3}$$

where the X_j's are left invariant on $(\mathbb{R}^N,\circ)$ and

$$X_j(0)=\frac{\partial}{\partial x_j^{(N_1)}},\quad j=1,\dots,N_1. \tag{4.4}$$

Then $\mathbb{G}=(\mathbb{R}^N,\circ,\delta_\lambda)$ is a *Carnot group* whose *homogeneous dimension* Q_0 is the natural number

$$Q_0:=N_1+2N_2+mN_m. \tag{4.5}$$

The vector fields $X_1,\dots,X_{N_1}$ are the *generators* of $\mathbb{G}$,

$$\Delta_{\mathbb{G}}:=\sum_{j=1}^{N_1}X_j^2 \tag{4.6}$$

is the *canonical sub-Laplacian* on $\mathbb{G}$ and the parabolic operator

$$\mathscr{L}=\Delta_{\mathbb{G}}-\partial_t\quad\text{in }\mathbb{R}^{N+1} \tag{4.7}$$

is called the *canonical heat operator* on $\mathbb{G}$. Obviously $\mathscr{L}$ can be written as in (3.25). Moreover, if we define

$$\mathbb{L}=\left(\mathbb{R}^{N+1},\circ,d_\lambda\right) \tag{4.8}$$

with $d_\lambda(x,t)=(D_\lambda x,\lambda^2 t)$ and the composition law $\circ$ given by

$$(x,t)\circ(x',t')=(x\circ x',t+t'), \tag{4.9}$$

then $\mathbb{L}$ is a homogeneous group, and the operator $\mathscr{L}$ in (4.7) satisfies condition (H1)*. We explicitly remark that the homogeneous dimension of $\mathbb{L}$ is $Q:=Q_0+2$.

In [1, page 70], it is proved that $\mathscr{L}$ also satisfies (H2).

Remark 4.2. The stationary part of the operator $\mathscr{L}$ in (4.7) is the sub-Laplacian $\Delta_{\mathbb{G}}$. For this kind of operator, the polynomial Liouville theorem in Theorem 2.8 was first proved in [10, Theorem 1.4].

Example 4.3 (B-Kolmogorov operators). Let us split $\mathbb{R}^N$ as follows:

$$\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^r \tag{4.10}$$

and denote by $x = (x^{(p)}, x^{(r)})$ its points. Let B be an $N \times N$ real matrix taking the following block form:

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ B_1 & 0 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_k & 0 \end{pmatrix}, \tag{4.11}$$

where B_j is an $r_j \times r_{j-1}$ matrix with rank r_j, and $r_0 = p \geq r_1 \geq \cdots \geq r_k \geq 1$, $r_0 + r_1 + \cdots + r_k = N$. Denote

$$E(t) = \exp(-tB) \tag{4.12}$$

and introduce in $\mathbb{R}^{N+1}$ the following composition law

$$(x,t) \circ (y,\tau) := (y + E(\tau)x, t + \tau). \tag{4.13}$$

The triplet

$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ, d_\lambda) \tag{4.14}$$

is a homogeneous Lie group with respect to the dilations

$$d_\lambda(x,t) = d_\lambda(x^{(p)}, x^{(r_1)}, \ldots, x^{(r_k)}, t) = (\lambda x^{(p)}, \lambda^3 x^{(r_1)}, \ldots, \lambda^{2k+1} x^{(r_k)}, \lambda^2 t) \tag{4.15}$$

(see [11]). The homogeneous dimension of $\mathbb{K}$ is

$$Q = p + 3r_1 + \cdots + (2k+1)r_k + 2. \tag{4.16}$$

We call $\mathbb{K}$ a *B-Kolmogorov-type group*.

Let us now consider the operator

$$\mathscr{K} = \Delta_{\mathbb{R}_p} + \langle Bx, D\rangle - \partial_t, \tag{4.17}$$

where $\Delta_{\mathbb{R}_p}$ denotes the usual Laplace operator in $\mathbb{R}^p$, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^N$, and $D = (\partial_{x_1}, \ldots, \partial_{x_N})$. In this case, we have

$$Y = \langle Bx, D\rangle - \partial_t. \tag{4.18}$$

The operator $\mathscr{K}$ satisfies (H1)* and (H2), and it is left translation invariant on $\mathbb{K}$ (see [1, 11]).

Remark 4.4. The matrix $E(t)$ in (4.13) takes the following triangular form:

$$E(t) = \begin{pmatrix} I_p & 0 \\ E_1(t) & I_r \end{pmatrix}, \tag{4.19}$$

where I_p and I_r are the identity matrix in $\mathbb{R}^p$ and $\mathbb{R}^r$, respectively. Then, the composition law in $\mathbb{K}$ has the following structure:

$$(x^{(p)}, x^{(r)}, t) \circ (y^{(p)}, y^{(r)}, \tau) = (x^{(p)} + y^{(p)}, x^{(r)} + y^{(r)} + E_1(\tau)x^{(p)}, t + \tau). \tag{4.20}$$

Remark 4.5. The *stationary* part of $\mathcal{H}$,

$$\mathcal{H}_0 = \Delta_{\mathbb{R}_p} + \langle Bx, D \rangle, \tag{4.21}$$

is contained in the class of degenerate Ornstein-Uhlenbeck operators studied by Priola and Zabczyk [12], where a Liouville theorem for *bounded* solutions is proved.

Example 4.6 (sub-Kolmogorov operators). Let $\mathbb{G} = (\mathbb{R}^p \times \mathbb{R}^q, \circ, d_\lambda^{(1)})$ be a Carnot group with first layer $\mathbb{R}^p$ and let $\mathbb{K} = (\mathbb{R}^p \times \mathbb{R}^r \times \mathbb{R}, \circ, d_\lambda^{(2)})$ be a Kolmogorov group. Let $\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$, $N = p + q + r$,

$$\mathbb{L} = \mathbb{G} \bigtriangleup \mathbb{K} \tag{4.22}$$

be the link of $\mathbb{G}$ and $\mathbb{K}$ (see [13, Section 5.2]).

Then, if Y is a derivative operator transverse to $\mathbb{G}$ (see [13, Definition 4.5]), and $X_1, \ldots, X_p$ are the generators of $\mathbb{G}$, the operator

$$\mathcal{L} = \sum_{j=1}^{p} X_j^2 + Y, \quad \text{in } \mathbb{R}^{N+1}, \tag{4.23}$$

satisfies (H1)* and (H2).

Example 4.7 (a nontranslations invariant operator). The operator

$$\mathcal{L} = \partial_{x_1}^2 + x_1^{2m+1}\partial_{x_2} - \partial_t \quad \text{in } \mathbb{R}^3 \tag{4.24}$$

$m \in \mathbb{N}$, satisfies hypotheses (H1) and (H2). The relevant dilation group is given by

$$d_\lambda(x_1, x_2, t) = (\lambda x_1, \lambda^{2m+3} x_2, \lambda^2). \tag{4.25}$$

Finally, it is easy to recognize that there is no Lie group structure in $\mathbb{R}^3$ leaving left translation invariant the operator $\mathcal{L}$.

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Alessia Elisabetta Kogoj: Dipartimento di Matematica, Università di Bologna,
Piazza di Porta San Donato 5, 40126 Bologna, Italy
Email address: kogoj@dm.unibo.it

Ermanno Lanconelli: Dipartimento di Matematica, Università di Bologna,
Piazza di Porta San Donato 5, 40126 Bologna, Italy
Email address: lanconel@dm.unibo.it