CONVERGENCE THEOREMS FOR FIXED POINTS OF DEMICONTINUOUS PSEUDOCONTRACTIVE MAPPINGS

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Let D be an open subset of a real uniformly smooth Banach space E. Suppose $T:\bar{D}\to E$ is a demicontinuous pseudocontractive mapping satisfying an appropriate condition, where $\bar{D}$ denotes the closure of D. Then, it is proved that (i) $\bar{D}\subseteq\mathcal{R}(I+r(I-T))$ for every $r>0$; (ii) for a given $y_0\in D$, there exists a unique path $t\to y_t\in\bar{D}$, $t\in[0,1]$, satisfying $y_t:=tTy_t+(1-t)y_0$. Moreover, if $F(T)\neq\emptyset$ or there exists $y_0\in D$ such that the set $K:=\{y\in D: Ty=\lambda y+(1-\lambda)y_0 \text{ for } \lambda>1\}$ is bounded, then it is proved that, as $t\to 1^-$, the path $\{y_t\}$ converges strongly to a fixed point of T. Furthermore, explicit iteration procedures with bounded error terms are proved to converge strongly to a fixed point of T.

1. Introduction

Let D be a nonempty subset of a real linear space E. A mapping $T:D\to E$ is called *a contraction mapping* if there exists $L\in[0,1)$ such that $\|Tx-Ty\|\leq L\|x-y\|$ for all $x,y\in D$. If $L=1$ then T is called *nonexpansive*. T is called *pseudocontractive* if there exists $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty,j(x-y)\rangle\leq\|x-y\|^2,\quad\forall x,y\in K,\tag{1.1}$$

where J is the normalized duality mapping from E to 2^{E^*} defined by

$$Jx:=\{f^*\in E^*:\langle x,f^*\rangle=\|x\|^2=\|f^*\|^2\}.\tag{1.2}$$

T is called *strongly pseudocontractive* if there exists $k\in(0,1)$ such that

$$\langle Tx-Ty,j(x-y)\rangle\leq k\|x-y\|^2,\quad\forall x,y\in K.\tag{1.3}$$

Clearly the class of nonexpansive mappings is a subset of class of pseudocontractive mappings. T is said to be *demicontinuous* if $\{x_n\}\subseteq D$ and $x_n\to x\in D$ together imply that $Tx_n\rightharpoonup Tx$, where $\to$ and $\rightharpoonup$ denote the strong and weak convergences, respectively. We denote by $F(T)$ the set of fixed points of T.

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Closely related to the class of pseudocontractive mappings is the class of accretive mappings. A mapping $A : D(A) \subseteq E \to E$ is called *accretive* if $T := (I - A)$ is pseudocontractive. If E is a Hilbert space, accretive operators are also called *monotone*. An operator A is called *m-accretive* if it is accretive and $\mathcal{R}(I + rA)$, the range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $\mathrm{cl}(D(A)) \subseteq \mathcal{R}(I + rA)$, for all $r > 0$, where $\mathrm{cl}(D(A))$ denotes the closure of the domain of A.

Let $z \in D$, then for each $t \in (0,1)$, and for a nonexpansive map T, there exists a unique point $x_t \in D$ satisfying the condition,

$$x_t = tTx_t + (1 - t)z \tag{1.4}$$

since the mapping $x \to tTx + (1 - t)z$ is a contraction. When E is a Hilbert space and T is a self-map, Browder [1] showed that $\{x_t\}$ converges strongly to an element of $F(T)$ which is nearest to u as $t \to 1^-$. This result was extended to various more general Banach spaces by Reich [10], Takahashi and Ueda [11], and a host of other authors. Recently, Morales and Jung [7] proved the existence and convergence of a continuous path to a fixed point of a continuous pseudocontractive mapping in reflexive Banach spaces. More precisely, they proved the following theorem.

Theorem 1.1 [7, Proposition 2(iv), Theorem 1]. *Suppose D is a nonempty closed convex subset of a reflexive Banach space E and $T : D \to E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition. Then for $z \in D$, there exists a unique path $t \to y_t \in D$, $t \in [0,1)$, satisfying the following condition,*

$$y_t = tTy_t + (1 - t)z. \tag{1.5}$$

Furthermore, suppose E is assumed to have a uniformly Gâteaux differentiable norm and is such that every closed convex and bounded subset of D has the fixed point property for nonexpansive self-mappings. If $F(T) \neq \emptyset$ or there exists $x_0 \in D$ such that the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0$ for $\lambda > 1\}$ is bounded, then as $t \to 1^-$, the path converges strongly to a fixed point of T.

From Theorem 1.1, one question arises quite naturally.

Question. Can the continuity of T be weakened to demicontinuity of T?

In connection with this, Lan and Wu [3] proved the following theorem in the Hilbert space setting.

Theorem 1.2 [3, Theorems 2.3 and 2.5]. *Let E be a Hilbert space. Suppose D is a nonempty closed convex subset of E and $T : D \to E$ is a demicontinuous pseudocontractive mapping satisfying the weakly inward condition. Then for $z \in D$, there exists a unique path $t \to y_t \in D$, $t \in (0,1)$, satisfying the following condition:*

$$y_t = tTy_t + (1 - t)z. \tag{1.6}$$

Moreover, if (i) D is bounded then $F(T) \neq \emptyset$ and $\{y_t\}$ converges strongly to a fixed point of T as $t \to 1^-$; (ii) D is unbounded and $F(T) \neq \emptyset$ then $\{y_t\}$ converges strongly to a fixed point of T as $t \to 1^-$.

Let D be a nonempty open and convex subset of a real uniformly smooth Banach space E. Suppose $T:\bar{D}\to E$ is a demicontinuous pseudocontractive mapping which satisfies

$$\text{for some } z\in D,\quad Tx-z\neq\lambda(x-z)\quad \text{for } x\in\partial D,\ \lambda>1, \tag{1.7}$$

where $\bar{D}$ is the closure of D.

It is our purpose in this paper to give sufficient conditions to ensure that $\bar{D}\subseteq(I+r(I-T))(\bar{D})$ for every $r>0$ and to prove the existence and convergence of a path to a fixed point of a demicontinuous pseudocontractive mapping in spaces more general than Hilbert spaces. More precisely, we prove that for a given $y_0\in D$, there exists a unique path $t\to y_t\in\bar{D}$, $t\in(0,1)$, satisfying $y_t:=tTy_t+(1-t)y_0$. Moreover, if $F(T)\neq\emptyset$ or there exists $y_0\in D$ such that the set $K:=\{y\in D: Ty=\lambda y+(1-\lambda)y_0 \text{ for } \lambda>1\}$ is bounded, then the path $\{y_t\}$ converges strongly to a fixed point of T. Furthermore, the sequence $\{x_n\}$ generated from $x_1\in K$ by $x_{n+1}:=(1-\lambda_n)x_n+\lambda_nTx_n-\lambda_n\theta_n(x_n-x_1)$, for all integers $n\geq1$, where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences satisfying appropriate conditions, converges strongly to a fixed point of T. Our theorems provide an affirmative answer to the above question in uniformly smooth Banach spaces and extend Theorem 1.2 to uniformly smooth spaces provided that the interior of D, $\text{int}(D)$, is nonempty.

2. Preliminaries

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau):=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1:\|x\|=1,\ \|y\|=\tau\right\},\quad \tau>0. \tag{2.1}$$

If there exist a constant $c>0$ and a real number $1<q<\infty$, such that $\rho_E(\tau)\leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are L_p and the Sobolev spaces W_p^m for $1<p<\infty$. A Banach space E is called *uniformly smooth* if $\lim_{\tau\to0}(\rho_E(\tau)/\tau)=0$. If E is a real uniformly smooth Banach space, then

$$\|x+y\|^2\leq\|x\|^2+2\langle y,j(x)\rangle+\max\{\|x\|,1\}\|y\|b(\|y\|) \tag{2.2}$$

holds for every $x,y\in E$ where $b:[0,\infty)\to[0,\infty)$ is a continuous strictly increasing function satisfying the following conditions:

(i) $b(ct)\leq cb(t)$, $\forall c\geq1$,

(ii) $\lim_{t\to0}b(t)=0$. (See, e.g., [8].)

Let D be a nonempty subset of a Banach space E. For $x\in D$, the *inward set* of x, $I_D(x)$, is defined by $I_D(x):=\{x+\lambda(u-x):u\in D,\lambda\geq1\}$. A mapping $T:D\to E$ is called *weakly inward* if $Tx\in \text{cl}[I_D(x)]$ for all $x\in D$, where $\text{cl}[I_D(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $D\subseteq E$ be closed convex and let Q be a mapping of E onto D. A mapping Q of E into E is said to be a *retraction* if $Q^2=Q$. If a mapping Q is a retraction, then $Qz=z$ for every $z\in R(Q)$, *range of* Q. A subset D of E is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto D. *If $E=H$, the metric projection P_D is a nonexpansive retraction from H to any closed convex subset D of H.*

In what follows, we will make use of the following lemma and theorems.

Lemma 2.1 [2]. *Let* $\{\lambda_n\}$, $\{\gamma_n\}$, *and* $\{\alpha_n\}$ *be sequences of nonnegative numbers satisfying* $\sum_1^\infty \alpha_n = \infty$ *and* $\gamma_n/\alpha_n \to 0$, *as* $n \to \infty$. *Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - 2\alpha_n \psi(\lambda_n) + \gamma_n, \quad n = 1, 2, \ldots, \tag{2.3}$$

be given where $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing function such that it is positive on* $(0,\infty)$ *and* $\psi(0) = 0$. *Then* $\lambda_n \to 0$, *as* $n \to \infty$.

Theorem 2.2 [6]. *Let E be a uniformly smooth Banach space and let D be an open subset of E. Suppose $T : \bar{D} \to E$ is a demicontinuous strongly pseudocontractive mapping which satisfies*

$$\text{for some } z \in D : Tx - z \neq \lambda(x - z) \quad \text{for } x \in \partial D,\ \lambda > 1. \tag{2.4}$$

Then T has a unique fixed point in $\bar{D}$.

Remark 2.3. We observe that, in Theorem 2.2, if, in addition, D is convex, then any weakly inward map satisfies condition (2.4).

Theorem 2.4 (Reich [10]). *Let E be uniformly smooth. Let $A \subset E \times E$ be accretive with* $\mathrm{cl}(D(A))$ *convex. Suppose A satisfies the range condition. Let* $J_t := (I + tA)^{-1}$, $t > 0$ *be the resolvent of A and assume that $A^{-1}(0)$ is nonempty. Then, for each* $x \in \mathcal{R}(I + rA)(\bar{D})$, $\lim_{t\to\infty} J_t x = Px \in A^{-1}(0)$, *where P is the sunny nonexpansive retraction of* $\mathrm{cl}(D(A))$ *onto* $A^{-1}(0)$.

Remark 2.5. From the proof of Theorem 2.4, we observe that we may replace the assumption that $A^{-1}(0) \neq \varnothing$ with the assumption that $x_t = J_t x$ is bounded, for each $x \in \mathcal{R}(I + tA)$ and $t > 0$.

3. Main results

We first prove the following results which will be used in the sequel.

Proposition 3.1. *Let D be an open subset of a real uniformly smooth Banach space E and let $T : \bar{D} \to E$ be a demicontinuous pseudocontractive mapping which satisfies condition (2.4). Let $A_T : \bar{D} \to E$ be defined by $A_T := I + r(I - T)$ for any $r > 0$. Then $\bar{D} \subseteq A_T[\bar{D}]$.*

Proof. Let $z \in \bar{D}$. Then it suffices to show that there exists $x \in \bar{D}$ such that $z = A_T(x)$. Define $g : \bar{D} \to E$ by $g(x) := (1/(1+r))(rT(x) + z)$ for some $r > 0$. Then clearly g is demicontinuous and for $x, y \in \bar{D}$ we have that $\langle g(x) - g(y), j(x - y)\rangle \leq (r/(1+r))\|x - y\|^2$. Thus, g is a strongly pseudocontractive mapping which satisfies condition (2.4). Therefore, by Theorem 2.2, there exists $x \in \bar{D}$ such that $g(x) = x$, that is, $z = A_T(x)$. The proof is complete. □

Corollary 3.2. *Let E be a real uniformly smooth Banach space and let $A : E \to E$ be demicontinuous accretive mapping. Then A is m-accretive.*

Proof. Set $T := (I - A)$. Then, we obtain that T is a demicontinuous pseudocontractive self-map of E. Clearly, condition (2.4) is satisfied. The conclusion follows from Proposition 3.1. □

Corollary 3.2 was proved by Minty [5] in a Hilbert space setting for continuous accretive mappings and this was extended to general Banach spaces by Martin [4].

We now prove the following theorems.

THEOREM 3.3. *Let D be an open and convex subset of a real uniformly smooth Banach space E. Let $T : \bar{D} \to E$ be a demicontinuous pseudocontractive mapping satisfying condition (2.4). Then for a given $y_0 \in D$, there exists a unique path $t \to y_t \in \bar{D}$, $t \in (0,1)$, satisfying*

$$y_t = tTy_t + (1-t)y_0. \tag{3.1}$$

Furthermore, if $F(T) \neq \emptyset$ or there exists $z \in D$ such that the set $K := \{y \in D : Ty = \lambda y + (1-\lambda)z \text{ for } \lambda > 1\}$ is bounded, then the path $\{y_t\}$ described by (3.1) converges strongly to a fixed point of T as $t \to 1^-$.

Proof. For each $t \in (0,1)$ the mapping T_t defined by $T_t x := tT(t_n)x + (1-t)y_0$ is demicontinuous and strongly pseudocontractive. By Theorem 2.2, it has a unique fixed point y_t in $\bar{D}$, that is, for each $t \in (0,1)$ there exists $y_t \in \bar{D}$ satisfying (3.1). Continuity of y_t follows as in [7]. Now we show the convergence of $\{y_t\}$ to a fixed point of T. Let $A := I - T$. Then A is accretive and by Proposition 3.1, $\bar{D} \subseteq (I + rA)(\bar{D})$ for all $r > 0$ and hence A satisfies the range condition. Moreover, from (3.1), $y_t + (t/(1-t))Ay_t = y_0$. But this implies that $y_t = (I + (t/(1-t))A)^{-1}y_0 = J_{(t/(1-t))}y_0$. Furthermore, since $A^{-1}(0) \neq \emptyset$ or the fact that K is bounded implies that $\{y_t\}$ is bounded (see, e.g., [7]), we have by Theorem 2.4 that $y_t \to y^* \in A^{-1}(0)$ and hence $y_t \to y^* \in F(T)$ as $t \to 1^-$. This completes the proof of the theorem. □

Remark 3.4. We note that, in Theorem 3.3, the requirement that T satisfies condition (2.4) may be replaced with the weakly inward condition. Furthermore, Theorem 3.3 extends [3, Theorems 2.3 and 2.5] to the more general Banach spaces which include $l_p, L_p, W_p^m, 1 < p < \infty$, spaces, provided that int(D) is nonempty.

For our next theorem and corollary, $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \theta_n = 0$;
(ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lim_{n\to\infty}(b(\lambda_n)/\theta_n) = 0$;
(iii) $\lim_{n\to\infty}((\theta_{n-1}/\theta_n - 1)/\lambda_n\theta_n) = 0$, $c_n = o(\lambda_n\theta_n)$.

THEOREM 3.5. *Let D be an open and convex subset of a real uniformly smooth Banach space E. Suppose $T : \bar{D} \to E$ is a bounded demicontinuous pseudocontractive mapping satisfying condition (2.4). Suppose $\bar{D}$ is a nonexpansive retract of E with Q as the nonexpansive retraction. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q((1-\lambda_n)x_n + \lambda_n Tx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)), \tag{3.2}$$

for all positive integers n, where $\{u_n\}$ *is a sequence of bounded error terms. If either* $F(T) \neq \emptyset$ *or the set* $K := \{x \in D : Tx = \lambda x + (1-\lambda)x_0 \text{ for } \lambda > 1\}$ *is bounded, then there exists* $d > 0$ *such that whenever* $\lambda_n \leq d$ *and* $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ *for all* $n \geq 0$, $\{x_n\}$ *converges strongly to a fixed point of T.*

Proof. By Theorem 3.3, $F(T) \neq \emptyset$. Let $x^* \in F(T)$. Let $r > 1$ be sufficiently large such that $x_0 \in B_{r/2}(x^*)$.

Claim 3.6. $\{x_n\}$ is bounded.

It suffices to show by induction that $\{x_n\}$ belongs to $B = \overline{B}_r(x^*)$ for all positive integers. Now, $x_0 \in B$ by assumption. Hence we may assume that $x_n \in B$ and set $M := 2r + \sup\{\|(I-T)x_i\| + \|x_i - u_i\|, \text{ for } i \leq n\}$. We prove that $x_{n+1} \in B$. Suppose x_{n+1} is not in B. Then $\|x_{n+1} - x^*\| > r$ and thus from (3.2) we have that $\|x_{n+1} - x^*\| \leq \|x_n - x^* - \lambda_n((I-T)x_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n)\| \leq \|x_n - x^*\| + \lambda_n\|(I-T)x_n + \theta_n(x_n - x_0) + (c_n/\lambda_n)(x_n - u_n)\| \leq r + M$. Moreover, from (3.2) and inequality (2.2), and using the fact that $\theta_n \leq 1$, we get that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||Q((1-\lambda_n)x_n + \lambda_n Tx_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)) - x^*|| \\
&\leq ||x_n - x^* - \lambda_n((I-T)x_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n)||^2 \\
&\leq ||x_n - x^*||^2 - 2\lambda_n\langle (I-T)x_n, j(x_n - x^*)\rangle \\
&\quad - 2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle - 2c_n\langle x_n - u_n, j(x_n - x^*)\rangle \\
&\quad + \max\{||x_n - x^*||, 1\}\lambda_n\left\|(I-T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right\| \\
&\quad \times b\left(\lambda_n\left\|(I-T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right\|\right) \\
&\leq ||x_n - x^*||^2 - 2\lambda_n\langle (I-T)x_n, j(x_n - x^*)\rangle \\
&\quad - 2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle - 2c_n\langle x_n - u_n, j(x_n - x^*)\rangle \\
&\quad + (r+1)\lambda_n M b(\lambda_n M).
\end{aligned} \tag{3.3}$$

Since T is pseudocontractive and $x^* \in F(T)$, we have $\langle (I-T)x_n, j(x_n - x^*)\rangle \geq 0$. Hence (3.3) gives

$$\begin{aligned}
||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 &- 2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle \\
&+ 2c_n||x_n - u_n|| \cdot ||x_n - x^*|| + (r+1)\lambda_n M^2 b(\lambda_n).
\end{aligned} \tag{3.4}$$

Choose $L > 0$ sufficiently small such that $L \leq r^2/(2\sqrt{D^*} + 2M)^2$, where $D^* = (r+1)M^2$. Set $d := \sqrt{L}$. Then since $\|x_{n+1} - x^*\| > \|x_n - x^*\|$ by our assumption, from (3.4) we get that $2\lambda_n\theta_n\langle x_n - x_0, j(x_n - x^*)\rangle \leq (r+1)M^2\lambda_n b(\lambda_n) + 2c_n Mr$ which gives $\langle x_n - x_0, j(x_n - x^*)\rangle \leq D^*L$, since $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq L = d^2$, for all $n \geq 1$ by our assumption.

Now adding $\langle x_0 - x^*, j(x_n - x^*)\rangle$ to both sides of this inequality, we get that

$$\begin{aligned}||x_n - x^*||^2 &\leq LD^* + \langle x_0 - x^*, j(x_n - x^*)\rangle \\ &\leq LD^* + ||x_0 - x^*|| ||x_n - x^*|| \leq LD^* + \frac{r}{2}||x_n - x^*||.\end{aligned} \tag{3.5}$$

Solving this quadratic inequality for $\|x_n - x^*\|$ and using the estimate $\sqrt{r^2/16 + LD^*} \leq r/4 + \sqrt{LD^*}$, we obtain that $\|x_n - x^*\| \leq r/2 + \sqrt{LD^*}$. But in any case, $\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \lambda_n \|(I-T)x_n + \theta_n(x_n - x_0) + (c_n/\lambda_n)(x_n - u_n)\|$ so that $\|x_{n+1} - x^*\| \leq r/2 + \sqrt{LD^*} + \lambda_n M \leq r$, by the original choices of L and λ_n, and this contradicts the assumption that x_{n+1} is not in B. Therefore, $x_n \in B$ for all positive integers n. Thus $\{x_n\}$ is bounded. Now we show that $x_n \to x^*$. Let $\{y_n\}$ be a subsequence of $\{y_t : t \in [0,1)\}$, such that $y_n := y_{t_n}$, $t_n = 1/(1+\theta_n)$. Then from (3.2) and inequality (2.2) and using the fact that $y_n \in \bar{D}$ for all $n \geq 0$, we get

$$\begin{aligned}||x_{n+1} - y_n||^2 &= ||Q((1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)) - y_n||^2 \\ &\leq ||x_n - y_n - \lambda_n((I-T)x_n + \theta_n(x_n - x_0)) - c_n(x_n - u_n)||^2 \\ &\leq ||x_n - y_n||^2 - 2\lambda_n\langle (I-T)x_n + \theta_n(x_n - x_0), j(x_n - y_n)\rangle \\ &\quad - 2c_n\langle x_n - u_n, j(x_n - y_n)\rangle \\ &\quad + \max\{||x_n - y_n||, 1\}\lambda_n \left|\left|(I-T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right|\right| \\ &\quad \times b\left(\lambda_n \left|\left|(I-T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right|\right|\right) \\ &\leq (1 - 2\lambda_n\theta_n)||x_n - y_n||^2 - 2\lambda_n\langle (I-T)x_n + \theta_n(y_n - x_0), j(x_n - y_n)\rangle \\ &\quad + 2c_n||x_n - u_n|| \cdot ||x_n - y_n|| \\ &\quad + \max\{||x_n - y_n||, 1\}\lambda_n \left|\left|(I-T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right|\right| \\ &\quad \times b\left(\lambda_n \left|\left|(I-T)x_n + \theta_n(x_n - x_0) + \frac{c_n}{\lambda_n}(x_n - u_n)\right|\right|\right).\end{aligned} \tag{3.6}$$

Since $Ty_n = y_n + \theta_n(y_n - x_0)$ and T is pseudocontractive, we get that $\langle (I-T)x_n + \theta_n(y_n - x_0), j(x_n - y_n)\rangle \geq 0$. Moreover, since $\{x_n\}$, $\{y_n\}$, and hence $\{Tx_n\}$, are bounded, there exists $M_0 > 0$ such that $\max\{\|x_n - y_n\|, 1, \|x_n - y_n\| \cdot \|x_n - u_n\|, \|(I-T)x_n + \theta_n(x_n - x_0) + (c_n/\lambda_n)(x_n - u_n)\|\} \leq M_0$. Therefore, (3.6) with property of b gives

$$||x_{n+1} - y_n||^2 \leq (1 - 2\lambda_n\theta_n)||x_n - y_n||^2 + M_0\lambda_n b(\lambda_n) + c_n M_0. \tag{3.7}$$

On the other hand, by the pseudocontractivity of T and the fact that $\theta_n(y_n - x_0) + (y_n - Ty_n) = 0$, we have that

$$\begin{aligned}||y_{n-1} - y_n|| &\leq \left|\left|y_{n-1} - y_n + \frac{1}{\theta_n}((I-T)y_{n-1} - (I-T)y_n)\right|\right| \\ &\leq \frac{\theta_{n-1} - \theta_n}{\theta_n}(||y_{n-1}|| + \|z\|) = \left(\frac{\theta_{n-1}}{\theta_n} - 1\right)(||y_{n-1}|| + \|z\|).\end{aligned} \tag{3.8}$$

However,

$$\|x_n - y_n\|^2 \le \|x_n - y_{n-1}\|^2 + \|y_{n-1} - y_n\|(\|y_{n-1} - y_n\| + 2\|y_{n-1} - x_n\|). \tag{3.9}$$

Therefore, these estimates with (3.7) give that

$$\|x_{n+1} - y_n\|^2 \le (1 - 2\lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + M_1\left(\frac{\theta_{n-1}}{\theta_n} - 1\right) + M_1\lambda_n b(\lambda_n) + c_n M_1, \tag{3.10}$$

for some $M_1 > 0$. Thus, by Lemma 2.1, $x_{n+1} - y_n \to 0$. Hence, since $y_n \to x^*$ by Theorem 3.3, we have that $x_n \to x^*$, this completes the proof of the theorem. □

With the help of Remark 2.3 and Theorem 3.5 we obtain the following corollary.

Corollary 3.7. *Let D be an open and convex subset of a real uniformly smooth Banach space E. Suppose $T : \bar{D} \to E$ is a bounded demicontinuous pseudocontractive mapping satisfying the weakly inward condition. Suppose $\bar{D}$ is a nonexpansive retract of E with Q as the nonexpansive retraction. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q((1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)), \tag{3.11}$$

for all positive integers n, where $\{u_n\}$ is a sequence of error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0$ for $\lambda > 1\}$ is bounded then, there exists $d > 0$ such that whenever $\lambda_n \le d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \le d^2$ for all $n \ge 0$, $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.8. For the case where E is q-uniformly smooth, where $q > 1$, and $t \le M$ for some $M > 0$, the function b in (2.2) is estimated by $b(t) \le ct^{q-1}$ for some $c > 0$ (see [9]). Thus, we have the following corollary.

Corollary 3.9. *Let D be an open and convex subset of a real q-uniformly smooth Banach space E. Suppose $T : \bar{D} \to E$ is a bounded demicontinuous pseudocontractive mapping satisfying condition (2.4). Suppose $\bar{D}$ is a nonexpansive retract of E with Q as the nonexpansive retraction and let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0,1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \theta_n = 0$;
(ii) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$, $\lim_{n\to\infty}(\lambda_n^{(q-1)}/\theta_n) = 0$;
(iii) $\lim_{n\to\infty}((\theta_{n-1}/\theta_n - 1)/\lambda_n\theta_n) = 0$, $c_n = o(\lambda_n\theta_n)$.

Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by

$$x_{n+1} = Q((1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)), \tag{3.12}$$

for all positive integers n, where $\{u_n\}$ is a bounded sequence of error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1 - \lambda)x_0$ for $\lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \le d$ and $c_n/\lambda_n\theta_n, \lambda_n^{(q-1)}/\theta_n \le d^2$ for all $n \ge 0$, $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.10. Examples of sequences $\{\lambda_n\}$ and $\{\theta_n\}$ satisfying conditions of Corollary 3.9 are as follows: $\lambda_n = 2(n+1)^{-a}$, $\theta_n = 2(n+1)^{-b}$, and $c_n = 2(n+1)^{-1}$ with $0 < b < a$ and $a+b < 1$ if $2 \le q < \infty$, and with $0 < b < a(q-1)$ and $a+b(q-1) < 1$ if $1 < q < 2$.

If in Theorem 3.5, T is a self-map of $\bar{D}$, then the projection operator Q is replaced with I, the identity map on E. Moreover, T satisfies condition (2.4). As a consequence, we have the following corollaries.

Corollary 3.11. *Let D be an open and convex subset of a real uniformly smooth Banach space E. Suppose $T : \bar{D} \to \bar{D}$ is a bounded demicontinuous pseudocontractive mapping. Suppose $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ are real sequences in $(0,1]$ satisfying conditions (i)–(iii) of Theorem 3.5 and $\lambda_n(1+\theta_n)+c_n \le 1$, $\forall n \ge 0$. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n), \tag{3.13}$$

for all positive integers n, where $\{u_n\}$ is a sequence of bounded error terms. If either $F(T) \neq \emptyset$ or the set $K := \{x \in D : Tx = \lambda x + (1-\lambda)x_0$ for $\lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \le d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \le d^2$ for all $n \ge 0$, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The conditions on λ_n, θ_n, and c_n imply that the sequence $\{x_n\}$ is well defined. Thus, the proof follows from Theorem 3.5. □

If in Theorem 3.5, D is assumed to be bounded, then the conditions $\lambda_n \le d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \le d^2$ for some $d > 0$ which guarantee the boundedness of the sequence $\{x_n\}$ are not needed. In fact, we have the following corollary.

Corollary 3.12. *Let D be an open convex and bounded subset of a real uniformly smooth Banach space E. Suppose $T : \bar{D} \to E$ is a bounded demicontinuous pseudocontractive mapping satisfying the weakly inward condition. Suppose $\bar{D}$ is a nonexpansive retract of E with Q as the nonexpansive retraction and let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0,1)$ satisfying conditions (i)–(iii) of Theorem 3.5. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q\big((1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_0) - c_n(x_n - u_n)\big), \tag{3.14}$$

for all positive integers n, where $\{u_n\}$ is a sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since D, and hence $\bar{D}$, is bounded we have that $\{x_n\}$ is bounded. Thus the conclusion follows from Theorem 3.5. □

Corollary 3.13. *Let D be an open convex and bounded subset of a real uniformly smooth Banach space E. Suppose $T : \bar{D} \to \bar{D}$ is a bounded demicontinuous pseudocontractive mapping. Let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0,1]$ satisfying conditions (i)–(iii) of Theorem 3.5 and $\lambda_n(1+\theta_n)+c_n \le 1$, $\forall n \ge 0$. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$*

by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_0) - c_n (x_n - u_n), \tag{3.15}$$

for all positive integers n, where $\{u_n\}$ is a sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.14. If in Theorem 3.5, D is bounded, T is a self-map, and $c_n \equiv 1$ for all $n \geq 1$, that is, the error term is ignored, then the following corollary holds.

Corollary 3.15. *Let D be an open convex and bounded subset of a real uniformly smooth Banach space E. Suppose $T : \bar{D} \to \bar{D}$ is a bounded demicontinuous pseudocontractive mapping. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0,1]$ satisfying conditions (i)–(iii) of Theorem 3.5 with $c_n \equiv 0$ for all $n \geq 1$ and $\lambda_n(1+\theta_n) \leq 1$, for all $n \geq 0$. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_0), \tag{3.16}$$

for all positive integers n. Then $\{x_n\}$ converges strongly to a fixed point of T.

The following convergence theorem is for the approximation of solution of demicontinuous accretive mappings.

Theorem 3.16. *Let D be an open and convex subset of a real uniformly smooth Banach space E. Suppose $A : \bar{D} \to E$ is a bounded demicontinuous accretive mapping which satisfies, for some $x_0 \in D$, $Ax \neq t(x - x_0)$ for all $x \in \partial D$ and $t < 0$. Suppose $\bar{D}$ is a nonexpansive retract of E with Q as the nonexpansive retraction and let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0,1]$ satisfying conditions (i)–(iii) of Theorem 3.5. Let a sequence $\{x_n\}$ be generated from $x_0 \in E$ by*

$$x_{n+1} = Q\big(x_n - \lambda_n (A x_n + \theta_n (x_n - x_0)) - c_n (x_n - u_n)\big), \tag{3.17}$$

for all positive integers n, where $\{u_n\}$ is a sequence of bounded error terms. Suppose either $N(A) \neq \varnothing$ ($N(A)$ is the null space of A)or the set $K := \{x \in D : (I - A)x = \lambda x + (1-\lambda)x_0$ for $\lambda > 1\}$ is bounded. Then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a zero of A.

Proof. Set $T := (I - A)$. Then, we have that for some $x_0 \in D$, $(I - T)x \neq t(x - x_0)$ for $x \in \partial D$ and $t < 0$. This implies that $Tx - x_0 = \lambda(x - x_0)$ for all $x \in \partial D$ and $\lambda > 1$. Moreover, $F(T) \neq \varnothing$ or the set $K = \{x \in D : Tx = \lambda x + (1-\lambda)x_0$, for $\lambda = (1-t) > 1\}$ is bounded. Therefore, by Theorem 3.5, $\{x_n\}$ converges strongly to $x^* \in F(T)$. But $F(T) = N(A)$. Hence, $\{x_n\}$ converges strongly to $x^* \in N(A)$. The proof of the theorem is complete. □

The following corollary follows from Theorem 3.16.

Corollary 3.17. *Let E be a real uniformly smooth Banach space and suppose $A : E \to E$ is a bounded demicontinuous accretive mapping. Let $\{\lambda_n\}$, $\{\theta_n\}$, and $\{c_n\}$ be real sequences in $(0,1]$ satisfying conditions (i)–(iii) of Theorem 3.5. Let a sequence $\{x_n\}$ be generated from*

$x_0 \in E$ by

$$x_{n+1} = x_n - \lambda_n (Ax_n + \theta_n (x_n - x_0)) - c_n (x_n - u_n), \tag{3.18}$$

for all positive integers n, where $\{u_n\}$ is a sequence of bounded error terms. If either $N(A) \neq \emptyset$ or the set $K := \{x \in E : (I - A)x = \lambda x + (1 - \lambda)x_0$ for $\lambda > 1\}$ is bounded, then there exists $d > 0$ such that whenever $\lambda_n \leq d$ and $c_n/\lambda_n\theta_n, b(\lambda_n)/\theta_n \leq d^2$ for all $n \geq 0$, $\{x_n\}$ converges strongly to a point of $N(A)$.

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