Hindawi Publishing Corporation
International Journal of Quality, Statistics, and Reliability
Volume 2009, Article ID 707583, 11 pages
doi:10.1155/2009/707583

Research Article

Process Monitoring with Multivariate p-Control Chart

Paolo C. Cozzucoli

Department of Economics and Statistics, University of Calabria, Via P. Bucci, cubo 0C, 87036 Rende, Italy

Correspondence should be addressed to Paolo C. Cozzucoli, cozzucoli@unical.it

Received 6 September 2008; Revised 4 May 2009; Accepted 2 June 2009

Recommended by Myong (MK) Jeong

We assume that the operator is interested in monitoring a multinomial process. In this case the items are classified into $(k+1)$ ordered distinct and mutually exclusive defect categories. The first category is used to classify the conforming defect-free items, while the remaining k categories are used to classify the nonconforming items in k defect grades, with increasing degrees of nonconformity. Usually the process is said to be capable if the overall proportion of nonconforming items is very small and remains low, or declines over time. Nevertheless, since we classify the nonconforming items into k distinct defect grades, the operator can also evaluate the overall level of defectiveness. This quality parameter depends on the k defect categories. Furthermore, we are interested in evaluating, over time, the proportion of nonconforming items in each category as well as the overall level of defectiveness. To achieve this goal, we propose (i) a normalized index that can be used to evaluate the capability of the process in terms of the overall level of defectiveness, and (ii) a two-sided Shewhart-type multivariate control chart to monitor the overall proportion of nonconforming items and the corresponding defectiveness level.

1. Introduction

In the modern industrial context, operators are usually interested in evaluating process quality. Assuming that the quality depends on several correlated quality characteristics, it is appropriate to use multivariate quality control methodology. If the quality characteristics are defined on nominal or ordinal scale, then the corresponding monitoring process is said to be a multivariate attribute process. For a general review of methods for monitoring multivariate attribute process see Topalidou and Psarakis [1]. Each item is inspected and classified as conforming or nonconforming, and different criteria for nonconforming classification may be used. For example, an item may be classified as absent, incidental, minor, or major defect, see Nelson [2]. Real example is shown in Taleb and Limam [3] and Taleb [4] where the porcelain industry items are classified as standard, second choice, third choice, or chipped.

In general, each item may be classified in $(k+1)$ ordered and mutually exclusive quality categories depending on the level of defect: the first category can be used to classify the conforming defect-free items, while the remaining k categories can be used to classify the nonconforming items in k distinct defect grades, with an increasing degree of nonconformity. In this case, the probabilistic model is the multinomial distribution, which is appropriate to define a statistical procedure to monitor this multivariate attribute process.

The most applied statistical method of monitoring the multinomial process is the chi-square control chart, originally proposed by Duncan [5]. This control chart is based on the chi-square sampling statistic to test the goodness of fit to the in control distribution and is a one-sided Shewhart-type control chart with only the upper approximate probabilistic control limit. The chi-square control chart is useful for identifying significant departures of almost one of the $(k+1)$ proportions from their in control values. The chart signals any changes in the quality target parameters, but it is unable to distinguish between increasing or decreasing levels of quality. If the chi-square sampling statistic plots out the upper control limit, then we may declare that the process is out of control, but cannot determine whether the defect proportion is increasing or not. To solve this limit, Marcucci [6] proposed a one-sided generalized p-chart, that is, specifically designed to indicate departures from the in control level only for the defect nonconforming proportions without considering the conforming proportion; but this chart is still one-sided and is designed to signal increases in the defect proportion and consequently identifies decreases in quality level only. Consequently, the chi-square control chart, proposed by Duncan [5], and the control chart, proposed by Marcucci [6], can be used only to identify process performance because they are able only to alert operators of any deterioration of the process so the potential damage can be alleviated or averted in time. From the management point of view, the ability to detect process improvements is very important since improvements can take place as a result of process stabilization, subprocess improvement, or learning effects. Continuous improvement of a high-quality process, although difficult and not straightforward, is an important issue to increase competition in the global market [7]. So in many manufacturing processes it is also becoming increasingly important to be able to detect moderate shifts in the performance of a process [8]. An example of complex process with very low fraction nonconforming is the manufacturing of integrated circuits; in this case, different defects can be observed [9], each of which can be categorized according to defectiveness grade. Therefore, in this perspective a useful quality control statistical method have to be able in evaluating whether the global quality level is increasing or decreasing. The need for innovating the statistical methodology for monitoring high quality processes has been stressed in Calvin [10], Bourke [8], Kaminsky et al. [11], Xie and Goh [12], Quesenberry [13], Xie and Goh [14], Xie et al. [7, 15], and Niaki and Abbasi [16].

So it is useful to define a two-sided control chart that uses, respectively, the upper control limit and the lower control limit to signal the deterioration or the improvement of the process quality. In the following, we define the sampling statistic that may be used as an index of the overall defectiveness level of the process and as a statistic necessary to determine a corresponding two-sided control chart with the approximate probabilistic control limits. In fact, the process is usually assumed to be capable if the proportion of nonconforming items is very small and remains low or declines over time. Therefore, because we have chosen to classify the nonconforming items into k different defect grades, the overall proportion of nonconforming items depends on the k categories, which are not necessarily independent. To get this purpose we propose (i) a normalized index that can be used to evaluate the overall defectiveness of the process, and (ii) a two-sided Shewhart-type multivariate control chart to monitor the overall defectiveness of nonconforming items. The corresponding sampling statistic is used to define the normalized index and the multivariate p-control chart. The paper is organized as follows: in Section 2 the overall defectiveness index is defined; in Section 3 a two-sided multivariate p-control chart with approximate probabilistic control limits is proposed; Section 4 presents some numerical examples useful for evaluating the performance of the chart in three different quality situations; the last section provides conclusions.

2. An Overall Defectiveness Index

For evaluating the overall defectiveness of the production, we draw from the process a sample of n items. Each sampling item may be classified only in one of the $(k + 1)$ ordered and mutually exclusive categories of quality defects. Let $\mathbf{D} = (D_0, D_1, \ldots, D_i, \ldots, D_k)$ be the vector of the $(k + 1)$ defect categories. The generic component D_i indicates the ith category of defect degree; D_0 is the free-defect category and D_k is the most serious category of defect degree. Since different defects bring to the process different losses of quality then, corresponding to the vector $\mathbf{D}$, we can define a vector of weights that are numerical evaluations of the defect degree found in the product; these weights may be selected on the following: dysfunction, dissatisfaction, economical loss, increasing costs, or demerit caused by defect. For example, if the vector $\mathbf{D}$ has only five categories, like *absent*, *minor*, *medium*, *major*, and *serious* defect, then we may associate weight zero to the first category, weight one to the last category, and other weights, between zero and one, to the other categories. For example, assuming as criterium the dysfunction present into the item, if the weight is fixed to 0.20 then this means that this type of defect implies a 20% of dysfunction. If arbitrary weights are used then we have to transform these data in such a manner as to obtain constants between zero and one.

Items are classified in each of the $(k + 1)$ quality defect categories. In this case the multivariate random variable $\mathbf{X} = (X_0, X_1, \ldots, X_i, \ldots, X_k)$ has a multinomial distribution with parameters n and probability vector $\mathbf{p} = (p_0, p_1, \ldots, p_i, \ldots, p_k)$, such that $0 \le p_i \le 1$ and $\sum_{i=0}^{k} p_i = 1$. Specifically, X_i is the number of items in the sample that are classified in the D_i defect category and p_i is the probability that an item may be classified in the D_i defect category. In the quality control procedure, p_i is the proportion of nonconforming items classified as defective in the ith class. Consequently, the multivariate random variable $\mathbf{X}$ has a multinomial random distribution with parameters $(n, \mathbf{p})$, where n is the sample size and $\mathbf{p}$ the probability vector.

If the operator has chosen to assign the same weight to every class of defects, then an overall defectiveness index may be the overall proportion of nonconforming items in the process; indirectly, the operator can use this index as a measure of the process capability. The process will be considered capable when conforming proportion is very high. Instead, it is reasonable to think that the operator assigns different weights for every type of defect, because they cause, in the process, different grades of dysfunction, dissatisfaction, economical loss, or demerit. In this case, we have to consider accurately this differentiation of weights if we are interested in evaluating and monitoring the real overall level of defectiveness in the process. This approach for classification of defective items was used in Lu et al. [17], to design the multivariate np chart. Further, these types of problems, particularly in the case of a process with a very high quality level, are also investigated in Xie et al. [7]. Moreover in Cassady et al. [18], using a normal approximation of sampling statistics, this scheme of item classification was applied to determine a 3-level control chart.

Let $\mathbf{d} = (d_0, d_1, \ldots, d_i, \ldots, d_k)$ be a vector of weights associated to the $\mathbf{D}$ vector of quality defect categories, where $0 \le d_i \le 1$, $d_i < d_{i+1}$, $d_0 = 0$ and $d_1 = 1$. In general, d_i indicates the amount of quality loss that the defect of ith class introduces into the system. Let us note that it is assumed that the defects are categorized according to the their effect on product quality and performance. If there are two types of defects with the same degree of defectiveness, evaluate by the constant d_i, then the possible cases are (i) only one defect of the two is identified in the selected item; (ii) both defects are identified in the selected item. In the first case the item is classified as belonging to the ith defect class; in the second case, when both defects are identified, the item should be classified as belonging to defect class with a higher degree of defectiveness of the ith defect class. Instead, if the operator assumes that the classes are determined by type of the defect and not by the degree of defectiveness then the appropriate probabilistic model to monitor the process quality degree is the multivariate binomial distribution and not the multinomial. So, for fixed $\mathbf{d}$ and $\mathbf{p}$ vectors, a normalized index of the overall defectiveness degree is

$$\delta = \sum_{i=0}^{k} d_i p_i. \tag{1}$$

To clarify, because $0 \le p_i \le 1$ and $0 \le d_i \le 1$, the index can take only values between zero and one, that is, $0 \le \delta \le 1$. It measures the weighted degree of overall defectiveness. In the extreme cases, the index takes a minimum when the produced items are all free of defects; that is, $\delta = 0$ if and only if $p_i = 0$, $i = 1, \ldots, k$; or it takes maximum when the produced items are all classified in the maximum defect class; that is, $\delta = 1$ if and only if $p_k = 1$.

Given $\mathbf{X} = (X_0, X_1, \ldots, X_i, \ldots, X_k)$, it is well known that the maximum likelihood-estimator of the parameters $\mathbf{p} = (p_0, p_1, \ldots, p_i, \ldots, p_k)$ is $\hat{p}_i = X_i/n$. Therefore, to monitor the overall defectiveness parameter δ, we can use the following sampling statistic:

$$\hat{\delta} = \sum_{i=0}^{k} d_i \hat{p}_i, \tag{2}$$

that is, an unbiased estimator of the index δ. The variance of $\hat{\delta}$ is

$$\sigma^2\left(\hat{\delta}\right) = \frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 p_i\right) - \left(\sum_{i=0}^{k} d_i p_i\right)^2\right]. \tag{3}$$

A consistent estimator of variance $\sigma^2(\hat{\delta})$ is

$$S^2\left(\hat{\delta}\right) = \frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 \hat{p}_i\right) - \left(\sum_{i=0}^{k} d_i \hat{p}_i\right)^2\right]. \tag{4}$$

In this paper, the main interest is in defining a statistical procedure for monitoring the in-control process hypothesis on the global parameter δ, that is,

$$H_0 : \delta = \delta_0, \quad \text{versus} \quad H_1 : \delta \ne \delta_0, \tag{5}$$

where δ_0 is the expected value of the sampling statistics $\hat{\delta}$ when the process is in control.

Nevertheless, since the parameter of global quality is defined as linear function of the components in the multinomial parameter vector $\mathbf{p}$, then it is also interesting to verify all partial hypothesis on the components of the parameter δ, that is,

$$H_0(i) : p_i = p_{i0}, \quad \text{versus} \quad H_1(i) : p_i \ne p_{i0}, \quad \text{for } i = 0, 1, \ldots, k, \tag{6}$$

where p_{i0} is the expected value of sampling proportion $\hat{p}_i$ when the process is in control.

The null hypothesis in (5) is verified if and only if all the null hypothesis in (6) are verified; so, in this case, it should be appropriate to use simultaneous inference procedure, see Miller [19].

By Gold [20, Theorem 3], the asymptotic simultaneous confidence interval, with at least $(1-\alpha)$ confidence level, for the overall defectiveness index is

$$\delta = \sum_{i=0}^{k} d_i p_i \in \left\{\sum_{i=0}^{k} d_i \hat{p}_i \pm \sqrt{\chi^2_{k,\alpha}}\sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 \hat{p}_i\right) - \left(\sum_{i=0}^{k} d_i \hat{p}_i\right)^2\right]}\right\}, \tag{7}$$

where $\chi^2_{k,\alpha}$ is the upper $(1-\alpha)$ quantile of the χ^2 distribution with (k) degrees of freedom.

By the multivariate Lindeberg-Lévy Central Limit Theorem, the vector $\hat{\mathbf{p}}$ has a $(k+1)$ asymptotic multivariate normal distribution, (see [21, page 108]); therefore, the estimator $\hat{\delta}$ has asymptotic normal distribution. Using a procedure based on Bonferroni's inequality, see Goodman [22], we can determine a shorter $(1-\alpha)$ simultaneous confidence interval for the overall defectiveness index, that is

$$\delta = \sum_{i=0}^{k} d_i p_i \in \left\{\sum_{i=0}^{k} d_i \hat{p}_i \pm z_{[1-(\alpha/2(k+1))]}\sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 \hat{p}_i\right) - \left(\sum_{i=0}^{k} d_i \hat{p}_i\right)^2\right]}\right\}, \tag{8}$$

where $z_{[1-(\alpha/2(k+1))]}$ is the upper $[1-(\alpha/2(k+1))]$ quantile of the standardized normal distribution. A further improvement in the length of the simultaneous confidence interval can be obtained using a procedure based on Šidák's inequality [23, 24]. Therefore the corresponding $(1-\alpha)$

simultaneous confidence interval for the overall defectiveness index is

$$\delta = \sum_{i=0}^{k} d_i p_i \in \left\{ \sum_{i=0}^{k} d_i \hat{p}_i \pm z_{(1-\alpha)^{1/(k+1)}} \sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 \hat{p}_i\right) - \left(\sum_{i=0}^{k} d_i \hat{p}_i\right)^2\right]} \right\}, \quad (9)$$

where $z_{(1-\alpha)^{1/(k+1)}}$ is the upper $((1-\alpha)^{1/(k+1)})$ quantile of the standardized normal distribution. In the nonsimultaneous approach, since for a large sample, the sample statistics converge to normal distribution, the $(1-\alpha)$ confidence interval is

$$\delta = \sum_{i=0}^{k} d_i p_i \in \left\{ \sum_{i=0}^{k} d_i \hat{p}_i \pm z_{(1-\alpha/2)} \sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 \hat{p}_i\right) - \left(\sum_{i=0}^{k} d_i \hat{p}_i\right)^2\right]} \right\}, \quad (10)$$

where $z_{(1-\alpha/2)}$ is the upper $(1-\alpha/2)$ quantile of the standardized normal distribution.

Since we want to design a control chart with control limits such that the overall error rate is not much larger than the nominal level α, then in the following we have to consider only the simultaneous confidence procedures and compare them with the nonsimultaneous approach. Besides, simultaneous procedures are suitable to solve the identification problem in multivariate quality control [25].

3. A Two-Sided Multivariate p Control Chart

If the operators are interested in monitoring a multinomial process $\mathbf{X}$ with items classified in $(k+1)$ defect classes, then, assuming equal weights of the losses in quality for the defect categories, the chi-square control chart may be used [6]. Suppose that we are interested at time t to test

$$H_0 : p_{it} = p_{i0}, \quad \text{versus} \quad H_1 : p_{it} \neq p_{i0}, \quad \text{for } i = 0, 1, \ldots, k, \quad (11)$$

where p_{it} is the proportion of ith defect category at monitoring time t and p_{i0} is the specified in control proportion. Notice that the hypothesis (6) is equivalent to the hypothesis (11). In the following, without loss of generality, we assume that the sample size n is constant in each monitoring period. Using a sample of size n, the control chart to test the hypothesis in (11) at time t is based on the Pearson goodness-of-fit statistic

$$Y_t = \sum_{i=0}^{k} \frac{(X_{it} - n p_{i0})^2}{n p_{i0}}, \quad (12)$$

where X_{it} is the number of items in the sample taken at time t classified in ith defect class. This statistic has asymptotic chi-square distribution with (k) degrees of freedom, (see [26, page 447]). Therefore, the asymptotic probabilistic upper control limit (UCL) is the $(1-\alpha)$ quantile of the χ^2 distribution with (k) degrees of freedom; that is UCL $= \chi^2_{(k-1),\alpha}$. If the sampling statistic is plotted out the UCL, then the process will be said to be out of control and the operator has to investigate it. Nevertheless, in practical situations, the p_{i0}, $i = 0, 1, \ldots, k$ are usually estimated using a set of preliminary samples taken in the in control base period. In this case, the correct statistical procedure based on the following sampling statistics:

$$Z_t = n_0 n_t \sum_{i=0}^{k} \frac{(\hat{p}_{it} - \hat{p}_{i0})^2}{X_{it} + X_{i0}} \quad (13)$$

is appropriate to test correctly the homogeneity of proportions between the base period and each monitoring period t, see Marcucci [6].

The control charts based on statistics (12) and (13) suffer from two limitations: first, the defect categories are assumed to have equal weights in terms of loss of quality when in the reality this is not true; second, the out of control signal is indistinct because the sampling statistic increases in case of a deterioration process and (or) in case of process improvement. Possible solutions to these limitations are (i) use different weights for the defect categories, specified in the vector $\mathbf{d}$, and (ii) design a two-sided control chart that is able to give correct information on the improvement or deterioration process. Following the Shewhart procedure, we can define this control chart using (2) as sampling statistics and the two simultaneous confidence interval extremes in (9) as upper and lower control limits. So, let $\mathbf{p_0} = (p_{00}, p_{10}, \ldots, p_{i0}, \ldots, p_{k0})$ be the specified parameter vector under the hypothesis that the process is in control. Therefore, the control chart has these asymptotic probabilistic control limits:

$$\mathrm{UCL} = \sum_{i=0}^{k} d_i p_{i0} + z_{(1-\alpha)^{1/(k+1)}} \sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 p_{i0}\right) - \left(\sum_{i=0}^{k} d_i p_{i0}\right)^2\right]},$$

$$\mathrm{CL} = \sum_{i=0}^{k} d_i p_{i0},$$

$$\mathrm{LCL} = \sum_{i=0}^{k} d_i p_{i0} - z_{(1-\alpha)^{1/(k+1)}} \sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2 p_{i0}\right) - \left(\sum_{i=0}^{k} d_i p_{i0}\right)^2\right]}. \quad (14)$$

Usually, in practical situations, the vector $\mathbf{p_0} = (p_{00}, p_{10}, \ldots, p_{i0}, \ldots, p_{k0})$ is unknown and it is necessary to estimate it using m preliminary samples of size n taken from the process in control, see Montgomery [9]. Let $\mathbf{X}_t = (X_{0t}, X_{1t}, \ldots, X_{it}, \ldots, X_{kt})$, $t = 1, 2, \ldots, m$, be a set of m preliminary samples of size n taken from the multinomial process $\mathbf{X}$ with parameters $(n, \mathbf{p})$. Specifically, X_{it} is the number of items in the tth sample that are classified in the D_i defect category. Therefore an unbiased estimator of parameter p_i is

$$\overline{\hat{p}}_i = \frac{1}{m}\sum_{t=1}^{m} \hat{p}_{it}, \quad i = 0,1,\ldots,k, \tag{15}$$

where $\hat{p}_{it} = X_{it}/n$, $i = 0,1,\ldots,k$; $t = 1,2,\ldots,m$, and the estimated control chart with asymptotic probabilistic control limits is

$$\begin{aligned}
\text{UCL} &= \sum_{i=0}^{k} d_i\overline{\hat{p}}_i + z_{(1-\alpha)^{1/(k+1)}}\sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2\overline{\hat{p}}_i\right) - \left(\sum_{i=0}^{k} d_i\overline{\hat{p}}_i\right)^2\right]},\\
\text{CL} &= \sum_{i=0}^{k} d_i\overline{\hat{p}}_i,\\
\text{LCL} &= \sum_{i=0}^{k} d_i\overline{\hat{p}}_i - z_{(1-\alpha)^{1/(k+1)}}\sqrt{\frac{1}{n}\left[\left(\sum_{i=0}^{k} d_i^2\overline{\hat{p}}_i\right) - \left(\sum_{i=0}^{k} d_i\overline{\hat{p}}_i\right)^2\right]}.
\end{aligned} \tag{16}$$

The sampling statistics (2), (12), and (13) are functions of $\hat{p}_{it}$ and since that the corresponding procedure test are asymptotic, then the sample size may be chosen following Cochran [27]; no more than twenty percent of expected events in each defect category should be less than five, and none of the expected events in each category should be less than one.

The proposed control chart signals the deterioration of the process when the sampling statistic is plotted out the UCL or the improvement of the process when the sampling statistic is plotted out the LCL. Analysis of the pattern of plotted statistics may be used to identify significant changes in the process parameters. We note that, for a specified vector $\mathbf{d}$, the sampling statistic $\hat{\delta}$ increases when at least one of the nonconforming proportions p_i, $i = 1,2,\ldots,k$, increases and decreases otherwise. Hence the control chart is able also to test the global hypothesis in (6) and (11).

The proposed statistical procedure is conservative because the overall coverage probability inside the control limits is at least $(1-\alpha)$ and therefore the level of significance is not larger than α. Consequently, the in control Average Run Length (ARL_0), that is, the average rate of false alarms, is expected to be larger than that of the chi-square control chart. Note that in the nonsimultaneous approach, the coverage probability inside the control limits is expected less than $(1-\alpha)$ and for that reason the level of significance is larger than α. Consequently the in control ARL_0 is expected to be smaller than that of the chi-square control chart. Nevertheless, the operator is more interested in evaluating the performance of control charts evaluating the out of control ARL_1, that is the average correct alarm rate to signal a true change.

Since the $\text{ARL}_1 = 1/(1-\beta)$ is a function of the probability of a type II error, then it is necessary to evaluate β; that is, the probability to have declared the process in control when it is not. Under a specified alternative hypothesis, the statistic (12) has a chi-square noncentral distribution with k degrees of freedom and the following noncentrality parameter [27]

$$\lambda = n\sum_{i=0}^{k} \frac{(p_{i1}-p_{i0})^2}{p_{i0}}, \tag{17}$$

where p_{i1}, $i = 0,1,\ldots,k$ are the specified alternative proportions. Numerical evaluation and comparisons of the ARL of the two-sided multivariate p-control chart, based on sampling statistics (2), will be done in the next section. Besides, we may note that in presence of an out of control signal, the operator is interested in identifying which defect categories has produced significant changes in the parameters of the in control process. In this case, estimating simultaneous confidence intervals for every parameter p_i, the operator may identify which parameters are changed significantly. Specifically, putting $d_i = 1$ and $d_j = 0$, $j \neq i$ in (2), we obtain the following simultaneous confidence interval for p_i, $i = 0,1,\ldots,k$:

$$p_i \in \left\{\hat{p}_i \pm z_{(1-\alpha)^{1/(k+1)}}\sqrt{\frac{\hat{p}_i(1-\hat{p}_i)}{n}}\right\}. \tag{18}$$

Some considerations on the choice of the weights in vector $\mathbf{d}$ are necessary. From the operator's point of view, the choice of the weights d_i will be made in the function of a corresponding degree of quality loss for every defect category; but, from the statistical perspective, it is necessary to consider also the influence of the weights d_i on the performance, in terms of ARL, of the corresponding control. In the following, assuming that the loss of quality due to the ith defect category is proportional to the loss of quality due to the others, we suggest the use of weights that are in terms of a geometric progression; that is, $d_i = kd_{i-1}$, where k is the common ratio of sequence. Numerical examples will be proposed in the next section to investigate the influence of the weights on the performance of the control chart.

4. Numerical Simulations

In this section, the performance of the control chart will be evaluated using three different cases of overall quality level: a *low*, *high*, and *very high* quality process. In particular, for the fixed sample size n and significance level α, fixed at 0.0027, the influence of the parameter k on the performance of the control chart will be evaluated through numerical simulation. Furthermore, we propose comparisons of performance of the control chart, designed by (a) simultaneous confidence interval based on Šidák's inequality, (b) upper and lower quantiles estimated using empirical cumulative distribution function, and (c) nonsimultaneous confidence interval.

4.1. Example One: Low-Quality Level. A first example is based on real data reported in Taleb and Limam [3] about the production of porcelain. The items are classified by experts, with respect to quality, into four defect categories: *standard*, *second choice*, *third choice*, and *chipped*. In this case, the loss

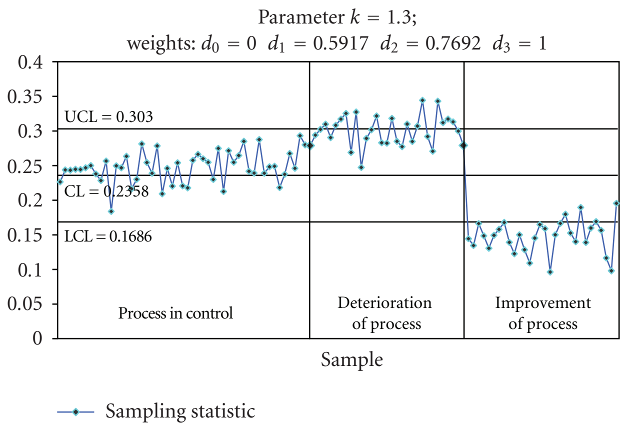

Figure 1: Two-sided multivariate p control chart (example one).

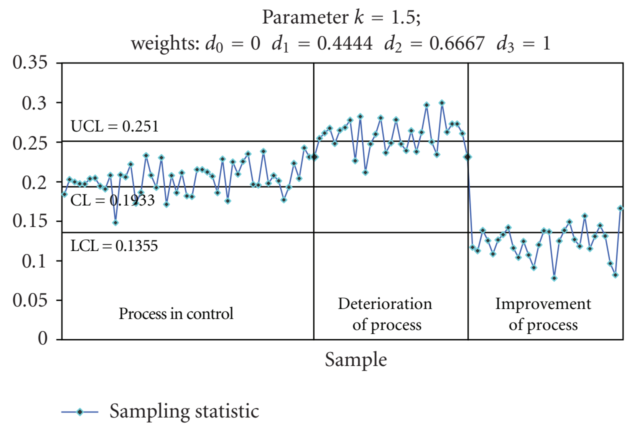

Figure 2: Two-sided multivariate p control chart (example one).

of quality may be evaluated in terms of economical loss; in fact, an item classified as standard or defect-free can be sold at the standard price, the second and third choice items sold at lower prices, while the chipped items cannot be sold at all. The following hypotheses, about the defect proportions in vector $\mathbf{p}$, are used:

$$\text{process in control: } p_0 = 0.65, \quad p_1 = 0.24, \quad p_2 = 0.07, \quad p_3 = 0.04, \tag{19}$$

$$\text{deterioration process: } p_0 = 0.5912, \quad p_1 = 0.24, \quad p_2 = 0.07, \quad p_3 = 0.0988, \tag{20}$$

$$\text{improvement process: } p_0 = 0.7996, \quad p_1 = 0.11, \quad p_2 = 0.07, \quad p_3 = 0.0204. \tag{21}$$

The in-control process (19) has a low-quality level because the proportion of defect-free items is fixed at 0.65. The hypothesis (20) indicates a deterioration in process since the maximum defect category is increased to 0.098; by contrast, hypothesis (21) indicates an improvement in process since the second and fourth defect categories are decreased, respectively, to 0.11 and 0.0204.

Usually, the operator is interested in identifying a deterioration in process and considers only the null hypothesis to choose the size of samples, but in our approach we want to develop a procedure that will also be able to identify an improvement in process. Therefore, for defining the appropriate sample size we have to consider the minimum proportion indicated in the process-improvement hypothesis. In this first example the minimum proportion is 0.0204; then by Cochran's rule we have to take a sample of size $n = 250$.

For numerical evaluations we have simulated samples from multinomial processes with parameters n and $\mathbf{p}$, respectively, defined in (19), (20), and (21). Using a random number generation procedure [28], for every hypothesis we have (i) simulated 10000 samples of size $n = 250$, (ii) calculated the sampling statistics $\hat{\delta} = \sum_{i=0}^{k} d_i \hat{p}_i$, and (iii) plotted it on the multivariate p control chart with control

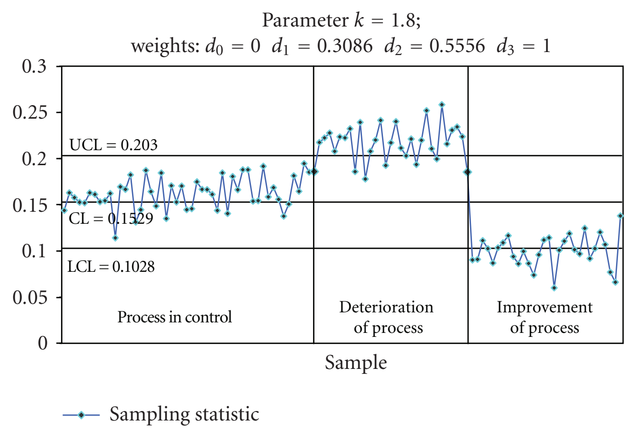

Figure 3: Two-sided multivariate p control chart (example one).

limits designed on the in-control process hypothesis. Only 110 values of 30000 simulated samples are plotted in the graphical representations of control charts, see Figures 1, 2, and 3; specifically, the first 50 values are taken from an in-control process, the next 30 values are taken from a deteriorated process and the last 30 values are taken from an improved process.

We note that if the parameter k, that is, chosen is small, near one, then the operator considers the defect categories similar in terms of loss of quality; instead, if the operator considers the highest categories more important than the lowest (in terms of loss of quality inducted in the process), then a higher parameter k has to be selected. We may note easily, from analysis of Figures 1–3, that the performance of control chart changes in conjunction with the value of parameter k. So we want to evaluate, using simulated data, the ARL_1 in function of different values of parameter k. In particular, we estimate the two one-sided ARL_1: ARL_1(upper) and ARL_1(lower). The ARL_1(upper) indicates the control chart's ability to identify correctly only deteriorations in the process; while ARL_1(lower) evaluates the control chart's ability to identify significant improvements in process. Combining the two one-sided ARL_1 we can estimate the ARL_1. The ARL_1 curves in function of the parameter

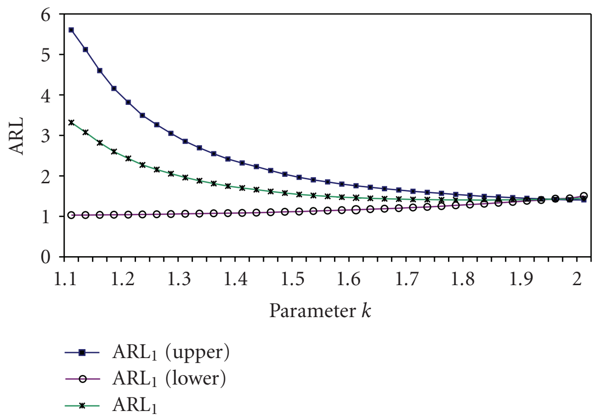

Figure 4: ARL_1 curves (example one).

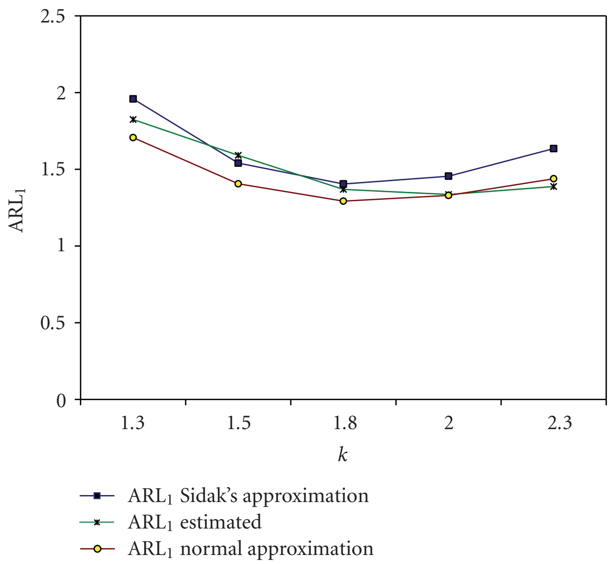

Figure 5: ARL_1 comparisons (example one).

k are shown in Figure 4. Figure 4 shows that ARL_1(upper) and ARL_1(lower) are, respectively, decreasing and increasing values of parameter k. Also, for comparison purposes of the performance of the control chart we have estimated the values of ARL for charts with control limits determined by (a) the simultaneous confidence interval based on Šidák's inequality, (b) the quantiles $q_{0.00135}$ and $q_{0.99865}$, estimated by the empirical cumulative distribution function, and (c) the non simultaneous confidence interval. Values of ARL are reported in Table 1 and graphical comparisons in Figure 5.

If we consider the results in Table 1 we have to conclude, since $ARL_1(\text{lower}) < ARL_1(\text{upper})$, that the proposed control chart is better to identify process improvement for $k \leq 1.8$. The same evidence is obtained from the analysis of the ARL_1 curves in Figure 4. In fact, the distance between the ARL_1(upper) curve and the ARL_1(lower) curve is high for small values of k and small for large values. However, the result is valid from operator's perspective. Note that it is reasonable to assume value of $k \leq 1.5$; in fact, in this case the operator assumes that each class of defect, in terms of defectiveness, is more important of 50% respect to the previous class.

Besides, to exemplify the methods for calculating the ARL reported in Table 1 and Figure 4, consider the case when the operator uses the control chart with simultaneous Šidák's approximation control limits, with $k = 1.3$. One thousand simulations have been carried out. Each experiment consisted in the generation of 10000 observations from a multinomial distribution with parameters $n = 250$ and probability vector defined in (19); that is, when the process is in control. For each experiment the Run Length in control (RL_0) has been calculated; that is, the number of observations where the sampling statistic defined in (2) assumes a value outside the control limits, identifying a false alarm. Then to estimate the ARL_0 the mean of the 1000 RL_0 simulation values has been calculated. Under the assumption of process in control the mean of the 1000 RL_0 was equal to 18. The false-alarm probability α is obtained by the ratio between the mean of RL_0 and the sample size; $\alpha = 18/10000 = 0.0018$; therefore $ARL_0 = 1/\alpha = 555$. Instead, using 1000 of simulations, each of which consists in 10000 observations from a multinomial distribution with parameters $n = 250$ and probability vector defined in (20); that is, when the process is out control. For each experiment the Run Length out of control (RL_1) has been calculated; that is, the number of observations where the sampling statistic defined in (2) assumes a value outside the control limits, identifying a really correct alarm. To estimate ARL_1 then the mean of the 1000 RL_1 simulation values has been calculated. Under the assumption of process out control the mean of the 1000 RL_1 was equal to 3503. The correct-alarm probability $(1 - \beta)$ is obtained by the ratio between the mean of the RL_1 and the sample size; $(1 - \beta) = 3503/10000 = 0.3503$; therefore $ARL_1 = 1/(1 - \beta) = 2.855$.

4.2. Example Two: High-Quality Level. A second example considers a process with a higher level of quality than that considered in case one; in fact, the defect-free proportion, that is, chosen is equal to 0.83. The following hypotheses, about the defect proportions in vector $\mathbf{p}$, are used:

$$\text{process in control: } p_0 = 0.83, \quad p_1 = 0.104, \quad p_2 = 0.04, \quad p_3 = 0.026, \tag{22}$$

$$\text{deterioration process: } p_0 = 0.7823, \quad p_1 = 0.104, \quad p_2 = 0.04, \quad p_3 = 0.0737, \tag{23}$$

$$\text{improvement process: } p_0 = 0.9389, \quad p_1 = 0.011, \quad p_2 = 0.04, \quad p_3 = 0.0101. \tag{24}$$

As in the first example, the hypothesis (23) indicates a deterioration in process and the hypothesis (24) indicates an improvement in process. Since the minimum proportion is 0.0101, then in accordance with Cochran's rule we have to take a sample of size $n = 500$. The same simulation procedure described in example one is used. From Figures 6, 7, and 8 the

Table 1: ARL comparisons (example one).

k	ARL for chart with simultaneous Šidák's approximation control limits				ARL for chart with exact estimated control limits				ARL for chart with nonsimultaneous approximation control limits			
	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1
$k = 1.3$	555	2.855	1.063	1.959	370	2.610	1.039	1.824	270	2.377	1.037	1.707
$k = 1.5$	417	1.963	1.118	1.541	370	2.109	1.073	1.591	286	1.734	1.077	1.406
$k = 1.8$	500	1.518	1.291	1.405	370	1.583	1.154	1.369	256	1.392	1.194	1.293
$k = 2$	555	1.406	1.505	1.456	370	1.440	1.231	1.336	303	1.313	1.347	1.330
$k = 2.3$	555	1.311	1.959	1.635	370	1.344	1.432	1.388	333	1.239	1.639	1.439

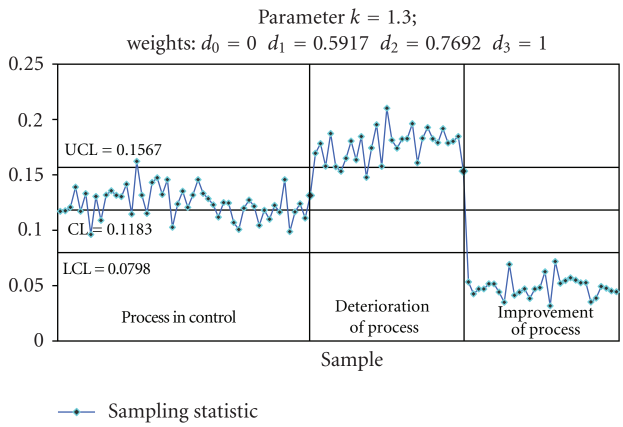

Figure 6: Two-sided multivariate p control chart (example two).

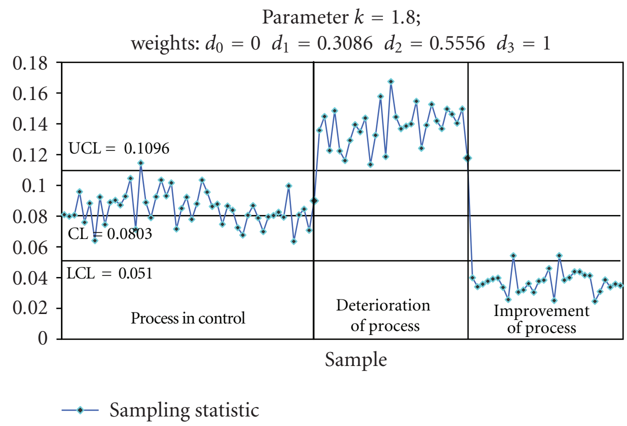

Figure 8: Two-sided multivariate p control chart (example two).

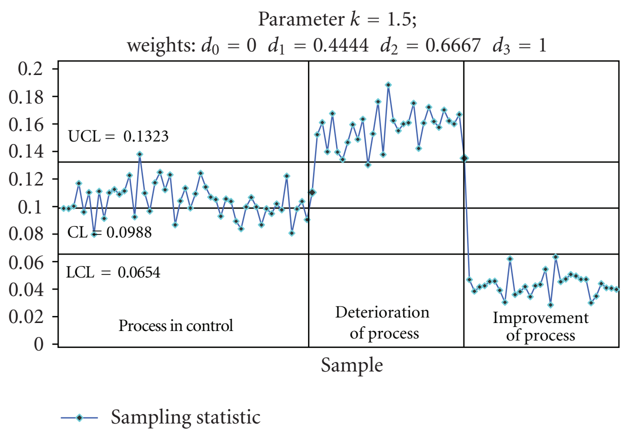

Figure 7: Two-sided multivariate p control chart (example two).

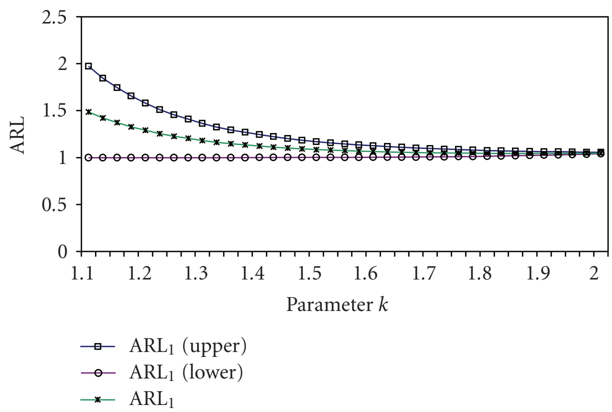

Figure 9: ARL_1 curves (example two).

performance of the control chart evidently depends on the value of parameter k. For the hypothesis under investigation, that is, to consider the process at a high level of quality, the chart seems to perform better. In fact, comparing ARL_1 curves in Figures 4 and 9 and data in Tables 1 and 2, we can conclude that the control chart performs better at a high level of quality than at a low level of quality, particularly when the statistical procedure is used to identify improvement in the process.

Table 2 and Figure 10 report for example two, the values of ARL for procedures based (a) on Šidák's inequality, (b) on empirical estimation and (c) on normal approximation.

If we consider the results in Table 2 we conclude that, as in the example one, since $ARL_1(\text{lower}) < ARL_1(\text{upper})$, then the proposed control chart is better to identify process improvement for $k \leq 2$. The same evidence is obtained from the analysis of the ARL_1 curves in Figure 9. In fact, the distance between the $ARL_1(\text{upper})$ curve and the $ARL_1(\text{lower})$ curve is high for small values of k and small for large values. The results, considering data in Table 2 and

Table 2: ARL comparisons (example two).

	ARL for chart with Šidák's approximation control limits				ARL for chart with exact estimated control limits				ARL for chart with normal approximation control limits			
k	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1
$k = 1.3$	455	1.365	1.000	1.183	370	1.428	1.00000	1.214	256	1.271	1.000	1.136
$k = 1.5$	455	1.169	1.001	1.085	370	1.192	1.00020	1.096	256	1.124	1.000	1.062
$k = 1.8$	435	1.076	1.015	1.046	370	1.085	1.00370	1.044	270	1.056	1.007	1.032
$k = 2$	526	1.056	1.04	1.048	370	1.065	1.01060	1.038	312	1.037	1.022	1.030
$k = 2.3$	625	1.037	1.115	1.076	370	1.037	1.02838	1.033	294	1.024	1.070	1.047

Table 3: ARL comparisons (example three).

	ARL for chart with Šidák's approximation control limits				ARL for chart with exact estimated control limits				ARL for chart with normal approximation control limits			
k	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1	ARL_0	ARL_1 upper	ARL_1 lower	ARL_1
$k = 1.3$	1250	1.000	1.080	1.040	370	1.000	1.034	1.017	385	1.000	1.042	1.021
$k = 1.5$	909	1.000	1.179	1.089	370	1.000	1.085	1.042	400	1.000	1.109	1.055
$k = 1.8$	769	1.000	1.460	1.230	370	1.000	1.211	1.106	526	1.000	1.302	1.151
$k = 2$	769	1.000	1.783	1.392	370	1.000	1.378	1.189	526	1.000	1.502	1.251
$k = 2.3$	909	1.000	2.539	1.770	370	1.000	1.586	1.293	455	1.000	1.999	1.500

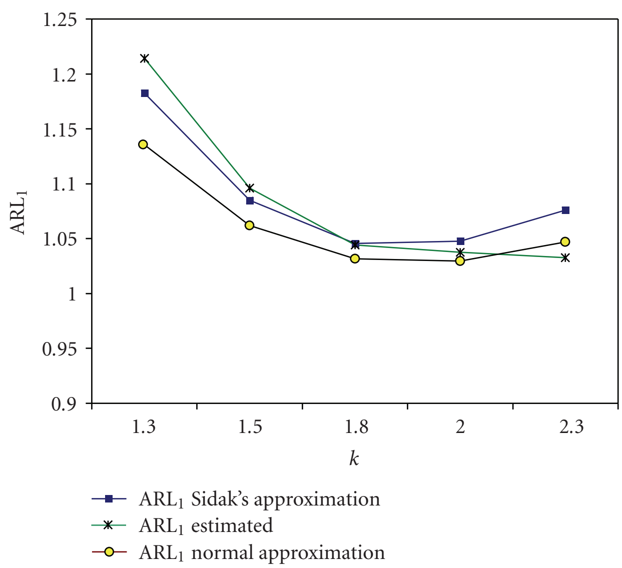

Figure 10: ARL_1 comparisons.

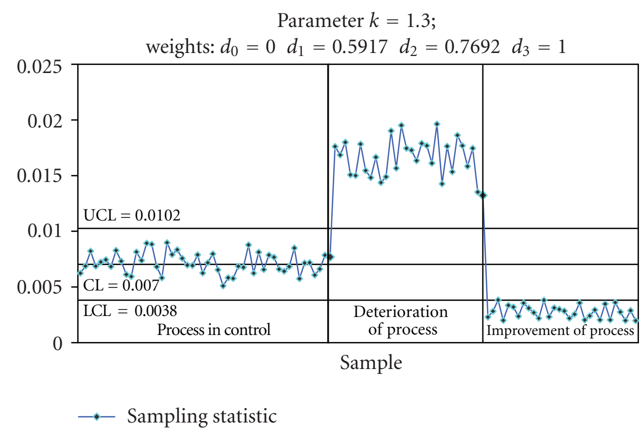

Figure 11: Two-sided multivariate p control chart (example three).

Figure 9, obtained in the case of high-quality level, are better than that obtained in the case of low-quality level.

4.3. Example Three: Very High-Quality Level. A third example considers a process with a much higher level of quality than that considered in cases one and two; in fact, the defect-free proportion, that is, chosen is equal to 0.99 and the maximum defective proportion is 0.001, that is, 1000 ppm. This level of quality will be classified as very high. The following hypotheses, regarding the defect proportions in vector $\mathbf{p}$, are used to evaluate the performance of the multivariate p control chart:

$$\text{process in control: } p_0 = 0.99, \quad p_1 = 0.005, \quad p_2 = 0.004, \quad p_3 = 0.001, \tag{25}$$

$$\text{deterioration process: } p_0 = 0.9805, \quad p_1 = 0.005, \quad p_2 = 0.004, \quad p_3 = 0.0105, \tag{26}$$

$$\text{improvement process: } p_0 = 0.9964, \quad p_1 = 0.011, \quad p_2 = 0.0015, \quad p_3 = 0.001. \tag{27}$$

The hypothesis (26) indicates a deterioration in process and the hypothesis (27) indicates an improvement in process,

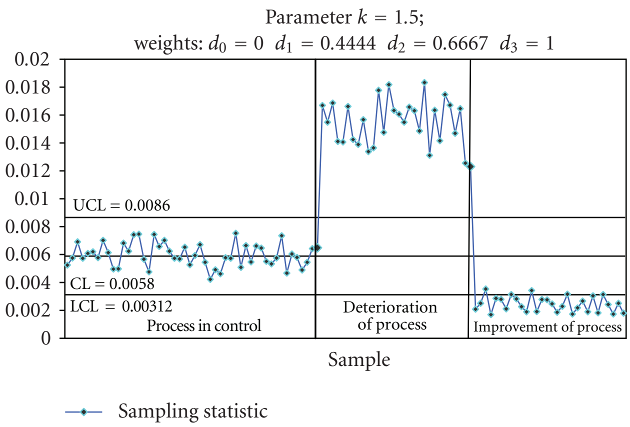

FIGURE 12: Two-sided multivariate p control chart (example three).

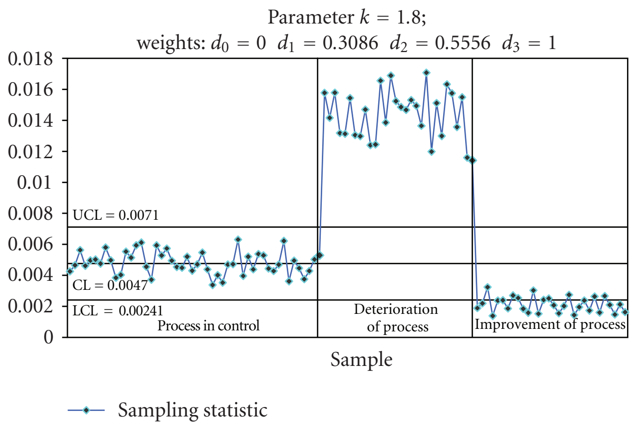

FIGURE 13: Two-sided multivariate p control chart (example three).

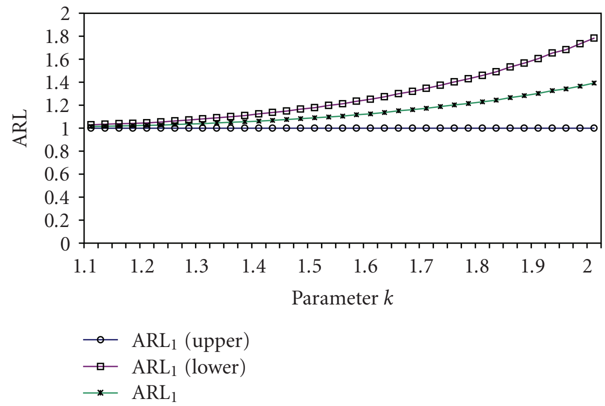

FIGURE 14: ARL_1 curves (example three).

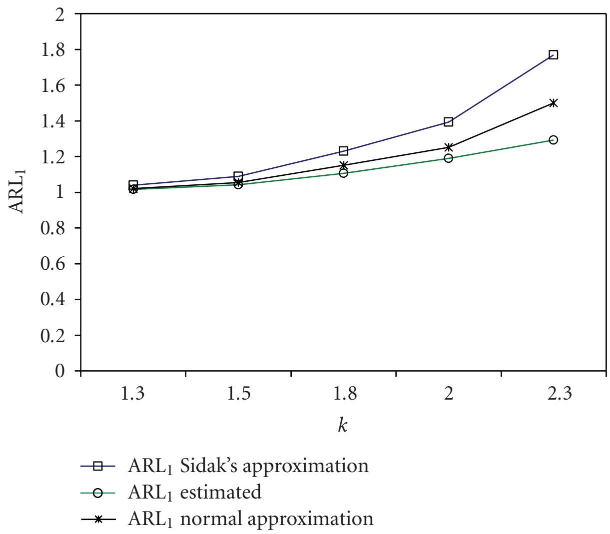

FIGURE 15: Comparisons of ARL_1 (example three).

since the minimum proportion is 0.001, then in accordance with Cochran's rule we have to take a sample of size $n = 5000$. The same simulation procedure described in example one is used. As in the previous examples, the performance of the control chart evidently depends on the value of parameter k. For the hypothesis under investigation, considering the ARL_1 values, the chart works much better than it does at the low and high quality levels. In fact, the ARL_1 in the example process with very high-quality level is always smaller than that in the example processes with high or low quality level. This indicates that our approach is more appropriate in the case of a process with a very high level of quality (see Figures 11, 12, 13 and 14).

Table 3 and Figure 15 report the values of ARL in the function of parameter k.

The comparisons of the ARL, reported in the Tables 1–3 point out that (i) ARL_1 is always better for the control chart with non simultaneous approximation control limits; (ii) ARL_0 is always better for the control chart with simultaneous Šidák's approximation control limits; that is, the false alarm probability is always smaller in the simultaneous approach respect to the nonsimultaneous approach. In general, a good multiattribute quality control procedure is one that provides a method for identifying which subset of the classes of defects are responsible when the process is determined to be out of control. The control chart proposed in this paper and based on simultaneous confidence intervals meets this criterium. On the contrary the non simultaneous approach is not able to solve the identification problem (see [25]).

5. Conclusions

In this article an index of weighted overall defectiveness of the process and a two-sided multivariate p control chart to monitoring the quality of the process are defined. Using simultaneous confidence interval, based on the Šidák's inequality, approximate control limits are determined. Specifically, the control chart is designed to identify changes in any of the defective class proportions and, differently of the one-sided chi-square control chart, is able to identify process deterioration or process improvement. Besides, if the control chart signals an out of control situation, then, estimating simultaneous confidence intervals for every parameters, the identification problem can be solved.

The sampling statistic used to estimate the overall defectiveness of the process and to design the corresponding control chart is function of the weights associated to the vector of quality defect categories. Therefore, the performance of the control chart is influenced by these values. In this paper we propose to use weights that are in terms of the geometric progression; the parameter k is the common ratio of thegeometric sequence. The performance of the control chart has been evaluated using simulated data from multinomial process in three different hypotheses: low, high, and very high quality level of the process. Properties of the ARL are examined and investigated by numerical simulations in function of different values of the parameter k. Some numerical comparisons, in terms of corresponding ARL, for control charts with different control limits, are proposed. The control chart is more effective in presence of very high quality level, specifically, in identifying improvement of the process.

An interesting methodological development is possible considering the multivariate binomial distribution as appropriate probabilistic model to monitor the process quality degree, because it is a more general distribution which contains the one used in this paper.

Acknowledgments

The author wishes to thank the editor and the anonymous referees for their thoughtful and detailed suggestions that improved the paper.

References

[1] E. Topalidou and S. Psarakis, "Review of multinomial and multiattribute quality control charts," *Quality and Reliability Engineering International*. In press.

[2] L. S. Nelson, "A chi-square control chart for several proportions," *Journal of Quality Technology*, vol. 19, no. 4, pp. 229–231, 1987.

[3] H. Taleb and M. Limam, "On fuzzy and probabilistic control charts," *International Journal of Production Research*, vol. 40, no. 12, pp. 2849–2863, 2002.

[4] H. Taleb, "Control charts applications for multivariate attribute processes," *Computers and Industrial Engineering*, vol. 56, no. 1, pp. 399–410, 2009.

[5] A. J. Duncan, "A chi-square chart for controlling a set of percentages," *Industrial Quality Control*, vol. 7, pp. 11–15, 1950.

[6] M. Marcucci, "Monitoring multinomial process," *Journal of Quality Technology*, vol. 17, no. 2, pp. 86–91, 1985.

[7] M. Xie, T. N. Goh, and V. Kuralmani, *Statistical Models and Control Charts for High Quality Processes*, Kluwer Academic Publishers, Boston, Mass, USA, 2002.

[8] P. D. Bourke, "Detecting a shift in fraction nonconforming using run-length control charts with 100% inspection," *Journal of Quality Technology*, vol. 23, no. 3, pp. 225–238, 1991.

[9] D. C. Montgomery, *Introduction to Statistical Quality Control*, John Wiley & Sons, New York, NY, USA, 2005.

[10] T. W. Calvin, "Quality control techniques for 'zero- defects'," *IEEE Transactions on Components, Hybrids, and Manufacturing Technology*, vol. 6, no. 3, pp. 323–328, 1983.

[11] F. C. Kaminsky, J. C. Benneyan, and R. D. Davis, "Statistical control charts based on a geometric distribution," *Journal of Quality Technology*, vol. 24, no. 2, pp. 63–69, 1992.

[12] M. Xie and T. N. Goh, "Improvement detection by control charts for high yield processes," *International Journal of Quality and Reliability Management*, vol. 10, pp. 24–31, 1993.

[13] C. P. Quesenberry, "Geometric Q charts for high quality processes," *Journal of Quality Technology*, vol. 27, no. 4, pp. 304–315, 1995.

[14] M. Xie and T. N. Goh, "The use of probability limits for process control based on geometric distribution," *International Journal of Quality and Reliability Management*, vol. 14, pp. 64–73, 1997.

[15] M. Xie, T. N. Goh, and L. Y. Chan, "A quality monitoring and decision-making scheme for automated production processes," *International Journal of Quality and Reliability Management*, vol. 16, pp. 148–157, 1999.

[16] S. T. A. Niaki and B. Abbasi, "On the monitoring of multiattributes high-quality production processes," *Metrika*, vol. 66, no. 3, pp. 373–388, 2007.

[17] X. S. Lu, M. Xie, T. N. Goh, and C. D. Lai, "Control chart for multivariate attribute processes," *International Journal of Production Research*, vol. 36, no. 12, pp. 3477–3489, 1998.

[18] C. R. Cassady and J. A. Nachlas, "Evaluating and implementing 3-level control charts," *Quality Engineering*, vol. 18, no. 3, pp. 285–292, 2006.

[19] R. G. Miller, *Simultaneous Statistical Inference*, Springer, New York, NY, USA, 2nd edition, 1981.

[20] R. Z. Gold, "Tests auxiliary to χ^2 tests in a Markov chain," *The Annals of Mathematical Statistics*, vol. 34, no. 1, pp. 56–74, 1963.

[21] R. J. Serfling, *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York, NY, USA, 1980.

[22] L. A. Goodman, "On simultaneous confidence intervals for multinomial populations," *Technometrics*, vol. 7, no. 2, pp. 247–254, 1965.

[23] Z. Šidàk, "Rectangular confidence regions for the means of multivariate normal distributions," *Journal of the American Statistical Association*, vol. 62, pp. 626–633, 1967.

[24] Z. Šidàk, "On multivariate normal probabilities of rectangles: their dependence on correlations," *The Annals of Mathematical Statistics*, vol. 39, pp. 1425–1434, 1968.

[25] A. J. Hayter and K.-L. Tsui, "Identification and quantification in multivariate quality control problems," *Journal of Quality Technology*, vol. 26, no. 3, pp. 197–208, 1994.

[26] V. K. Rohatgi, *An Introduction to Probability Theory and Mathematical Statistics*, John Wiley & Sons, New York, NY, USA, 1976.

[27] W. G. Cochran, "Some methods for strengthening the common χ^2 tests," *The Annals of Mathematical Statistics*, vol. 10, no. 4, pp. 417–451, 1954.

[28] N. A. Heckert and J. Filliben, "NIST Handbook 148: DATAPLOT Reference Manual," National Institute of Standards and Technology Handbook Series, 2003.