A REACTION-DIFFUSION EQUATION WITH MEMORY

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ABSTRACT. We consider a one-dimensional reaction-diffusion type equation with memory, originally proposed by W.E. Olmstead *et al.* to model the velocity u of certain viscoelastic fluids. More precisely, the usual diffusion term u_{xx} is replaced by a convolution integral of the form $\int_0^\infty k(s)u_{xx}(t-s)ds$, whereas the reaction term is the derivative of a double-well potential. We first reformulate the equation, endowed with homogeneous Dirichlet boundary conditions, by introducing the integrated past history of u . Then we replace k with a time-rescaled kernel k_ε , where $\varepsilon > 0$ is the relaxation time. The obtained initial and boundary value problem generates a strongly continuous semigroup $S_\varepsilon(t)$ on a suitable phase-space. The main result of this work is the existence of the global attractor for $S_\varepsilon(t)$, provided that ε is small enough.

1. Introduction. Let $(0, \ell) \subset \mathbb{R}$ be a bounded interval. For $u = u(x, t) : (0, \ell) \times \mathbb{R} \rightarrow \mathbb{R}$, we consider the equation

$$u_t - \int_0^\infty k(s)u_{xx}(t-s)ds + \phi(u) = f, \quad t > 0, \quad (1.1)$$

where $k : [0, \infty) \rightarrow [0, \infty)$ is a suitable memory kernel, the function ϕ is a smooth nonlinearity with (at most) cubic growth, and f is a given external force, which is supposed to be independent of time for the sake of simplicity.

Equation (1.1), endowed with the boundary conditions

$$u(0, t) = u(\ell, t) = 0, \quad t \geq 0, \quad (1.2)$$

has been proposed in [14] to model the behavior of certain viscoelastic fluids. There, u represents the velocity of the fluid and $\phi(\xi) = \xi^3 - R\xi$, R being the Rayleigh number. The kernel k is assumed to be of Jeffreys type, namely,

$$k(s) = (1 - \sigma) \frac{e^{-\frac{s}{\varepsilon}}}{\varepsilon} + \sigma \delta(s),$$

where $\varepsilon > 0$ is the relaxation time, $\sigma \in [0, 1]$ is the ratio of retardation to relaxation times, and δ is the Dirac mass at zero. Observe that for $\sigma = 1$ we recover the

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