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RESEARCH PAPER

Stochastic user equilibrium in the presence of state dependence

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Abstract
We consider the following two state-dependent effects at the level of route choice: inertia to change and, as a consequence of experience, lower perception variance for the currently used route. A heteroscedastic extreme value model embodying heterogeneity across alternatives in the mean of the random terms is used. Estimations based on stated preference data confirm the presence of both state-dependent effects. We introduce a new class of stochastic user equilibrium (SUE) models that take state-dependent effects into account. The class includes conventional SUE as special case. The equilibrium conditions are formulated as fixed-point states of deterministic day-to-day assignment processes. At the equilibrium (1) no user can improve her/his utility by unilaterally changing route, and (2) if each user shifts from her/his current route to her/his newly chosen route the observed route flows do not change. The existence of the equilibrium is guaranteed under usually satisfied conditions. A modified method of successive averages is proposed for solution. Examples related to a two arc network and to the Nguyen-Dupuis network illustrate the model.

Keywords Route choice · State dependence · Inertia · Heteroschedastic extreme value · Stochastic user equilibrium

1 Introduction

The unrealistic assumptions of the user equilibrium (UE) principle proposed by Wardrop (1952) have long been recognized. The principle predicts traffic equilibrium flows while accounting for the effects of congestion that arise due to users sharing arcs. The assumptions at its basis are that all travellers are rational in terms of preferring the lower travel cost routes, have perfect knowledge

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of network travel costs, and are able to identify the minimum travel cost route (Sheffi 1985).

Daganzo and Sheffi (1977) have proposed the stochastic user equilibrium (SUE) principle to relax the perfect knowledge assumption of UE. SUE incorporates a random perception error term in the route cost function to capture users' imperfect knowledge of travel costs (notice that costs are assumed to be certain, i.e., uncertainty is on the demand side only, not on the supply side). The demand of every origin–destination (OD) pair is, therefore, split up among the various routes. The SUE principle states that:

> "at equilibrium no user can improve her perceived utility by unilaterally changing route",

Mathematically, SUE is formulated as the problem of seeking flows satisfying the condition:

> "each user chooses the route with the maximum perceived utility".

The utility is a theoretical construct that, on the one hand, allows to include monetary cost items (such as tolls and charges) in the user's cost function in addition to travel time, on the other, it achieves consistency with the formulation of the route choice models used which are based on random utility maximisation.

The first route choice model proposed for SUE is probit (Daganzo and Sheffi 1977; Maher and Hughes 1997) because of the flexible pattern of correlation which is particularly suitable in the light of route overlappings. Additionally, probit can model the variability in perception variances across routes which can stem from different trip lengths and different levels of knowledge on network conditions (e.g. routes with low congestion for which users have a more certain perception). However, probit SUE is computationally burdensome because choice probabilities are not in closed form and require simulation.

After Daganzo and Sheffi (1977), research on SUE has shown significant developments. We make a distinction between (1) developments that have retained the utility maximisation principle, and (2) developments based on a relaxation of this principle.

Developments in the first area have essentially tried to achieve computational efficiency while mimicking the good variance–covariance matrix properties of probit.

Logit SUE suffers from the i.i.d (independently and identically distributed) error term assumption, but has the advantages of uniqueness of the solution under mild conditions [based on the equivalence with a convex optimization problem; Fisk (1980)], and availability of efficient arc-based algorithms to compute equilibrium flows.

To overcome the limitations of the independently distributed assumption, researchers have proposed to either modify the deterministic component of the additive (dis)utility function of the logit model, or the distribution of the error terms while maintaining the Gumbel distribution for the individual error term. The models modifying the deterministic component include the c-logit (Cascetta et al. 1996; Zhou et al. 2012) and the path size logit (Ben-Akiva and Bierlaire 1999; Chen et al. 2012). The models modifying the distribution of the error terms are based on the generalised extreme value (GEV) theory by McFadden (1978). They include the cross-nested logit (Vovsha 1997; Vovsha and Bekhor 1999; Bekhor and Prashker 1999), the pair combinatorial logit (Chu 1989; Bekhor and Prashker 1999; Gliebe et al. 1999; Pravinvongvuth and Chen 2005), and the generalised nested logit (Bekhor and Prashker 2001; Wen and Koppelman 2001).

To overcome the limitations of the identically distributed assumption, Chen has proposed OD-specific perception variance (Chen et al. 2012). Other authors have departed from logit and used the weibit model developed by Castillo et al. (2008) to model route-specific perception variance dependent on trip cost. Differently from the logit, in weibit the utility has a multiplicative form and the error term follows a Weibull distribution. This model has the advantage that, for independently distributed error terms, the probabilities are in closed form. The weibit has been proposed by Kitthamkesorn and Chen (2013) for SUE in combination with a path size element to obviate the independence assumption. Xu et al. (2015) have attempted to capture the strengths of both the logit and the weibit by developing, and applying to SUE, a closed-form hybrid choice model which considers simultaneously absolute and relative cost differences (which are accommodated, respectively, within the logit model and within the weibit model).

Recent years have seen the appearance of distribution-free approaches as alternatives to the probit SUE. Ahipaşaoğlu et al. (2015) have used the cross-moment choice model, introduced by Mishra et al. (2012) and Ahipaşaoğlu et al. (2013), and have developed a new SUE model that uses the mean and covariance information on route utilities but does not assume the particular form of the distribution. The choice probabilities are computed for the joint distribution that maximises expected utility. The equilibrium flows are computed by solving a convex optimization problem, which also guarantees uniqueness, in lieu of the simulation-based probit model. The approach, however, does not incorporate additional information on the marginal distributions such as skewness or heavy tails. This limitation is overcome by the approach based on the marginal distribution choice model, proposed by Natarajan et al. (2009) and Mishra et al. (2014), which is used for SUE by Ahipaşaoğlu et al. (2016).

Developments in the second area have relaxed the utility maximisation principle on the assumption that this falls short in representing users' decision making processes.

Chorus (2010, 2012) has proposed, on the basis also of empirical evidence, the principle of random regret minimisation. The user is assumed to compare a considered route with all other routes for their respective costs. If the considered route has a lower cost than another route there is no regret. If the considered route has a higher cost than another route then the regret for the considered route equals the cost difference. In this model, the error represents unobserved regret which is in turn function of cost differences, while in the classical random utility maximisation model the error represents unobserved costs. Bekhor et al. (2012) have applied random regret minimisation to SUE.

Another alternative to random utility maximisation is reference-dependence theory (Tverski and Kahneman 1991). According to this theory, carriers of utilities are not states but gains and losses relative to a reference point, with losses valued more heavily than gains (loss aversion). De Borger and Fosgerau (2008), Hess et al. (2008) and Delle Site and Filippi (2011) have formulated reference-dependent route choice models and estimated random utility versions of these models based on logit assumptions for the error terms. All have found evidence of asymmetrical responses with respect to gains and losses in travel time and money attributes of route alternatives.

Subsequent work has applied this new paradigm to SUE: Delle Site and Filippi (2011) have formulated reference-dependent SUE with exogenously given reference points, while Delle Site et al. (2013) have considered SUE with endogenous reference points. Endogenous reference points are based on the idea first proposed for deterministic UE and risky choices within a prospect theory framework by Xu et al. (2011). Reference-dependent theory has been recently applied to deterministic activity-travel equilibria by Li et al. (2016).

The contributions of the present paper are in this second area. The paper moves from the assumption that route choice is affected by state-dependence effects.

The first state-dependent effect is inertia: users exhibit a propensity to continue on the currently chosen route. A body of studies has provided experimental evidence supporting the assumption [see Srinivasan and Mahmassani (2000), for a review]. Inertia can be explained with habit persistence of users with satisfactory choices, and the psychological switching cost consequent to the absence of experience of alternatives. Inertia in route choice is commonly modelled by an alternative-specific constant associated with the currently chosen route [among the others: Cantillo et al. (2007), Cascetta and Cantarella (1991), Srinivasan and Mahmassani (2000); see Xie and Liu (2014), for a different approach].

The second state-dependent effect relates to the interpersonal heterogeneity in the valuation of the unobserved factors affecting the choice: as a consequence of experience, this heterogeneity is lower for the currently chosen route. Within a discrete choice random utility framework, this translates into a lower perception variance for the currently chosen route. Models that consider route-specific variance have been reviewed earlier in this introduction. However, the weibit model is not suitable for state-dependent effects because of the trip cost dependent variance. Bhat (1995) has considered the heteroschedastic extreme value model in the context of mode choice.

The paper addresses the following two research questions: (a) whether the second state-dependent effect is also confirmed by experimental evidence, and (b) how to define an equilibrium condition for the network in the presence of state-dependence.

Route choice is modelled based on random utility maximisation embodying heterogeneity across alternatives in the mean (first effect) and in the variance (second effect) of the error terms. The heteroschedastic extreme value model by Bhat (1995) is adopted because of the approximated closed-form probabilities.

For equilibrium, the starting point is the equivalence between SUE and steady states of discrete-time deterministic day-to-day assignment processes [see, among the others, Horowitz (1984), Cantarella and Cascetta (1995), Cantarella and Watling (2016), Lo et al. (2016), Delle Site (2017)]. The adjective deterministic refers to a process where route flows are regarded as deterministic variables. Since state dependence is inherently a dynamic phenomenon, equilibrium conditions in the presence of state dependence are formulated as fixed-point states of deterministic day-to-day assignment processes. The equilibrium model that is obtained is the state-dependent counterpart of the reference-dependent SUE formulated in Delle Site et al. (2013).

The paper is organised as follows. Section 2 presents the mathematical model with the formulation of the equilibrium conditions. Section 3 presents estimation results for the route choice model, which provide experimental evidence of the presence of the second state-dependent effect, as well as numerical results relating to the equilibrium of two (one smaller and one larger) illustrative networks. Directions for further research conclude.

2 Network equilibrium

2.1 Network representation and assumptions

The model is formulated according to the notation listed in the "Appendix".

Consider a strongly connected road transportation network. The associated graph is the couple of sets of nodes and directed arcs. Let A be the arc set and a the arc index. Origins (O) and destinations (D) constitute a subset of nodes. Let W be the set of OD pairs and w the OD pair index. Let R^w be the set of simple routes of OD pair w, and r the route index.

Consider day n. For each route $r \in R^w$, $F_r^{w(n)}$ denotes the corresponding route flow. We denote by $f_a^{(n)}$ the flow on arc a. The arc flows are obtained from the route flows by:

$$f_a^{(n)} = \sum_{w\in W} \sum_{r\in R^w} \delta_{a,r}^w \cdot F_r^{w(n)} \quad a \in A, \tag{1}$$

where $\delta_{a,r}^w$ is the element of the arc-route incidence matrix of OD pair w whose value is 1 if route r includes arc a, is 0 otherwise.

The demand flow of the OD pair w is denoted by d^w. Travel demand is fixed and does not change over time. We have the demand constraints:

$$d^w = \sum_{r\in R^w} F_r^{w(n)} \quad w \in W. \tag{2}$$

The feasible route flows are all the non-negative $F_r^{w(n)}$ satisfying the demand constraints (2).

Let $T_r^{w(n)}$ denote the travel time on route r of OD pair w. Let $t_a^{(n)}$ denote the travel time on arc a. The arc travel times are continuous functions of the arc flows:

$$t_a^{(n)} = t_a\left(f_a^{(n)},\ a \in A\right) \quad a \in A. \tag{3}$$

The route travel times are obtained from the arc travel times by the standard arc-additive model:

$$T_r^{w(n)} = \sum_a \delta_{a,r}^w \cdot t_a^{(n)} \quad r \in R^w, w \in W. \tag{4}$$

2.2 Deterministic day-to-day assignment process

The deterministic day-to-day assignment process requires modelling of users' behaviour in terms of the following.

Forecasting process	How users forecast the travel times that they will experience today, given the travel times experienced yesterday
Choice process	How users make a choice today, given the forecasted travel times, the time-independent monetary cost, and the choice made yesterday

Users behave according to a stochastic perception of travel (dis)utility. The systematic utility $V_{r|j}^{w(n)}$ of route r is assumed dependent on the travel time experienced the day before $T_r^{w(n-1)}$, and the time-independent monetary cost M_r^w. In addition, there is the term accounting for inertia: the systematic utility of the newly chosen route r is higher if this route is the route j chosen the day before. Thus, the systematic utility updating recursive equations are:

$$V_{r|j}^{w(n)} = \beta_T \cdot T_r^{w(n-1)} + \beta_M \cdot M_r^w + \eta \cdot I_{r|j}^w \quad r,j \in R^w, w \in W, \tag{5}$$

$$I_{r|j}^w = \begin{matrix} 1 j = r \\ 0 j \neq r \end{matrix}, \tag{6}$$

where β_T, β_M and η are estimation coefficients, $I_{r|j}^w$ the indicator function. The third term accounts for inertia: it is equal to η if the newly chosen route r is equal to the route j chosen the day before (because $I_{r|j}^w = 1$), it is equal to 0 otherwise (because $I_{r|j}^w = 0$).

The estimation coefficients β_T, β_M and η have the economic meaning of marginal utilities of, respectively, travel time, monetary cost and inertia.

The choice process considers the conditional probabilities $P_{r|j}^{w(n)}$ of choosing route r at day n having chosen route j the day before.

Conditional probabilities are based on additive random utility models. The utility of a route is the sum of the systematic part plus a random term. The random terms summarise factors that are unobserved by the modeler and account for inter-personal heterogeneity of preferences.

For each OD pair $w \in W$, the vector of the conditional on j random terms $\varepsilon_{r|j}^{w(n)}$, $r \in R^w$, is assumed to follow a heteroschedastic extreme value distribution, i.e., the random terms follow a type I extreme value distribution and are independent, but not identically distributed across route alternatives. The variance of each term $\varepsilon_{r|j}^{w(n)}$, $r \in R^w$, is given by $\left(\theta_{r|j}^{w} \cdot \pi\right)^2 \Big/ 6$ where $\theta_{r|j}^{w}$ is the scale parameter. We set the scale parameter equal to $\theta < 1$ when the newly chosen route is the route chosen the day before, we set the scale parameter equal to 1 otherwise:

$$\theta_{r|j}^{w} = \begin{matrix} \theta j = r \\ 1 j \neq r \end{matrix}. \tag{7}$$

With this normalization, required for identification problems and to set the overall scale of utility, the variance of the route chosen the day before is estimated relative to the normalized variance of the other routes which is equal to $\pi^2/6$ (Train 2003).

Conditional on the choice of j the day before, users who choose route r at day n are those who perceive this route to maximize their utility. The conditional choice probabilities are given by (Bhat 1995):

$$P_{r|j}^{w(n)} = \int_{t=-\infty}^{t=\infty} \left[\prod_{r' \in R^w,\ r' \neq r} \mathrm{e}^{-\mathrm{e}^{-\left(V_{r|j}^{w(n)} - V_{r'|j}^{w(n)} + \theta_{r|j}^{w} \cdot t\right)/\theta_{r'|j}^{w}}} \right] \cdot \mathrm{e}^{-\mathrm{e}^{-t}} \cdot \mathrm{e}^{-t} \mathrm{d}t \quad r, j \in R^w, \quad w \in W. \tag{8}$$

Probabilities have not an exact closed form. They can be computed by Gauss-Laguerre quadrature (Press et al. 1986), which easily yields the approximated closed-form expression:

$$P_{r|j}^{w(n)} = \sum_{s=1}^{S} \omega_s \cdot \Phi\left(x_s\right) = \sum_{s=1}^{S} \omega_s \cdot \mathrm{e}^{-\sum_{r' \in R^w, r' \neq r} \mathrm{e}^{\frac{V_{r|j}^{w(n)} - V_{r'|j}^{w(n)}}{\theta_{r'|j}^{w}}} \cdot x_s^{\theta_{r|j}^{w} / \theta_{r'|j}^{w}}} \qquad r, j \in R^w, w \in W, \tag{9}$$

where x_s is a node and ω_s is a weight, $s = 1, \ldots, S$. Nodes and related weights are in Rabinowitz and Weiss (1959).

It is seen from formulas (8) and (9), and the way the scale parameters enter the probability formula, that the normalization of the scale parameters does not affect the probabilities.

The model reduces to a multinomial logit with inertia when the variances are equal across alternatives. Route overlapping effects can be accounted for at the level of the systematic term as in the common-factor logit model proposed by Cascetta et al. (1996) and in the path size logit proposed by Ben-Akiva and Bierlaire (1999).

The conditional probabilities are dependent, via the utility updating Eqs. (5) and (6), on the travel times experienced the day before and, therefore, on the route flows of the day before $F_r^{w(n-1)}, r \in R^w, \quad w \in W$:

$$P_{r|j}^{w(n)} = P_{r|j}^{w}\left(F_r^{w(n-1)},\ r \in R^w,\ w \in W\right) \quad r,j \in R^w, w \in W. \tag{10}$$

The route flow updating recursive eqns are:

$$F_r^{w(n)} = \sum_{j \in R^w} P_{r|j}^{w}\left(F_r^{w(n-1)},\ r \in R^w,\ w \in W\right) \cdot F_j^{w(n-1)} \quad r \in R^w, \quad w \in W. \tag{11}$$

2.3 State-dependent stochastic user equilibrium

The steady states associated with the process (11) are governed by the fixed-point eqns:

$$F_r^{w*} = \sum_{j \in R^w} P_{r|j}^{w}\left(F_r^{w*},\ r \in R^w,\ w \in W\right) \cdot F_j^{w*} \quad r = 1, \ldots |R^w| - 1; w \in W, \tag{12}$$

$$F_{|R^w|}^{w*} = d^w - \sum_{r=1}^{|R^w|-1} F_r^{w*} \quad w \in W, \tag{13}$$

where $|R^w|$ denotes the cardinality of the set R^w. Equations (13) are justified to satisfy demand constraints.

The fixed point Eqs. (12) and (13) represent states of equilibrium which can be seen as generalization of SUE: we will refer to them as State-Dependent Stochastic User Equilibrium (SDSUE).

In words, the SDSUE represents the conditions where:

"no user can improve her/his utility by unilaterally changing route", and

"if each user shifts from her/his currently used route to her/his newly chosen route, the observed route flows do not change".

For clarification of the statement, the matrix of transition flows, i.e. the flows of shifters from the currently used route to the newly chosen route, is shown in

$F_1^{w*} \cdot P_{1\|1}^{w}$	$F_1^{w*} \cdot P_{2\|1}^{w}$	$\cdot$	$F_1^{w*} \cdot P_{\|R^w\|\|1}^{w}$	F_1^{w*}
$F_2^{w*} \cdot P_{1\|2}^{w}$	$F_2^{w*} \cdot P_{2\|2}^{w}$	$\cdot$	$F_2^{w*} \cdot P_{\|R^w\|\|2}^{w}$	F_2^{w*}
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$F_{\|R^w\|}^{w*} \cdot P_{1\|\|R^w\|}^{w}$	$F_{\|R^w\|}^{w*} \cdot P_{2\|\|R^w\|}^{w}$	$\cdot$	$F_{\|R^w\|}^{w*} \cdot P_{\|R^w\|\|\|R^w\|}^{w}$	$F_{\|R^w\|}^{w*}$
F_1^{w*}	F_2^{w*}	$\cdot$	$F_{\|R^w\|}^{w*}$	totals/totals

Fig. 1 Transition flow matrix at SDSUE for OD pair *w*

Fig. 1. In each column we have a newly chosen route, in each row a currently used route. On the main diagonal (north–west to south–east) we find the number of users who stay on the given route. The fixed-point Eqs. (12) express the column sums.

It is possible to state existence conditions for SDSUE using Brouwer's fixed-point theorem: existence is satisfied, under the assumptions here, because the feasible set of route flows is non-empty, compact and convex, and the map of the fixed-point eqns is continuous.

Based on the theory of discrete dynamical systems (Parker and Chua 1989; Galor 2010), to ensure local stability of the steady states, i.e. of the fixed points, it is necessary and sufficient that the Jacobian matrix, computed in the fixed point, of the transition functions in the right-hand side of Eq. (11) has all eigenvalues in absolute values less than 1. This condition constraints the eigenvalues to lie in the interior of a circle of unit radius in the complex (Argand) plane. This means that if and only if these conditions are satisfied, the system, upon a sufficiently small perturbation, converges back to the fixed point.

It is convenient to re-write in compact form the fixed-point problem using a vector notation (vectors in bold):

$$\boldsymbol{F} = \boldsymbol{\Psi}(\boldsymbol{F}), \ \boldsymbol{F} \geq 0, \tag{14}$$

where $\mathbf{F}$ is the $\left[\left(\sum_{w\in W}|R^w|\right)\times 1\right]$ vector of route flows, and $\boldsymbol{\Psi}$ the $\left[\left(\sum_{w\in W}|R^w|\right)\times 1\right]$ vector mapping representing the functions in the right-hand sides of Eqs. (12) and (13).

The SDSUE collapses to a conventional SUE (Daganzo and Sheffi 1977) when $\eta = 0$ and the variances are constant across alternatives.

In fact, when this condition occurs, choice probabilities are not affected by the currently used route:

$$P^w_{r|j} = P^w_r \quad r, j \in R^w, \ w \in W. \tag{15}$$

The SDSUE fixed-point problem (12) and (13) reduces then to the conventional SUE fixed-point problem:

$$F^w_r = q^w \cdot P^w_r \quad r, j \in R^w, \ w \in W, \tag{16}$$

which can be re-written in compact form:

$$\boldsymbol{F} = \tilde{\boldsymbol{q}} \circ \boldsymbol{P}, \tag{17}$$

where $\tilde{\mathbf{q}}$ is the $\left[\left(\sum_{w\in W}|R^w|\right)\times 1\right]$ route-based expanded version of the demand vector $q = \left[q^1, \dots, q^w\right]$, $\mathbf{P}$ is the $\left[\left(\sum_{w\in W}|R^w|\right)\times 1\right]$ vector mapping of probabilities, and "$\circ$" denotes the Hadamard, i.e. componentwise, product

($x \circ y$ is the vector whose i-th component is $x_i \cdot y_i$).

2.4 Solution algorithm

To solve the problem we use a heuristic approach based on the method of successive averages (MSA). The formulation of SDSUE as a fixed point in the route flows suggests a route flow-based MSA. A route-based algorithm is in any case the only viable option since the route utilities are not additive in the constituent arcs because of the route-specific mean (monetary cost and inertia term) and route-specific variance of the random terms. The main steps are, in compact notation, as follows.

Step 1. *Initialisation*.
Set arc flows equal to zero.
Compute arc and route travel times.
Set the current route of each OD pair equal to the route $j°$ with minimum time in free-flow conditions.
Set iteration counter: k=1.
Compute the conditional probability vector $\mathbf{P}_k = \left[P^w_{r|j°,k},\ r \in R^w, w \in W\right]$ where the current route is set, for each OD pair, as above.
Set initial route flows: $\mathbf{F}_k = \widetilde{\mathbf{q}} \circ \mathbf{P}_k$

Step 2. *Convergence check*.
Compute arc flows when route flows are equal to $\mathbf{F}_k$. Compute arc and route travel times.
Compute all conditional probability vectors, $P^w_{r|j},\ r, j \in R^w, w \in W$.
Compute $\mathbf{\Psi}(\mathbf{F}_k)$.
If $\|\mathbf{F}_k - \mathbf{\Psi}(\mathbf{F}_k)\|_\infty < \gamma$
then stop and provide outputs,
otherwise go to step 3.

Step 3. *Updating of route flows*.
Set route flows: $\mathbf{F}_{k+1} = \mathbf{F}_k + \frac{1}{k} \cdot [\mathbf{\Psi}(\mathbf{F}_k) - \mathbf{F}_k]$
Increment iteration counter: $k = k + 1$.
Go to step 2.

The symbol $\|\mathbf{x}_\infty\|$ denotes the infinity, or maximum, norm of the vector $\mathbf{x}$ with components x_i, i.e. $\|\mathbf{x}_\infty\| = \max_i x_i \quad \|\mathbf{x}\|_\infty = \max_i |x_i|$. The convergence tolerance is expressed by γ.

Notice that in the first step only one conditional probability vector is computed, the one conditional on the current route set equal, for each OD pair, to the minimum travel time route in free-flow conditions. In contrast, in the second step, for each OD pair, the algorithm computes recursively all conditional probability vectors, i.e. one probability vector for each available current route.

The algorithm generates a sequence of feasible route flows, i.e. satisfying both the demand and non-negativity constraints. At each iteration the solution $\mathbf{F}_{k+1}$ is the average of the first k solutions $\mathbf{\Psi}(\mathbf{F}_k)$, hence the name successive averages. As in the MSA for conventional SUE, the convergence can be slow because of the decreasing flow correction which depends on the factor $1/k$.

Enumeration of routes is required, therefore, in large networks the route set may need to be reduced selectively to a subset of routes that is justified behaviourally [reviews of selection criteria are in Bekhor and Toledo (2005), Cascetta (2009), Prato (2009).

When $\eta = 0$ and the variances are constant across alternatives, the algorithm reduces to the conventional route flow-based MSA for SUE. It is possible to prove that the MSA algorithm converges to SUE if probabilities are logit (Powell and

Table 1 Estimation results (t-stat in brackets)

	MNL1	MNL2	HEV
Travel time (minutes) coefficient β_T	-0.10276 (-9.150)	-0.11434 (-9.587)	-0.1617 (-5.284)
Travel cost (EUR/trip) coefficient β_C	-1.48148 (-12.226)	-1.468 (-12.073)	-2.202 (-6.181)
Inertia coefficient η	–	0.5083 (4.480)	–
Currently used route scale parameter θ	–	–	0.4324 (5.189)
Non-currently used route scale parameter	–	–	1 (fixed)
Adjusted ρ^2	0.41479	0.43337	0.42435

Sheffi 1982). Conditions of convergence of MSA to SUE for other random utility models are investigated in Cantarella and Velonà (2010).

3 Numerical results

3.1 Estimation results

Data on route choice behavior are from a stated preference survey which took place in Rome and was used in a previous work to estimate a reference-dependent logit (Delle Site and Filippi 2011). Three models have been estimated (Table 1): standard multinomial logit (MNL1), multinomial logit with inertia (MNL2), and heteroschedastic extreme value (HEV). The estimation was carried out using NLOGIT statistical software.

3.2 State-dependent stochastic user equilibrium: illustrative example—two arc network

We consider a two-arc network representing a town-centre route and a bypass route, both without toll.

Table 2 SDSUE with MNL2: sensitivity to inertia coefficientη

η	Town-centre route		Bypass route		Time expenditure
	Flow (veh/h)	Time (min)	Flow (veh/h)	Time (min)	(h)
0 (SUE)	560	3.96	640	2.79	66.7
0.4	554	3.92	646	2.80	66.3
0.8	548	3.90	652	2.80	66.0
1.2	543	3.88	657	2.80	65.8
1.6	540	3.87	660	2.80	65.6

Table 3 SDSUE with MNL2 and inertia coefficient η= 0.5083: transition route flows

	Newly chosen route		Total
	Town-centre route	Bypass route	
Currently used route			
Town-centre route	328	224	552
Bypass route	224	424	648
Total	552	648	1200

We assume a total demand of 1200 veh/h. For supply, BPR volume-delay functions derived empirically for similar routes are used. The functions (in hours) are $T = 0.057 \cdot \left[1 + (F/800)^{5.2}\right]$ for the town-centre route, and $T = 0.045 \cdot \left[1 + 0.68 \cdot \left(\frac{F}{1230}\right)^{4.6}\right]$ for the bypass route.

Table 2 shows the SDSUE results when route choice is MNL2 (inertia effect). As the magnitude of the state-dependence effect increases (the inertia coefficient η increases), the flow on the town-centre route, which is the route where travel time is higher, decreases. At the same time the time expenditure decreases.

Table 3 shows, in the case = 0.5083 (the estimation value), the matrix of transition route flows. The route flows are found in the bottom line. The last column on the right provides the flows having each a given current route. The row sums equal the column sums. Notice that the matrix is symmetric, which is an obvious property with two alternatives only.

The Table is useful to comprehend the intuition of SDSUE. Of the 552 veh/h (=328 + 224) which are found on the town centre route, 328 veh/h stay on this route, while the remaining 224 veh/h are those shifting to the bypass route. Of the 648 veh/h (224 + 424) which are found on the bypass route, 224 veh/h shift to the

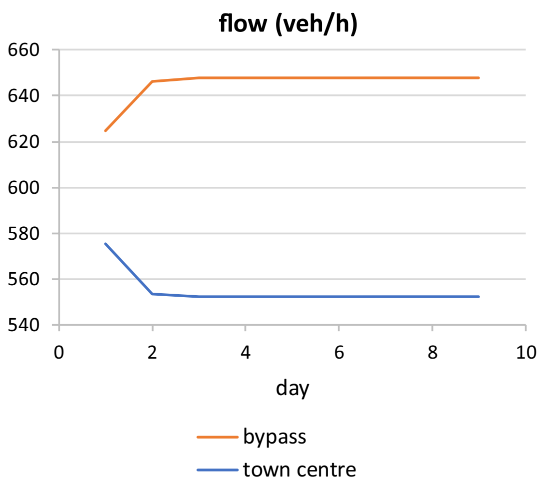

Fig. 2 Day-to-day dynamics

Table 4 SDSUE with HEV: sensitivity to the scale parameter of the currently used route θ

θ	Town-centre route		Bypass route		Time expenditure
	Flow (veh/h)	Time (min)	Flow (veh/h)	Time (min)	(h)
1 (SUE)	547	3.89	653	2.80	66.0
0.75	543	3.88	657	2.80	65.8
0.5	540	3.86	660	2.80	65.6
0.25	535	3.84	665	2.81	65.4

Table 5 SDSUE with HEV and scale parameter of currently used route $\theta = 0.4324$: transition route flows

	Newly chosen route		Total
	Town-centre route	Bypass route	
Currently used route			
Town-centre route	209	330	539
Bypass route	330	331	661
Total	539	661	1200

town-centre route, while the remaining 424 stay on the bypass route. Therefore, if the 224 + 224 veh/h update their route to their newly chosen route, the total flow on each route does not change.

We have computed the two eigenvalues λ_1 and λ_2 of the Jacobian matrix, at the fixed point, of the transition functions for the two flows. We have obtained $\lambda_1 = 0.489$ and $\lambda_2 = -0.824$. Since both eigenvalues are real and inside the unit circle, the SDSUE is a stable node, i.e., the system, upon a sufficiently small perturbation, converges back without any oscillation to this state (Parker and Chua 1989; Cantarella 1993). Figure 2 shows the day-to-day dynamic behavior, when the initial point is a stochastic network loading with conventional logit and free-flow travel times. The system converges monotonically to the SDSUE fixed point, a manifestation of global monotonic convergence.

Table 4 shows the SDSUE results when route choice is HEV (heteroschedasticity effect). The formulas of probabilities of Eqs. (8) have been computed with 20 nodes ($S = 20$). This value for S has proved sufficient based on the approximation obtained in terms of sum of probabilities (equal to unity). Table 4 shows that as the magnitude of the state-dependence effect increases (the scale parameter of the currently used route decreases), the flow on the town-centre route, which is the route where travel time is higher, decreases, and the time expenditure decreases too.

Table 5 shows, in the case the scale parameter of the currently used route is 0.4324 (the estimation value), the symmetric matrix of transition route flows.

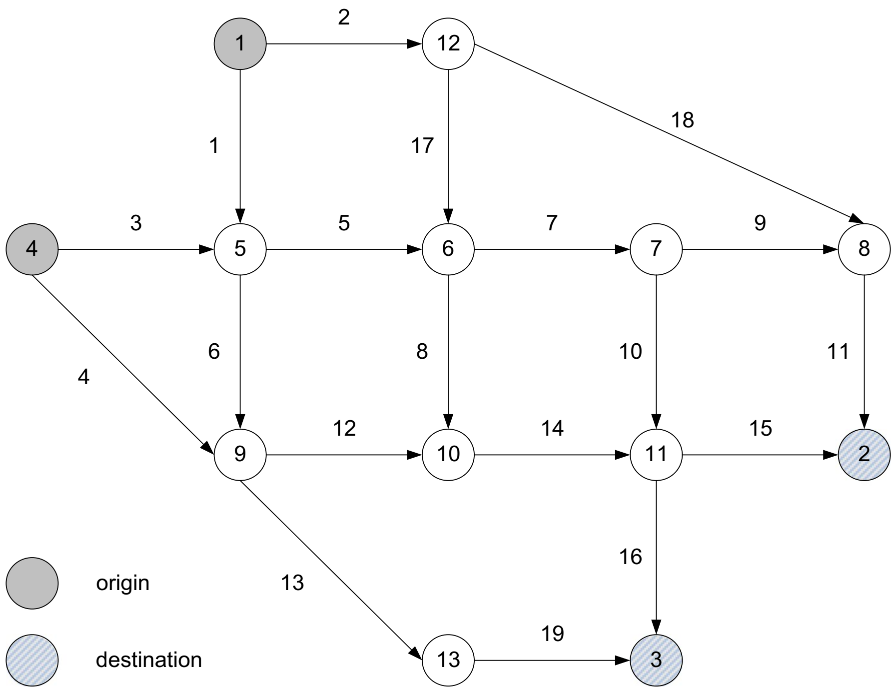

Fig. 3 Nguyen-Dupuis network

Table 6 Arc-route incidence relationship for the Nguyen-Dupuis network

OD pair	Route	Arc sequence	OD pair	Route	Arc sequence
(1,2)	1	2-18-11	(1,3)	9	2-17-8-14-16
	2	2-17-8-14-15		10	2-17-7-10-16
	3	2-17-7-10-15		11	1-6-13-19
	4	2-17-7-9-11		12	1-6-12-14-16
	5	1-6-12-14-15		13	1-5-8-14-16
	6	1-5-8-14-15		14	1-5-7-10-16
	7	1-5-7-10-15			
	8	1-5-7-9-11			
(4,2)	15	4-12-14-15	(4,3)	20	4-13-19
	16	3-6-12-14-15		21	4-12-14-16
	17	3-5-8-14-15		22	3-6-13-19
	18	3-5-7-10-15		23	3-6-12-14-16
	19	3-5-7-9-11		24	3-5-8-14-16
				25	3-5-7-10-16

Table 7 Arc characteristics of the Nguyen-Dupuis network

Arc	Free-flow travel time	Capacity	Arc	Free-flow travel time	Capacity
1	7	300	11	9	500
2	9	200	12	10	550
3	9	200	13	9	200
4	12	200	14	6	400
5	3	350	15	9	300
6	9	400 $\gamma = 1$	16	8	300
7	5	500	17	7	200
8	13	250	18	14	300
9	5	250	19	11	200
10	9	300			

Table 8 SDSUE with MNL2: transition and total route flows for OD pair (1,3)

	Newly chosen route						Total
	9	10	11	12	13	14	
Currently used route							
9	2.9	3.4	7.1	5.5	3.5	6.8	29.2
10	3.4	10.9	13.6	10.5	6.7	13.0	58.1
11	7.0	13.6	46.8	21.7	13.8	27.0	129.9
12	5.3	10.4	21.6	27.6	10.6	20.7	96.2
13	3.4	6.6	13.7	10.6	11.2	13.1	58.6
14	6.6	13.0	26.9	20.7	13.2	42.8	123.2
Total	28.6	57.9	129.7	96.6	59	123.4	≈ 495

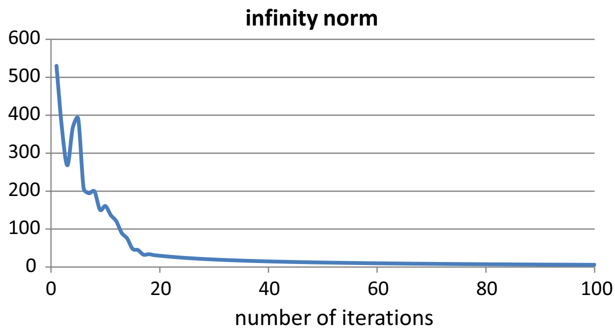

Fig. 4 SDSUE with MNL2: convergence of the algorithm

3.3 State-dependent stochastic user equilibrium: illustrative example—Nguyen-Dupuis network

In the second example, the Nguyen-Dupuis network (Nguyen and Dupuis 1984)

Table 9 SDSUE with MNL2: sensitivity of route and arc flows to inertia coefficient η

Route flows				Arc flows			
Route	$\eta=0$ (SUE)	$\eta=0.5083$	$\eta=1.5$	Arc	$\eta=0$ (SUE)	$\eta=0.5083$	$\eta=1.5$
OD pair (1,2)				1	694.2	694.1	694.5
1	251.8	260.4	278.6	2	460.8	460.9	460.5
2	15.1	14.7	13.6	3	472.0	470.9	468.8
3	29.9	29.0	26.5	4	435.5	436.6	438.7
4	73.8	69.6	62.2	5	740.7	740.6	739.4
5	47.9	47.4	45.8	6	425.5	424.5	423.9
6	30.3	29.6	27.3	7	755.9	752.6	747.1
7	60.8	59.8	57.7	8	193.8	188.4	174.3
8	150.4	149.5	148.4	9	369.0	367.5	364.9
OD pair (1,3)				10	386.8	385.1	382.2
9	30.3	29.2	25.8	11	620.8	627.9	643.5
10	59.8	58.0	53.8	12	496.6	496.5	497.9
11	129.6	129.9	131.2	13	364.4	364.6	364.6
12	95.0	96.2	99.3	14	690.5	684.9	672.1
13	60.1	58.6	55.4	15	451.7	444.6	429.0
14	120.2	123.2	129.5	16	625.6	625.4	625.4
OD pair (4,2)				17	209.0	200.4	181.9
15	133.5	133.3	132.9	18	251.8	260.4	278.6
16	46.3	45.3	43.8	19	364.4	364.6	364.6
17	29.3	28.1	25.4				
18	58.6	57.5	56.1				
19	144.8	148.5	154.3				
OD pair (4,3)							
20	173.1	174.2	175.2				
21	128.8	129.1	130.6				
22	61.6	60.5	58.2				
23	45.2	45.3	45.6				
24	28.7	28.3	26.8				
25	57.5	57.7	58.6				

is used. The network, which includes 13 nodes, 19 directed arcs and 4 OD pairs, is shown in Fig. 3.

The arc-route incidence relationship is shown in Table 6. There is a total of 25 routes. The OD demand flows are $q_{1,2}=660$, $q_{1,3}=495$, $q_{4,2}=412.5$, $q_{4,3}=495$ (as assumed in Xu et al. 2011). The following BPR volume-delay functions are used: $t_a = t_a^{\circ} \cdot \left[1 + 0.15 \cdot \left(\frac{f_a}{c_a}\right)^4\right]$, where the free-flow travel time t_a° and the capacity c_a are given, for each arc, in Table 7 (the values in the Table are from Xu et al. 2011). No arc is subject to a toll. The SDSUE is found using the MSA algorithm of Sect. 2.4.

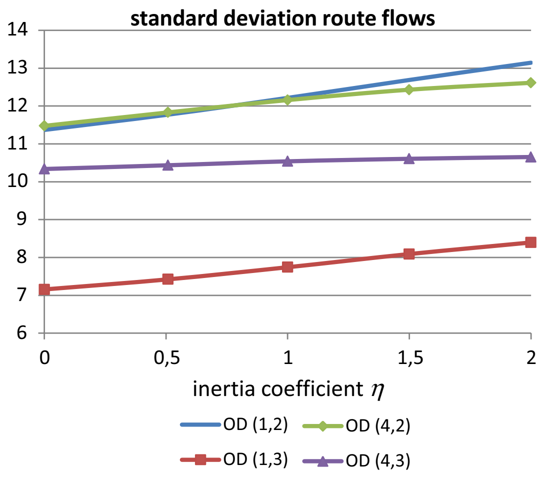

Fig. 5 SDSUE with MNL2: variation of standard deviation of route flows with inertia coefficient η

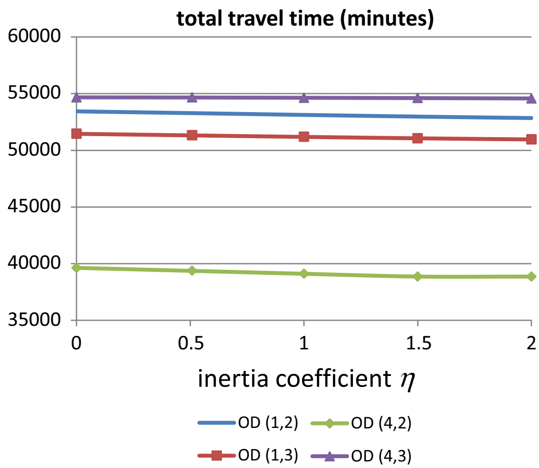

Fig. 6 SDSUE with MNL2: variation of total travel time spent on the network with inertia coefficient η

3.3.1 Inertia effect

This section presents the SDSUE results when route choice is MNL2 (inertia effect). Table 8 shows the SDSUE route flows together with the transition flows for OD pair (1–3). There can be small differences between the row sums and the column sums due to the approximate convergence of the computations. In all cases, these deviations are less than unity because the algorithm uses a convergence tolerance. Notice that in this network also, the matrix of transition flows is symmetric.

Table 10 SDSUE with HEV: transition and total route flows for OD pair (1,3)

	Newly chosen route						Total
	9	10	11	12	13	14	
Currently used route							
9	0.0	2.9	7.2	5.4	3.1	7.0	25.6
10	2.9	0.9	16.5	12.4	7.0	16.0	55.7
11	7.2	16.5	19.3	30.6	17.1	39.4	130.1
12	5.4	12.4	30.5	7.9	12.8	29.5	98.5
13	3.0	6.9	17.0	12.8	1.1	16.5	57.3
14	7.0	16.0	39.5	29.7	16.6	17.6	126.4
Total	25.5	55.6	130.0	98.8	57.7	126.0	≈ 495

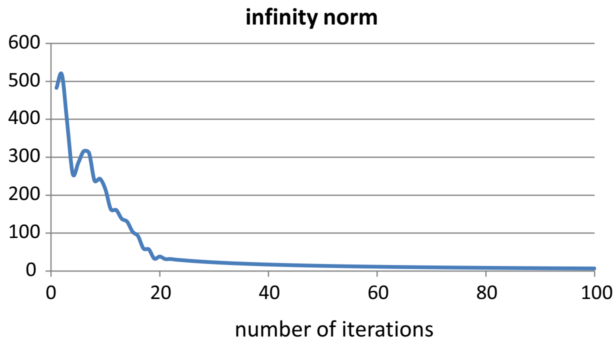

Fig. 7 SDSUE with HEV: convergence of the algorithm

Figure 4 shows the convergence of the MSA algorithm. Initially, the infinity norm changes non-monotonically as the number of iterations increases, then it decreases monotonically and with a decreasing rate. The number of iterations required for convergence is 602.

The results of the sensitivity analysis with respect to inertia are shown in Table 9. The Table shows the SDSUE route and arc flows. In the Table, three values for the inertia are considered: the case $\eta = 0$ which is the conventional SUE, i.e., no inertia effect; the base case with the estimation value $\eta = 0.5083$; and a case with a marked inertia ($\eta = 1.5$).

It is possible to detect the following pattern when the distribution of route flows for a given OD pair is considered. If the standard deviation of route flows is computed for each OD pair, the standard deviation increases with the inertia effect. This means that, as the inertia effect increases, the OD flow is distributed across routes with higher variability. This pattern is shown in Fig. 5, where the standard deviation of route flows is computed after having normalised route flows as percentage values of the corresponding OD flows. The same pattern had been found in the two-arc network example (Table 2). Another pattern which can be detected relates to the total travel time spent on the network.

Table 11 SDSUE with HEV: sensitivity of route and arc flows to scale parameter of the currently used route θ

Route flows				Arc flows			
Route	$\theta = 0.25$	$\theta = 0.4324$	$\theta = 1$ (SUE)	Arc	$\theta = 0.25$	$\theta = 0.4324$	$\theta = 1$ (SUE)
OD pair (1,2)				1	694.6	694.9	695.3
1	277.8	279.9	283.0	2	460.2	460.1	459.6
2	6.7	7.2	7.4	3	469.3	469.0	468.5
3	27.6	26.6	24.6	4	438.2	438.5	439.0
4	66.0	64.6	63.9	5	738.3	738.5	737.5
5	44.9	45.7	45.6	6	425.7	425.4	426.4
6	28.7	27.7	25.4	7	749.1	748.1	748.2
7	58.4	58.3	57.9	8	171.7	170.6	166.2
8	149.9	150.1	152.4	9	365.0	364.6	365.2
OD pair (1,3)				10	384.0	383.4	382.9
9	25.6	25.6	25.2	11	642.8	644.5	648.0
10	56.4	55.7	55.7	12	499.6	499.5	501.1
11	130.0	130.1	131.0	13	364.4	364.4	364.3
12	98.5	98.5	99.1	14	671.3	670.2	667.3
13	58.6	57.3	56.5	15	429.7	428.0	424.5
14	125.7	126.4	127.5	16	625.6	625.6	625.7
OD pair (4,2)				17	182.5	180.3	176.9
15	134.2	133.9	134.1	18	277.8	279.9	282.8
16	46.3	45.3	45.3	19	364.4	364.4	364.3
17	25.6	26.1	25.8				
18	57.2	57.2	58.3				
19	149.1	149.9	148.9				
OD pair (4,3)							
20	174.5	174.3	173.5				
21	129.5	130.3	131.4				
22	59.8	59.6	59.8				
23	46.1	45.6	45.5				
24	26.4	26.5	25.9				
25	58.6	58.7	58.9				

This quantity decreases for each OD pair as the inertia effect increases. This is shown in Fig. 6. The same pattern had been found in the two-arc network example (Table 2).

3.3.2 Heteroscedasticity effect

This section presents the SDSUE results when route choice is HEV (heteroscedasticity effect). To reach a satisfactory approximation in the probability sums,

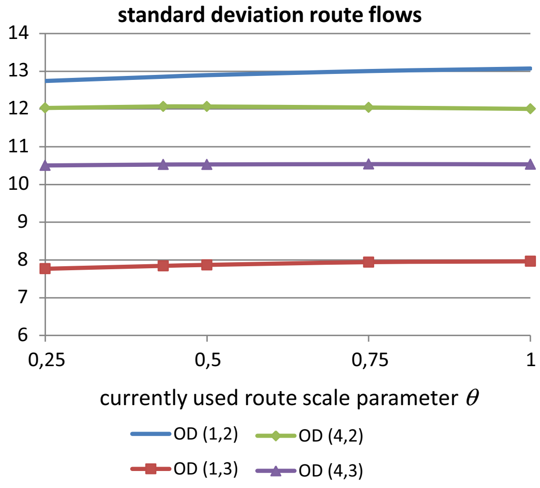

Fig. 8 SDSUE with HEV: variation of standard deviation of route flows with scale parameter of the currently used route θ

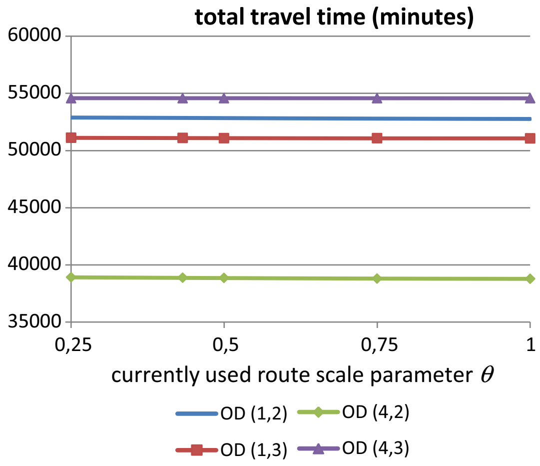

Fig. 9 SDSUE with HEV: variation of total travel time spent on the network with scale parameter of the currently used route θ

the formulas of probabilities of Eqs. (9) have been computed with 40 nodes ($S = 40$).

Table 10 shows the SDSUE route flows together with the symmetric matrix of transition flows for OD pair (1-3) for the estimation value of the scale parameter of the currently used route $\theta = 0.4324$. As in the inertia effect case, there can be small differences between the row sums and the column sums and between the symmetric elements due to the approximate convergence of the computations.

Figure 7 shows the convergence of the MSA algorithm. Similarly to the inertia effect case, the infinity norm changes initially non-monotonically as the number of iterations increases, then it decreases monotonically and with a decreasing rate. The number of iterations required for convergence, with $\gamma = 1$, is 691.

The results of the sensitivity analysis with respect $\theta = 0.25$ to the scale parameter of the currently used route are shown in Table 11. The Table shows the SDSUE route and arc flows. In the Table, three values for the heteroscedasticity are considered: a case with $\theta=0.25$; the base case with the estimation value $\theta=0.4324$; and the case $\theta=1$ which is the conventional SUE, i.e. no heteroscedasticity effect.

Figure 8 shows the standard deviation of route flows which is computed after having normalised route flows as percentage values of the corresponding OD flows. Differently from the inertia effect case, it is not possible to detect a monotonic pattern with the parameter controlling the magnitude of the heteroscedasticity effect. Figure 9 shows that the total travel time spent on the network is practically constant with respect to the heteroscedasticity effect.

4 Conclusion

In this paper, a state-dependent model is considered for route choice. Experimental evidence based on stated-preference data confirms the presence of inertia and heteroscedasticity effects in route choice. The state-dependent stochastic user equilibrium, in short SDSUE, is formulated as a fixed-point problem in the route flows. SDSUE is equivalent to the steady states of a deterministic day-to-day assignment process. Conventional SUE is a special case of SDSUE. It is proved that the solution of SDSUE exists under conditions usually satisfied in practice. Uniqueness of the solution is an open problem.

Further research is needed on alternative SDSUE formulations that use other choice models able to capture the heteroscedasticity effect in lieu of the heteroscedastic extreme value model, such as the probit model.

A comparison has been performed between SDSUE and conventional SUE for a simple two-arc network and for the Nguyen-Dupuis network. Results indicate that SDSUE produces a compatible traffic flow pattern compared to the conventional SUE, with magnitudes of the differences in flows dependent on the parameter controlling the state-dependent effect. The numerical illustration has considered the route choice model as a function of travel time only, but the formulation proposed is more general because it accommodates a full utility function with the possibility to include additional explanatory variables, in particular those accounting for route overlappings. For convenience, the applications have considered the inertia and the heteroscedasticity effect separately, but it would be reasonable to investigate also the simultaneous presence of the two effects in route choice and the attendant SDSUE results.

The SDSUE is a condition where, from a macro-level perspective, the observed route flows do not change, whereas, from a micro-level perspective, each user updates her currently used route to her newly chosen route. Thus, there is an inherent day-to-day dynamics in SDSUE, as it also occurs in the reference-dependent SUE (Delle Site et al. 2013). The numerical results are indicative that the magnitudes of the flow shifts are subject to a symmetry condition, whereby the number of shifters from route i to route j equals the number of shifters from route j to route i.

The insights presented are based on a number of assumptions that are common to basic equilibrium problems: static assignment, fixed demand and single user class in terms of marginal disutility of travel time. Extension of the framework here to more general problems will require additional research.

Appendix

Notation

The following notation (in alphabetical order) is used. The symbols are, wherever possible, self-explaining because based on the initials of the associated quantity. Quantities (flows and travel times) that are related to arcs are lowercase, quantities that are related to routes are uppercase.

a	Arc
A	Arc set
C_a	Capacity of arc
d^w	Demand of OD pair w
f_a	Flow of arc a
F_r^w	Flow of route r of OD pair w
$\mathbf{F}$	Vector of route flows
k	Iteration counter
$I_{r\|j}^w$	Indicator function related to the identity of route r at day n and route j the day before for OD pair w
j	Route
M_r^w	Monetary cost of route r of OD pair w
n	Day
$P_{r\|j}^{w(n)}$	Conditional probability of choosing route r at day n having chosen route j the day before for OD pair w
r	Route
R^w	Route set of OD pair w
$t_a^{(n)}$	Travel time of arc a at day n
t_a^0	Free-flow travel time of arc a
$T_r^{w(n)}$	Travel time of route r of OD pair w at day n
$V_{r\|j}^{w(n)}$	Conditional systematic utility of route r of OD pair w at day n having chosen route j the day before
w	OD pair
W	OD pair set
β_T	Estimation coefficient of route travel time
β_M	Estimation coefficient of route monetary cost

γ Algorithm convergence tolerance
$\delta^w_{a,r}$ Entry of the arc-route incidence matrix
η Estimation coefficient of the inertia term
θ Scale parameter of the route chosen the day before
λ Eigenvalue of the Jacobian, computed in the fixed point, of the transition functions
$\boldsymbol{\Psi}$ Route-flow based fixed-point map

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