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The stable set polytope of claw-free graphs with stability number at least four. I. Fuzzy antihat graphs are $\mathcal{W}$-perfect

A. Galluccio, C. Gentile, P. Ventura

Istituto di Analisi dei Sistemi ed Informatica "A. Ruberti", Consiglio Nazionale delle Ricerche (IASI-CNR), viale Manzoni 30, 00185 Roma, Italy

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ABSTRACT

Fuzzy antihat graphs are graphs obtained as 2-clique-bond compositions of fuzzy line graphs with three different types of three-cliqued graphs. By the decomposition theorem of Chudnovsky and Seymour [2], fuzzy antihat graphs form a large subclass of claw-free, not quasi-line graphs with stability number at least four and with no 1-joins.

A graph is $\mathcal{W}$-perfect if its stable set polytope is described by: nonnegativity, rank, and lifted 5-wheel inequalities. By exploiting the polyhedral properties of the 2-clique-bond composition, we prove that fuzzy antihat graphs are $\mathcal{W}$-perfect and we move a crucial step towards the solution of the longstanding open question of finding an explicit linear description of the stable set polytope of claw-free graphs.

1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of maximum weight. Let $\alpha(G, w)$ denote the maximum weight of a stable set of G; we refer

E-mail addresses: galluccio@iasi.cnr.it (A. Galluccio), gentile@iasi.cnr.it (C. Gentile), ventura@iasi.cnr.it (P. Ventura).

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to $\alpha(G) = \alpha(G, \mathbb{1})$ ($\mathbb{1}$ being the vector of all ones) as the *stability number* of G. The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G. A linear system $Ax \leqslant b$ is said to be *defining* for $STAB(G)$ if $STAB(G) = \{x \in \mathbb{R}^V\colon\ Ax \leqslant b\}$. Since the stable set problem is NP-hard, it is unlikely to find a defining linear system of $STAB(G)$ for general graphs. Nevertheless the study of the stable set polytope of *claw-free graphs*, i.e., graphs such that the neighbourhood of each node has stability number at most two, attracts the attention of the scientific community since early seventies when the pioneering work of Edmonds on the matching polytope [6] was translated for the stable set polytope of line graphs (a line graph $L(G)$ of a graph G is obtained by considering the edges of G as nodes of $L(G)$ and two nodes of $L(G)$ are adjacent if and only if the corresponding edges of G have a common endnode). At that time it seemed natural to look for a linear description of the stable set polytope for classes of graphs that properly contain line graphs such as claw-free graphs or *quasi-line graphs*, i.e., graphs such that the neighbourhood of each node can be partitioned into two cliques. Notice that the class of claw-free graphs properly contains the class of quasi-line graphs. A number of conjectures were posed on the inequalities that are facet defining for $STAB(G)$ when G is claw-free [14,29], but an explicit linear description of $STAB(G)$ is not known yet.

The study of the stable set polytope of claw-free graphs revived in late 80's after Grötschel, Lovász and Schrijver proved the equivalence of the separation and the optimization problems over polyhedra [15]. They also noted that claw-free graphs constitute an anomaly in this respect [16]. Indeed, a defining linear system for the stable set polytope is known for almost all classes of graphs for which a polynomial time algorithm to solve the weighted stable set problem is known. This is true for bipartite graphs, line graphs [6], series-parallel graphs [19], and perfect graphs. On the contrary, for claw-free graphs, a polynomial time algorithm to solve the weighted stable set problem is known since 1980 [20,21] but no linear description of $STAB(G)$ is at hand (see also [27]).

A breakthrough to start to understand the structure of claw-free graphs came out with the decomposition theorem of Chudnovsky and Seymour [2,3]. This theorem states that the class of claw-free graphs is the union of different classes of graphs that have very specific features. In particular, Chudnovsky and Seymour proved that every claw-free graph that does not admit a 1-join satisfies one of the following conditions: it has stability number at most 3, or it is a fuzzy circular interval graph, or it can be obtained by "properly composing" five types of graphs, called *strips*: fuzzy linear interval strips (also called fuzzy Z_1-strips), fuzzy Z_2-strips, fuzzy Z_3-strips, fuzzy Z_4-strips, and fuzzy Z_5-strips.

We call *fuzzy line* the graphs that are composition of fuzzy linear interval strips and denote them by $\mathcal{Q}^\ell$. Then we denote by $\mathcal{Q}^c$ the set of quasi-line graphs that are fuzzy circular interval and by $\mathcal{C}^s$ the class of *striped graphs*, claw-free graphs obtained by composing fuzzy Z_i-strips, $i = 1, 2, 3, 4, 5$. Thus the Chudnovsky–Seymour decomposition states that every claw-free graph with stability number at least 4 and without 1-joins belongs to $\mathcal{Q}^c$ or to $\mathcal{C}^s$. This result partially explains why it was so hard to deal with

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$STAB(G)$ for all claw-free graphs simultaneously and suggests that, in order to find a linear description of $STAB(G)$ for claw-free graphs, it is convenient to study the facet defining inequalities for each of the subclasses identified by the decomposition separately.

A linear inequality $\sum_{j\in V}\pi_j x_j \leqslant \pi_0$ is said to be a *rank inequality* for $STAB(G)$ if there exists a subset $U \subseteq V$ such that $\pi_i = 1$ for each $i \in U$, $\pi_i = 0$ for each $i \in V \setminus U$ and $\pi_0 = \alpha(G[U])$ where $G[U]$ is the subgraph of G induced by U.

A defining linear system for $STAB(G)$, when $G \in \mathcal{Q}^\ell$, was given by Chudnovsky and Seymour [1] and consists of nonnegativity and rank inequalities. In 2008 Eisenbrand et al. [7] provided a linear description of $STAB(G)$ when $G \in \mathcal{Q}^c$. Their result shows that rank inequalities are not sufficient to describe $STAB(G)$ as soon as G is not fuzzy line. Indeed, a special class of inequalities with two different nonzero coefficients (*clique-family inequalities* [24]) has to be added to rank inequalities in order to describe $STAB(G)$ when G is quasi-line.

In this paper we investigate the polyhedral properties of the following strips: fuzzy Z_2-strips, fuzzy Z_3-strips, and fuzzy Z_4-strips, and their composition with fuzzy line graphs. Since all these strips share the common feature of being *three-cliqued*, namely their node set is partitionable into three cliques, we refer to graphs that are "composition" of fuzzy line graphs with fuzzy Z_i-strips, $i = 2, 3, 4$, as *fuzzy antihat graphs*.

We consider a family $\mathcal{W}$ of inequalities consisting of: nonnegativity, rank, and lifted 5-wheel inequalities (for formal definitions of these inequalities see the end of Section 2) and we say that a graph is $\mathcal{W}$-perfect if its stable set polytope is described only by inequalities in $\mathcal{W}$. Finally we prove that fuzzy antihat graphs are $\mathcal{W}$-perfect.

When a Z_5-strip is induced in a claw-free graph G, the inequalities in $\mathcal{W}$ are not sufficient to describe $STAB(G)$ and new facet defining inequalities for $STAB(G)$ come into play [9]. This case will be investigated in a companion paper [12] where it will be provided a complete linear description of the stable set polytope of striped graphs.

In Sections 2 and 3 we recall the basic definitions and some polyhedral results. In Section 4 we give some properties of the stable set polytope of claw-free graphs that contain homogeneous pairs of cliques. In Section 5 we provide the minimal linear description of the stable set polytope of an important subclass of three-cliqued graphs. In Sections 6, 7 and 8, we provide the minimal linear description of the stable set polytope of fuzzy closed Z_i-strips, $i = 2, 3, 4$, respectively. Finally, in Section 9, we prove that fuzzy antihat graphs are $\mathcal{W}$-perfect, i.e., the minimal linear description of the stable set polytope of fuzzy antihat graphs consists of: nonnegativity, rank, and lifted 5-wheel inequalities.

2. Basic definitions

Let $G = (V, E)$ be a simple, connected graph with node set $V(G)$ and edge set $E(G)$. Two nodes u and v are adjacent (nonadjacent) if $uv \in E(G)$ ($uv \notin E(G)$). The *neighbourhood* of v, written $N_G(v)$ or $N(v)$, is the set of nodes of $V(G)$ that are adjacent to v and the closed neighbourhood $N[v]$ is the set $N(v) \cup \{v\}$. Two adjacent nodes u and v are *twins* if $N[u] = N[v]$. The neighbourhood of a set $S \subseteq V$, denoted by $N(S)$, is the set

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of nodes of $V \setminus S$ that are adjacent to at least one node in S. The *closed neighbourhood* of a set $A \subset V$ is $N[A] = \bigcup_{v \in A} N[v] = A \cup N(A)$.

We also denote by $G \setminus A$ the subgraph of G induced by $V \setminus A$ where $A \subseteq V$ and by $G + e$ ($G - e$, G/e) the subgraph of G obtained by adding (deleting, contracting, respectively) the edge e. Given two subsets of nodes $U, Z \subset V$, we say that U is *Z-complete (Z-anticomplete)* if every node $u \in U$ is adjacent (nonadjacent) to every node $z \in Z$. Obviously, U is Z-complete (Z-anticomplete) if and only if Z is U-complete (U-anticomplete).

A *k-path* is a chordless path with k nodes and it is denoted by P_k. A *k-hole* $C_k = (v_1, v_2, \dots, v_k)$ is a chordless cycle of length k; a *k-antihole* $\overline{C}_k$ is the complement of a k-hole. A *k-antiwheel* $W = (h : \overline{C}_k)$ is a graph consisting of a k-antihole $\overline{C}_k$ and a node h (*hub* of W) adjacent to every node of $\overline{C}_k$. If $k = 3$, the 3-antiwheel is called *claw* and denoted by $(y : w_1, w_2, w_3)$, where y is the centre of the claw. If $k = 5$, then $\overline{C}_5$ is isomorphic to C_5 and we refer to W as a *5-wheel*. A node is *simplicial* if its neighbourhood induces a clique, i.e., a complete subgraph. An edge ab is *simplicial* if $N(a) \setminus \{b\}$ and $N(b) \setminus \{a\}$ are both cliques.

A clique-cutset of G is a clique whose removal disconnects G. A graph $G = (V, E)$ admits a 1-join if V can be partitioned into two sets V_1 and V_2 and, for $i = 1, 2$, there are subsets A_i of V_i such that: $A_1 \cup A_2$ is a clique, $V_1 \setminus A_1$ and $V_2 \setminus A_2$ are nonempty, and the only edges between V_1 and V_2 are those between A_1 and A_2. Clearly, if G admits a 1-join then $A_1 \cup A_2$ is a clique-cutset of G.

Suppose that V_0, V_1, and V_2 are a partition of V and, for $i = 1, 2$, there are subsets A_i, B_i of V_i satisfying the following:

- $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ are cliques, and no node of V_0 is adjacent to $V_i \setminus (A_i \cup B_i)$ for $i = 1, 2$,
- for $i = 1, 2$, $A_i \cap B_i = \emptyset$ and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty,
- for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is not adjacent to v_2, or $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$.

The triple (V_0, V_1, V_2) is called a *generalized* 2-*join* in [3].

A *strip* (H, a_0, b_0) is a claw-free graph H with two nonadjacent simplicial nodes $a_0, b_0 \in V(H)$.

A *closed strip* $(H, a_0 b_0)$ is the graph $H + a_0 b_0$ where (H, a_0, b_0) is a strip. A *contracted closed strip* $H/a_0 b_0$ is obtained from $(H, a_0 b_0)$ by contracting the edge $a_0 b_0$ into the node z_0.

Given two strips (G_i, a_0^i, b_0^i), for $i = 1, 2$, let A_i, B_i denote the set of nodes of $G_i \setminus \{a_0^i, b_0^i\}$ adjacent in G_i to a_0^i, b_0^i respectively. The strip composition defined by Chudnovsky and Seymour in [1] produces a new graph G by deleting the four nodes a_0^i, b_0^i, for $i = 1, 2$, and by completely joining the nodes of A_1 with those of A_2 and the nodes of B_1 with those in B_2. Clearly, this new graph G admits the generalized 2-join

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$(V_0, V(G_1) \setminus \{a_0^1, b_0^1\}, V(G_2) \setminus \{a_0^2, b_0^2\})$ where $V_0 = (A_1 \cap B_1) \cup (A_2 \cap B_2)$. Note that, by claw-freeness, $N[A_i \cap B_i] = N[a_0^i] \cup N[b_0^i]$ for $i = 1, 2$.

Observation 1. *The closed strip (G, a_0b_0) is obtained as a strip composition of (H, v_1^1, v_2^1) and the 4-path (v_1^2, a_0, b_0, v_2^2). The contracted closed strip G/a_0b_0 is obtained as a strip composition of (H, v_1^1, v_2^1) and the 3-path (v_1^2, z_0, v_2^2).*

In [11] we introduced the following composition.

Definition 2. Let G_1 and G_2 be two disjoint graphs. Let (a_0^i, b_0^i) be an ordered pair of nodes such that $a_0^i b_0^i$ is a simplicial edge of G_i and let $A_i = N(a_0^i) \setminus \{b_0^i\}$ and $B_i = N(b_0^i) \setminus \{a_0^i\}$, $i = 1, 2$.

The 2-*clique-bond composition* of G_1 and G_2 along (a_0^1, b_0^1) and (a_0^2, b_0^2) is the graph G obtained by deleting the nodes a_0^i and b_0^i, for $i = 1, 2$, and joining every node in A_1 with every node in A_2 and every node of B_1 with every node of B_2.

Under the restriction that $N[A_i \cap B_i] = N[a_0^i] \cup N[b_0^i]$ for $i = 1, 2$, the 2-clique-bond produces graphs that admit generalized 2-joins where the role of V_0 is played by the set $(A_1 \cap B_1) \cup (A_2 \cap B_2)$. In the following we say that an edge $a_0^i b_0^i$ is *super simplicial* if it is simplicial and moreover it satisfies $N[A_i \cap B_i] = N[a_0^i] \cup N[b_0^i]$. It is not difficult to check that the 2-clique-bond composition preserves claw-freeness when performed along ordered pairs corresponding to super simplicial edges. Thus, the 2-clique-bond composition applied on claw-free graphs along such ordered pairs produces the same graphs as the strip composition: the only difference is that the former applies on closed strips while the latter applies on strips. In [11] we provide examples of graphs obtained by 2-clique-bond composition that do not admit a generalized 2-join.

For basic results on the stable set polytope we refer to textbooks such as [23,16,27]. In particular, we will use the following concepts: n vectors $x_1, x_2, \ldots, x_n$ are affinely independent if and only if the vectors $(1, x_1), (1, x_2), \ldots, (1, x_n)$ are linearly independent. A polyhedron contained in $\mathbb{R}^n$ has dimension p if and only if it contains $p + 1$ affinely independent vectors. Note that $STAB(G)$ has dimension $n = |V(G)|$ as the n vectors of the canonical base of $\mathbb{R}^n$ plus the zero vector constitute $n + 1$ affinely independent vectors in $STAB(G)$.

Given a vector $\beta \in \mathbb{R}^{|V|}$ and a subset $U \subseteq V$, define $\beta_U \in \mathbb{R}^{|V|}$ as the subvector of β restricted to the elements of U and let $\beta(U) = \sum_{i \in U} \beta_i$. A linear inequality $\sum_{j \in V(G)} \beta_j x_j \leqslant \beta_0$ is *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. For short, we also denote a linear inequality $\beta^T x \leqslant \beta_0$ as (β, β_0). A valid inequality for $STAB(G)$ *defines* a facet of $STAB(G)$ if and only if it is satisfied as an equality by $|V(G)|$ affinely independent incidence vectors of stable sets of G. A stable set S is *tight* for (β, β_0) if $\beta(S) = \beta_0$ and S *violates* (β, β_0) if $\beta(S) > \beta_0$. Given a valid inequality (β, β_0) of $STAB(G)$, its *supporting graph* G_β is the subgraph of G induced by the nodes with nonzero coefficients in (β, β_0). The *nonnegativity inequalities* $x_v \geqslant 0$, $v \in V(G)$, are known to be facet defining for $STAB(G)$ and we refer to them as trivial inequalities. Basic properties of the stable set polytope (see [8,25,22]) establish that the nonnegativity inequalities define the only facets of $STAB(G)$ containing the zero vector and that any other facet defining inequality (β, β_0) has $\beta \geqslant 0$, and $\beta_0 > 0$. This implies that for each nontrivial facet defining inequality $\beta^T x \leqslant \beta_0$, there exist n linearly independent vectors x_i satisfying $\beta^T x_i = \beta_0$ for $i = 1, \dots, n$. As a consequence there exists at least one tight stable set for (β, β_0) containing v, for each node $v \in V(G)$. Moreover, it is not difficult to see that if (β, β_0) is facet defining for $STAB(G)$ with supporting graph G_β, then (β, β_0) is also facet defining for $STAB(G_\beta)$.

A *clique inequality* (*5-hole inequality*) is a rank inequality where the subgraph $G[U]$ is a clique (a 5-hole, respectively). Given a 5-wheel $W = (h : v_1, v_2, v_3, v_4, v_5)$, the inequality $\sum_{i=1}^{5} x_{v_i} + 2x_h \leqslant 2$ is called 5-*wheel inequality* and it is facet defining for $STAB(W)$.

For the sake of completeness, we recall the definition of the *sequential lifting* procedure defined in [25] that will be often mentioned in the following sections. Let $\mathscr{S}(G)$ denote the family of the stable sets of $G = (V, E)$. If $\sum_{j \in V \setminus \{v\}} \beta_j x_j \leqslant \beta_0$ is a facet defining inequality of $STAB(G \setminus \{v\})$, then the inequality

$$\sum_{j \in V \setminus \{v\}} \beta_j x_j + \beta_v x_v \leqslant \beta_0 \quad \text{with} \quad \beta_v = \beta_0 - \max_{S \in \mathscr{S}(G \setminus N[v])} \beta(S) \tag{1}$$

is facet defining for $STAB(G)$. This inequality is called *sequential lifting of* $(\beta_{V \setminus \{v\}}, \beta_0)$ and β_v is called the *lifting coefficient* of v. Starting from a facet defining inequality in a class $\mathcal{C}$ for a lower dimensional polytope, say $STAB(G')$ with G' being an induced subgraph of G, the lifting procedure is usually applied sequentially: nodes in the set $V(G) \setminus V(G')$ are lifted one after the other and a separate optimization problem has to be solved to determine each lifting coefficient. The resulting inequality depends on the order in which the variables are lifted, but, in all cases, it is a facet defining inequality for the higher dimensional polytope $STAB(G)$. Inequalities obtained in this way are called *lifted* $\mathcal{C}$ *inequalities*.

3. Preliminary results

In this section we present some general results on the stable set polytope of a graph G. The first lemma will be often used in the remainder of the paper. Its proof follows from the full dimensionality of $STAB(G)$ [27].

Lemma 3. *Let (β, β_0) be a facet defining inequality of $STAB(G)$. Then, for any valid inequality (γ, γ_0) that is not a positive scalar multiple of (β, β_0), there exists a stable set S such that $\beta(S) = \beta_0$ and $\gamma(S) < \gamma_0$.*

Moreover, it is not difficult to observe the following:

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Proposition 4. *Let (β, β_0) be a nontrivial facet defining inequality of STAB(G). If $u, v \in V(G)$ with $N[u] \subseteq N[v]$, then $\beta_u \leqslant \beta_v$. In particular, if u and v are twins, then $\beta_u = \beta_v$.*

Proof. As (β, β_0) is facet defining there exists a tight stable set S containing v. Since $S \setminus \{v\} \cup \{u\}$ is a stable set, $\beta(S \setminus \{v\} \cup \{u\}) = \beta(S) - \beta_v + \beta_u \leqslant \beta_0 = \beta(S)$ and the claim follows. □

Next, we present a few results on the stable set polytope of graphs with stability number two.

Definition 5. Let G be a graph and H an induced subgraph of G with $\alpha(H) = 2$. For any set $K \subseteq V(H)$, let $\tilde{N}_H(K)$ denote the set of all nodes $v \in V(H) \setminus K$ for which $N_H(v) \supseteq K$. If K induces a clique or $K = \emptyset$ then the inequality

$$2x(K) + x(\tilde{N}_H(K)) \leqslant 2 \tag{2}$$

is the *clique-neighbourhood inequality* generated by K.

Notice that, for any maximal clique K, $\tilde{N}_H(K) = \emptyset$ and the associated clique-neighbourhood inequality becomes a clique inequality. Rank inequalities with right hand side two are also particular clique-neighbourhood inequalities where $\tilde{N}_H(K) = V(H)$ and $K = \emptyset$. Finally, lifted 5-wheel inequalities are clique-neighbourhood inequalities where the nodes in K are copies of the hub of a 5-wheel. In the following we simply write $\tilde{N}(K)$ when $H = G$ and thus $\alpha(G) = 2$.

The next result is attributed to W. Cook in [28].

Theorem 6. *Let G be a graph with $\alpha(G) = 2$. Then STAB(G) is described by:*

- *nonnegativity inequalities,*
- *clique-neighbourhood inequalities.*

Moreover, a clique-neighbourhood inequality is facet defining for STAB(G) if and only if no connected component of $\overline{G}[\tilde{N}(K)]$ is bipartite.

As an easy consequence of Theorem 6 we have that:

Corollary 7. *Let $G = (V, E)$ be a graph. Let (β, β_0) be a clique-neighbourhood inequality generated by $K \subset V$ that is not a clique inequality. If $\tilde{N}_{G_\beta}(K)$ is partitionable into two cliques then (β, β_0) is not facet defining.*

Corollary 8. *Let $G = (V, E)$ be a claw-free graph with a super simplicial edge a_0b_0 and let z_0 be the node obtained by the contraction of a_0b_0. Let (β, β_0) be a clique-neighbourhood inequality generated by K that is facet defining for STAB(G) (STAB(G/a_0b_0)). Then no node in $N[a_0] \cup N[b_0]$ ($N[z_0]$, respectively) belongs to K.*

Proof. Suppose conversely that there exists a node $h \in K \cap (N[a_0] \cup N[b_0])$. By Corollary 7, h is different from a_0 and b_0 and it does not belong to $N(a_0) \cap N(b_0)$ because all these nodes have a neighbourhood that is partitioned into two cliques. The same holds for the node z_0.

Let A and B denote the sets $N(a_0)$ and $N(b_0)$, respectively, and assume, without loss of generality, that $h \in K \cap (A \setminus B)$. Since G is claw-free and $\alpha(G_\beta) = 2$, it follows, by Theorem 6, that each connected component of $\overline{G_\beta}[\tilde{N}(K)]$ contains an odd hole C of length at least 5. Let T indicate the nodes of $C \setminus (A \cup \{a_0\})$. Clearly T is a clique since otherwise $(h : a_0, u, v)$ would be a claw for each pair of nonadjacent nodes $u, v \in T$. Thus, the odd hole C partitions into the cliques T and $C \cap (A \cup \{a_0\})$, a contradiction. □

We now show a property of clique-neighbourhood inequalities that will be used later:

Lemma 9. *Let G be a claw-free graph that does not contain a $(2t+1)$-antiwheel with $t \geqslant 3$. Let (β, β_0) be a clique-neighbourhood inequality generated by K that is facet defining for $STAB(G)$ and is different from a clique inequality. If $K \neq \emptyset$ and $\overline{G_\beta}[\tilde{N}(K)]$ is connected then (β, β_0) is a lifted 5-wheel inequality.*

Proof. Since the inequality (β, β_0) is facet defining, no connected component of $\overline{G_\beta}[\tilde{N}(K)]$ is bipartite and, in particular, $\overline{G_\beta}[\tilde{N}(K)]$ does not contain isolated nodes. Since G is claw-free, $\overline{G_\beta}[\tilde{N}(K)]$ is triangle-free, and so $\overline{G_\beta}[\tilde{N}(K)]$ contains a $(2t+1)$-hole with $t \geqslant 2$. Then, by hypothesis, $t = 2$, i.e., there exists a 5-hole C contained in $\overline{G_\beta}[\tilde{N}(K)]$. As 5-holes are self-complementary, C induces a 5-hole also in G_β.

Now we prove that (β, β_0) can be obtained from the (lifted) 5-wheel inequality induced by $C \cup K$ by sequentially lifting all the nodes of $V(G_\beta) \setminus (C \cup K)$. This amounts to show that all nodes in $V(G_\beta) \setminus (C \cup K)$ can be lifted with coefficient one.

Now, let $W = \{w_1, w_2, \dots, w_s\}$ be the longest sequence of nodes of $V(G_\beta) \setminus (C \cup K)$ that can be lifted with coefficient 1 starting from the (lifted) 5-wheel inequality defined by $C \cup K$. Suppose that $s < |V(G_\beta) \setminus (C \cup K)|$, i.e., $V(G_\beta) \setminus (C \cup K \cup W)$ is nonempty.

Since $V(G_\beta) \setminus (C \cup W) \subseteq \tilde{N}_{G_\beta}(K)$, each node $v \in V(G_\beta) \setminus (C \cup W)$ is adjacent to all nodes with coefficient 2 and, moreover, $\alpha(G_\beta \setminus N[v]) \leqslant 1$. Thus, according to equation (1) and by the maximality of W, the lifting coefficient of v is 2 for each $v \in V(G_\beta) \setminus (C \cup W)$. It follows that $V(G_\beta) \setminus (C \cup K \cup W)$ is $(C \cup K \cup W)$-complete, i.e., $\overline{G_\beta}[\tilde{N}(K)]$ is not connected, a contradiction. Hence, $s = |V(H) \setminus (C \cup K)|$ and the lemma follows. □

In [1] Chudnovsky and Seymour give the following definition of fuzzy linear interval graphs:

Definition 10. A graph $G = (V, E)$ is said to be fuzzy linear interval if:

1. there is a map ϕ from V to a line L, and
2. there is a family $\mathcal{I}$ of intervals of L (none including another) such that no point of L is an end of more than one interval, so that

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3. for $u, v \in V$, if $uv \in E$ then $\{\phi(u), \phi(v)\}$ is a subset of one interval of $\mathcal{I}$, and if $uv \notin E$ then $\phi(u)$ and $\phi(v)$ are both ends of any interval of $\mathcal{I}$ containing both of them (and in particular, if $\phi(u) = \phi(v)$ then u and v are adjacent).

Moreover, if $[a, b]$ is an interval of $\mathcal{I}$ such that $\phi^{-1}(a)$ and $\phi^{-1}(b)$ are both nonempty subsets of V and at least one of the sets $\phi^{-1}(a)$ and $\phi^{-1}(b)$ has more than one member, then the interval $[a, b]$ is said to be *fuzzy*.

A fuzzy linear interval graph with two nonadjacent simplicial nodes a_0 and b_0 is a fuzzy linear interval strip (G, a_0, b_0) and graphs that are strip compositions of fuzzy linear interval strips are called *fuzzy line graphs*.

Chudnovsky and Seymour provided a linear description of the stable set polytope of fuzzy line graphs (see [29] for an alternative proof):

Theorem 11. (Chudnovsky and Seymour [1]) *If G is a fuzzy line graph, then $STAB(G)$ is described by nonnegativity and rank inequalities.*

Next proposition concerns the coefficients of the endnodes of simplicial edges in facet defining inequalities of $STAB(G)$.

Proposition 12. (Galluccio et al. [10]) *Let G be a graph and let (β, β_0) be a nontrivial facet defining inequality of $STAB(G)$. If uv is a simplicial edge of G_β, then $\beta_u = \beta_v$.*

Finally, we consider two different polyhedral compositions. The first one was introduced by Chvátal [4] and concerns the stable set polytope of graphs composed via clique-cutsets. More precisely:

Theorem 13. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. If $G_1 \cap G_2$ is a complete graph, then the defining linear system of $STAB(G_1 \cup G_2)$ is given by the union of the defining linear systems of $STAB(G_1)$ and $STAB(G_2)$.*

As a corollary of the previous result we have the following:

Corollary 14. *Let G be a graph and let (β, β_0) be a facet defining inequality for $STAB(G)$. Then*

i) *G_β does not contain a clique-cutset;*
ii) *if $u \in V(G_\beta)$ is a simplicial node in G_β, then G_β is a maximal clique of G.*

Proof. As observed in Section 2, (β, β_0) is also a facet defining inequality for $STAB(G_\beta)$.

i) Suppose by contradiction that G_β contains a clique Q such that $G_\beta \setminus Q$ is disconnected. Let H_1 and H_2 be the two (possibly disconnected) nonempty subgraphs of G_β

obtained by deleting Q. For $i = 1, 2$, let G_i be the graph induced by $V(H_i) \cup Q$. Then $G_\beta = G_1 \cup G_2$, $G_1 \cap G_2$ is the complete graph induced by Q, and, by Theorem 13, G_β is a subgraph of either G_1 or G_2, a contradiction.

ii) Suppose there exists a node $v \in V(G_\beta) \setminus N_{G_\beta}(u)$ different from u. Then $N_{G_\beta}(u)$ is a clique-cutset in G_β, contradicting i). □

From ii) of Corollary 14 and Proposition 12, it follows:

Corollary 15. *Let G be a graph with a simplicial edge uv. Then the only facet defining inequalities for $STAB(G)$ with different coefficients on u and v are the clique inequalities $x(N[u] \setminus \{v\}) \leqslant 1$ and $x(N[v] \setminus \{u\}) \leqslant 1$.*

It is then convenient to give names to the facet defining inequalities indicated in Proposition 12 and to some particular facet defining inequalities for $STAB(G/a_0b_0)$:

Definition 16. Let G be a graph with a simplicial edge a_0b_0 and let $A = N(a_0) \setminus \{b_0\}$ and $B = N(b_0) \setminus \{a_0\}$. We call *even* a facet defining inequality of $STAB(G)$ with nonzero coefficients on a_0 and b_0 that is different from $x_{a_0} + x_{b_0} \leqslant 1$ and we call *odd* a facet defining inequality of $STAB(G/a_0b_0)$ with nonzero coefficient on z_0 that is different from $x(A \cup \{z_0\}) \leqslant 1$, $x(B \cup \{z_0\}) \leqslant 1$ and $x_{z_0} \geqslant 0$.

The second composition we consider is the 2-clique-bond composition described in Definition 2. In order to present the major polyhedral features of this composition we need the following definition describing the facet defining inequalities obtained as composition of inequalities of smaller polytopes:

Definition 17. Let G be the 2-clique-bond composition of G_1 and G_2 along (a_0^1, b_0^1) and (a_0^2, b_0^2). Let z_0^i be the node resulting from the contraction of $a_0^i b_0^i$, $i = 1, 2$.

Let $\beta^i x \leqslant \beta_0^i$ be an even inequality of $STAB(G_i)$ and let $\beta^j x \leqslant \beta_0^j$ be an odd inequality of $STAB(G_j/a_0^j b_0^j)$ such that $\beta^i_{a_0^i} = \beta^i_{b_0^i} = \beta^j_{z_0^j} = 1$, for $i, j \in \{1, 2\}$ and $i \neq j$.

An inequality of the form

$$\sum_{v \in V(G_i \setminus \{a_0^i, b_0^i\})} \beta_v^i x_v + \sum_{v \in V((G_j/a_0^j b_0^j) \setminus \{z_0^j\})} \beta_v^j x_v \leqslant \beta_0^i + \beta_0^j - 1 \tag{3}$$

is said to be an *even-odd combination* of (β^i, β_0^i) and (β^j, β_0^j) (see Fig. 1 for an example).

Note that the conditions $\beta^i_{a_0^i} = \beta^i_{b_0^i} = \beta^j_{z_0^j} = 1$, for $i, j = 1, 2$ and $i \neq j$, are not restrictive because, by Proposition 12, $\beta^i_{a_0^i} = \beta^i_{b_0^i}$.

Theorem 18. (Galluccio et al. [11]) *Let G_i be a graph with a simplicial edge $a_0^i b_0^i$, $i = 1, 2$, and let G be the 2-clique-bond composition of G_1 and G_2 along (a_0^1, b_0^1) and (a_0^2, b_0^2). The following system is defining for $STAB(G)$:*

- *nonnegativity inequalities;*
- *clique inequalities induced by $A_1 \cup A_2$ and $B_1 \cup B_2$;*
- *facet defining inequalities of $STAB(G_i)$ with zero coefficients on a_0^i and b_0^i, $i = 1, 2$;*
- *even-odd combinations of facet defining inequalities of $STAB(G_i)$ and $STAB(G_j/a_0^j b_0^j)$ for each $i, j = 1, 2$ and $i \neq j$.*

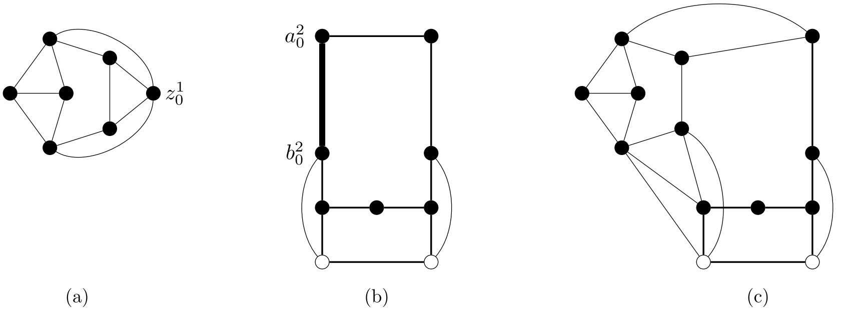

Fig. 1. (a) The odd inequality $\sum_{u\in\bullet} x_u \leqslant 2$; (b) the even inequality $\sum_{u\in\bullet} x_u \leqslant 3$; (c) $\sum_{x\in\bullet} x_u \leqslant 4$, the even-odd combination of (a) and (b).

The above result shows explicitly how to combine the facet defining inequalities of four polytopes related to G_1 and G_2 in order to obtain a defining linear system for $STAB(G)$ when G is the 2-clique-bond composition of G_1 and G_2. The next lemma shows that the class of rank facet defining inequalities of the stable set polytope is closed under even-odd combinations.

Lemma 19. (Chudnovsky and Seymour [11]) *Even-odd combinations of rank inequalities that are facet defining for $STAB(G_i)$ and $STAB(G_j/a_0^j b_0^j)$, $i, j = 1, 2$ and $i \neq j$, are rank inequalities that are facet defining for $STAB(G)$.*

Notice that the 2-clique-bond composition requires an edge to be simplicial but, in order to preserve the claw-freeness of the resulting graph, in the rest of the paper we always assume that the edges involved in the 2-clique-bond composition are super simplicial. This requirement guarantees that in claw-free graphs the 2-clique-bond composition is equivalent to the strip composition described by Chudnovsky and Seymour.

4. Homogeneous pairs of cliques

In [3], a *homogeneous pair of cliques* in a graph G is defined as a pair (A, B) such that: i) A, B are cliques in G and $A \cap B = \emptyset$; ii) no vertex of $G \setminus (A \cup B)$ has both a neighbour and a non-neighbour in A, and the same for B; iii) $|A| \geqslant 2$ or $|B| \geqslant 2$. A homogeneous pair of cliques is then a particular case of the homogeneous pair defined by Chvátal and Sbihi in [5].

Definition 20. Let $G = (V, E)$ be a claw-free graph. A pair of nodes $\{u, v\} \subset V$ is said to be *fuzzy* if one of the following holds:

a) if $uv \in E$ then $G - uv$ is claw-free,
b) if $uv \notin E$ then $G + uv$ is claw-free.

Chudnovsky and Seymour define the *thickening* as a procedure to build homogeneous pair of cliques in claw-free graphs. We slightly extend their definition [3] to include deletion/addition of the single edge uv.

Definition 21. Given a graph H and a set F of disjoint fuzzy pairs of nodes of $V(H)$, a *thickening* of H on F is a graph G satisfying the following:

- for every $v \in V(H)$ there is a nonempty clique $X_v \subseteq V(G)$ and the family $\{X_v \mid v \in V(H)\}$ is a partition of $V(G)$;
- if $uv \notin E(H)$ and $\{u, v\} \notin F$, then X_u is X_v-anticomplete in G;
- if $uv \in E(H)$ and $\{u, v\} \notin F$, then X_u is X_v-complete in G;
- if $\{u, v\} \in F$, then
  either X_u is neither X_v-complete nor X_v-anticomplete in G
  or X_u is X_v-complete (X_v-anticomplete) if and only if $uv \notin E(H)$ ($uv \in E(H)$).

Observe that if $F = \emptyset$, then G is obtained only by substituting cliques for nodes in H (see [4]). Observe also that a thickening on the fuzzy pair $\{u, v\}$ of F such that $|X_u| \geqslant 2$ or $|X_v| \geqslant 2$ produces a homogeneous pair of cliques (X_u, X_v) in G.

We say that a graph G is *fuzzy* if it is obtained from H by performing a thickening on a (possibly empty) set of fuzzy pairs. In order to investigate the stable set polytope of a fuzzy graph G it is convenient to deal with facet defining inequalities (β, β_0) whose supporting graph G_β is minimal in some respect. For instance, we may assume that G_β does not contain twins, because twins have the same coefficient in any facet defining inequality of $STAB(G)$ by Proposition 4. Furthermore, G_β has no clique-cutset because of item i) of Corollary 14. The structure of G_β can be further specified by considering the following lemma of Eisenbrand et al.:

Lemma 22. (Eisenbrand et al. [7]) *Let (β, β_0) be a facet defining inequality of $STAB(G)$. Then there exists a graph G', obtained from G by removing some edges, such that (β, β_0) is also facet defining for $STAB(G')$ and no homogeneous pair of cliques of G' contains an induced C_4.*

As a consequence, we may assume that the homogeneous pairs of cliques contained in G_β do not contain any induced C_4. In his thesis, King proved an interesting property of homogeneous pair of cliques that do not contain C_4's:

Lemma 23. (King [17]) *Let (X_u, X_v) be a homogeneous pair of cliques with $X_u = \{u_1, u_2, \dots, u_{|X_u|}\}$ and $X_v = \{v_1, v_2, \dots, v_{|X_v|}\}$. If (X_u, X_v) contains no induced C_4, then the nodes of X_u and X_v can be ordered so that:*

- $N(u_i) \cap X_v \supseteq N(u_j) \cap X_v$ *for* $1 \leqslant i \leqslant j \leqslant |X_u|$,
- $N(v_i) \cap X_u \supseteq N(v_j) \cap X_u$ *for* $1 \leqslant i \leqslant j \leqslant |X_v|$.

To our purposes we can further reduce the number of nodes of $X_u \cup X_v$ by eliminating twins. This allows us to identify a few types of homogeneous pairs of cliques that can appear in the supporting graph of a minimal facet defining inequality.

Lemma 24. *Let (β, β_0) be a facet defining inequality such that G_β contains a homogeneous pair of cliques (X_u, X_v) with $|X_u| = p$ and $|X_v| = q$, $p \geqslant q \geqslant 1$. If (X_u, X_v) contains no twins and no induced C_4, then $p \in \{q, q+1\}$ and the nodes in $X_u \cup X_v$ can be ordered so that:*

1. *if $p = q$ then either $N(u_i) \cap X_v = \{v_1, v_2, \dots, v_{q-i+1}\}$ for $i = 1, \dots, p$, and $N(v_i) \cap X_u = \{u_1, u_2, \dots, u_{p-i+1}\}$ for $i = 1, \dots, q$, or $N(u_i) \cap X_v = \{v_1, v_2, \dots, v_{q-i}\}$ for $i = 1, \dots, p$, and $N(v_i) \cap X_u = \{u_1, u_2, \dots, u_{p-i}\}$ for $i = 1, \dots, q$;*
2. *if $p = q+1$ then $N(u_i) \cap X_v = \{v_1, v_2, \dots, v_{q-i+1}\}$ for $i = 1, \dots, p$, and $N(v_i) \cap X_u = \{u_1, u_2, \dots, u_{p-i}\}$ for $i = 1, \dots, q$.*

Proof. Assume that the nodes of (X_u, X_v) are ordered according to Lemma 23. Since X_u contains no twins, there do not exist two nodes of X_u with the same adjacencies in X_v. It follows that $N(u_i) \cap X_v \supset N(u_{i+1}) \cap X_v$ and, in particular, that $|N(u_i) \cap X_v| > |N(u_{i+1}) \cap X_v|$, for $i = 1, \dots, p-1$.

Then, as $0 \leqslant |N(u_i) \cap X_v| \leqslant q$ for all $i = 1, \dots, p$, for the pigeon hole principle, we have that $p \leqslant q+1$. Moreover $|N(u_i) \cap X_v| - |N(u_{i+1}) \cap X_v| = 1$, for $i = 1, \dots, p-1$. Indeed, assume by contradiction that there exist $u_i, u_{i+1} \in X_u$ such that $N(u_i) \cap X_v = \{v_1, \dots, v_j\}$, $N(u_{i+1}) \cap X_v = \{v_1, \dots, v_k\}$ and $j - k \geqslant 2$. Let $v_h \in X_v$ with $j > h > k$. Then $N(v_h) \cap X_u$ is either $\{u_1, \dots, u_i\} = N(v_j) \cap X_u$ or $\{u_1, \dots, u_i, u_{i+1}\} = N(v_k) \cap X_u$, contradicting the hypothesis that X_v contains no twins.

Now, it is not difficult to verify that the only feasible configurations are those listed in the statement. □

We call *canonical* the homogeneous pair of cliques satisfying Lemma 24. See Fig. 2 to see examples of the three types of canonical homogeneous pair of cliques.

Summarizing the previous results, hereafter we consider facet defining inequalities whose supporting graph contains no twins and each of its homogeneous pair of cliques is canonical. For short, we say that such facet defining inequalities are *minimal*.

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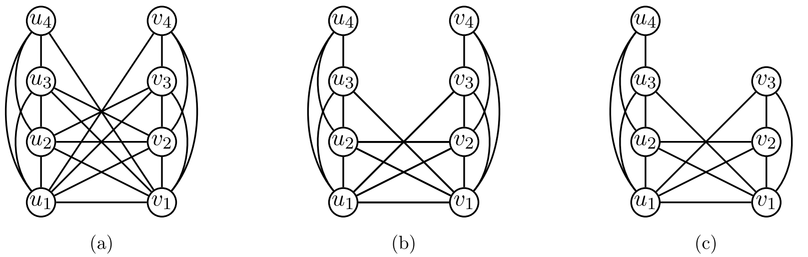

Fig. 2. In (a) and (b) homogeneous pairs of cliques (X_u, X_v) with $p = q = 4$; in (c) a homogeneous pair of cliques with $p = 4$ and $q = 3$.

5. Nice three-cliqued graphs

In this section we deal with graphs whose node set can be covered by three cliques.

Definition 25. Let H be a graph whose node set can be covered by three cliques A, B, and C such that $A \cap C = \emptyset$ and $B \cap C = \emptyset$. The graph G obtained from H by adding two nodes a_0 and b_0 such that $N(a_0) = A$ and $N(b_0) = B$ is a *three-cliqued strip* and is denoted by $G = (A, B, C, a_0, b_0)$.

A *nice three-cliqued graph* is a closed three-cliqued strip $G = (A, B, C, a_0b_0)$ such that $\alpha(G \setminus \{a_0, b_0\}) \leqslant 2$.

The previous definition of three-cliqued strip is slightly more general than the one given by Chudnovsky and Seymour [3]. In fact, we allow A and B to intersect. Notice also that three-cliqued graphs (even if nice) are in general not claw-free. The following results concern the linear description of the stable set polytope of nice three-cliqued graphs.

Theorem 26. *Let $G = (A, B, C, a_0b_0)$ be a nice three-cliqued graph such that a_0b_0 is super simplicial. Then $STAB(G)$ is described by nonnegativity and clique-neighbourhood inequalities. In particular, the inequalities with nonzero coefficients on the nodes a_0 and b_0 are rank inequalities.*

Proof. Suppose conversely that there exists a nontrivial facet defining inequality (β, β_0) of $STAB(G)$ that is not a clique-neighbourhood inequality. Since clique inequalities are a special kind of clique-neighbourhood inequalities, we may assume that (β, β_0) is not a clique inequality.

First observe that $\alpha(G) \leqslant 3$. Denote by G_β the supporting graph of (β, β_0). Let $A_\beta = A \cap V(G_\beta)$, $B_\beta = B \cap V(G_\beta)$, and $C_\beta = C \cap V(G_\beta)$. Then $C_\beta \neq \emptyset$, since otherwise G_β is partitionable into two cliques, i.e., $\alpha(G_\beta) \leqslant 2$, and, by Theorem 6, (β, β_0) is a clique-neighbourhood inequality. Contradiction.

Moreover, as a_0b_0 is simplicial, $\beta_{a_0} = \beta_{b_0}$ by Corollary 15. If $\beta_{a_0} = \beta_{b_0} = 0$, then $\alpha(G_\beta) \leqslant 2$ and (β, β_0) is a clique-neighbourhood inequality by Theorem 6. So assume that $\beta_{a_0} = \beta_{b_0} \neq 0$, i.e., a_0 and b_0 belong to $V(G_\beta)$. Then there exist two nonadjacent nodes $a_1 \in A_\beta \setminus B_\beta$ and $b_1 \in B_\beta \setminus A_\beta$, since otherwise $A_\beta \cup B_\beta$ would be a clique-cutset in G_β, contradicting claim i) of Corollary 14.

As (β, β_0) is not the clique inequality defined by $A_\beta \cup \{a_0\}$, there exists, by Lemma 3, a stable set S that is tight for (β, β_0) and that does not intersect $A_\beta \cup \{a_0\}$. It follows that $b_0 \in S$, otherwise $S \cup \{a_0\}$ would be a stable set violating (β, β_0). As a consequence, $S = \{b_0, \tilde{c}\}$ with $\tilde{c} \in C_\beta$ and, by Proposition 12, also $S' = \{a_0, \tilde{c}\}$ is tight.

If there exists $a \in (A_\beta \setminus B_\beta) \setminus N(\tilde{c})$, then $\{b_0, \tilde{c}, a\}$ violates (β, β_0), a contradiction. Symmetrically, if there exists $b \in (B_\beta \setminus A_\beta) \setminus N(\tilde{c})$, then $\{a_0, \tilde{c}, b\}$ violates (β, β_0), a contradiction. It follows that $N(\tilde{c}) \supseteq (A_\beta \setminus B_\beta) \cup (B_\beta \setminus A_\beta)$.

Consider now the inequality $(\gamma, 2)$ obtained from the 5-hole inequality induced by $(a_0, b_0, b_1, \tilde{c}, a_1)$ by sequentially lifting the other nodes of G_β as follows: first the nodes in $(A_\beta \setminus B_\beta) \cup (B_\beta \setminus A_\beta)$ receive coefficient 1 because their non-neighbourhood in the supporting graph of the inequality that is lifted is a nonempty clique; then the nodes in $A_\beta \cap B_\beta$ receive coefficient 1 because their non-neighbourhood is $\tilde{c}$ (since $a_0 b_0$ is super simplicial); finally, we lift with coefficient 1 the nodes in $C_\beta \setminus \{\tilde{c}\}$ whose non-neighbourhood is a clique and with coefficient 0 all the remaining nodes in C_β.

Suppose now that there exists a stable set S'' that is tight for (β, β_0) and not tight for $(\gamma, 2)$. Then $S'' = \{u, v\}$ with $u \in C_\beta \setminus C_\gamma$ and $v \in A_\beta \cap B_\beta$. As $\gamma_u = 0$ then $G_\gamma \setminus N[u]$ contains a stable set of size 2: either $\{a, b_0\}$ with $a \in A_\beta \setminus B_\beta$ or $\{b, a_0\}$ with $b \in B_\beta \setminus A_\beta$. Let us assume without loss of generality that the former case occurs, and so $\beta_u + \beta_a + \beta_{b_0} \leqslant \beta_0$. Since $\{b_0, \tilde{c}\}$ is tight for (β, β_0), $\beta_{\tilde{c}} + \beta_{b_0} = \beta_0$ and so, $\beta_{\tilde{c}} \geqslant \beta_u + \beta_a$, i.e., $\beta_{\tilde{c}} > \beta_u$ and $\beta_{\tilde{c}} > \beta_a$. Therefore $\{\tilde{c}, v\}$ is a stable set that violates (β, β_0). A contradiction.

By Lemma 3 it follows that (β, β_0) is a positive scalar multiple of $(\gamma, 2)$ and the thesis follows. $\square$

Theorem 27. *Let $G = (A, B, C, a_0 b_0)$ be a nice three-cliqued graph and let z_0 denote the node resulting from the contraction of $a_0 b_0$. Then $STAB(G/a_0 b_0)$ is described by nonnegativity and clique-neighbourhood inequalities. In particular, z_0 has coefficient zero or one in every facet defining inequality of $STAB(G/a_0 b_0)$.*

Proof. As $\alpha(G \setminus \{a_0, b_0\}) \leqslant 2$ and $G \setminus N[z_0]$ is a clique, it follows that $\alpha(G/a_0 b_0) \leqslant 2$. Thus we can apply Theorem 6 to obtain the first part of the thesis. Now we prove that every clique-neighbourhood inequality with nonzero coefficient on z_0 has coefficient 1 on such a node. Indeed, select a clique $K \ni z_0$ and consider the clique-neighbourhood inequality generated by K: $2x(K) + x(\tilde{N}(K)) \leqslant 2$. Since $\tilde{N}(K) \subseteq A \cup B$ is partitioned into two cliques, it follows, by Corollary 7, that the clique-neighbourhood inequality generated by K is not facet defining unless $\tilde{N}(K) = \emptyset$, i.e., the clique-neighbourhood inequality is actually a clique inequality. Hence, $z_0 \notin K$ in any facet defining inequality with coefficients $\{1, 2\}$ and the thesis follows. $\square$

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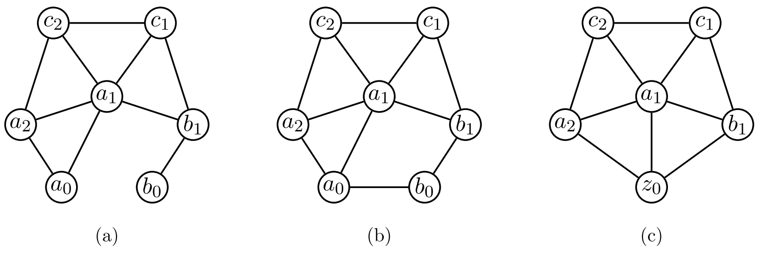

Fig. 3. (a) A nice three-cliqued strip $G = (A, B, C, a_0, b_0)$; (b) the corresponding closed strip (G, a_0b_0); (c) the contracted graph G/a_0b_0.

Observe now that there exist non-rank facet defining inequalities with nonzero coefficient on z_0: for instance, consider the nice three-cliqued graph (A, B, C, a_0b_0) with $A = \{a_1, a_2\}$, $B = \{b_1\}$, $C = \{c_1, c_2\}$, and edges as in Fig. 3 (b). It is easy to verify that the graph obtained by contracting the edge a_0b_0 into a single node z_0 supports a 5-wheel inequality with coefficient 1 on node z_0 (see Fig. 3). Notice that G is not claw-free.

In the next sections we shall prove that this situation never occurs in claw-free three-cliqued strips Z_i, $i = 2, 3, 4$.

6. Fuzzy Z_2-strips

In this section we specialize the results obtained so far to a special class of claw-free nice three-cliqued graphs: the closed fuzzy Z_2-strips.

Definition 28. Let G^* be a three-cliqued strip $(A^*, B^*, C^*, a_0, b_0)$ such that the following conditions hold:

1) $A^* = \{a_1, a_2, \dots, a_n\}$, $B^* = \{b_1, b_2, \dots, b_n\}$, and $C^* = \{c_1, c_2, \dots, c_n\}$ are three pairwise disjoint cliques,
2) for $1 \leqslant i, j \leqslant n$, a_i and b_j are adjacent if and only if $i = j$,
3) for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, c_i is adjacent to a_j, b_j if and only if $i \neq j$.

A fuzzy Z_2-strip (A, B, C, a_0, b_0) is obtained from G^* by deleting a (possibly empty) set of nodes $Y \subseteq A^* \cup B^* \cup C^*$, such that $A = A^* \setminus Y$, $B = B^* \setminus Y$, $C = C^* \setminus Y$, $|C| \geqslant 2$, and possibly performing a thickening on the following pairs:

- $\{a_i, c_i\}$ for at most one value of $i \in \{1, \dots, n\}$, with $b_i \in Y$,
- $\{b_i, c_i\}$ for at most one value of $i \in \{1, \dots, n\}$, with $a_i \in Y$,
- $\{a_i, b_i\}$ for at most one value of $i \in \{1, \dots, n\}$, with $c_i \in Y$.

In the following, we denote by $\hat{u}$ a generic node in X_u.

Lemma 29. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_2-strip. Then G and G/a_0b_0 do not contain a $(2t + 1)$-antihole, $t \geqslant 3$, as an induced subgraph.*

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Proof. Suppose conversely that G contains a $(2t+1)$-antihole H with $t \geqslant 3$ and let the nodes of $H = (v_0, v_1, \dots, v_{2t})$ be ordered so that two nodes are not adjacent if and only if they are consecutive on H (sums are taken modulo $2t+1$). Suppose first that $a_0 \in H$. Then a_0 is adjacent to exactly $2t-2$ nodes of H and, consequently, H contains either $2t-3$ (if $b_0 \in H$) or $2t-2$ (if $b_0 \notin H$) nodes of A. In both cases, H would contain a clique of size $2t-2$, contradicting the hypothesis that $t \geqslant 3$. Hence $a_0 \notin H$ and symmetrically $b_0 \notin H$.

Suppose now that $z_0 \in H$ and let $z_0 = v_0$. Then v_1 and v_{2t} belong to C and, moreover, $\{v_2, v_3, \dots, v_{2t-1}\} \subseteq A \cup B$. To preserve the nonadjacency of consecutive nodes in H, the nodes in $H \setminus \{v_{2t}, v_0, v_1\}$ belong alternatively to A and B. Without loss of generality, let $v_2 = \hat{a}_i \in X_{a_i}$ for some $i \in \{1, \dots, n\}$ (where $n = |A^*| = |B^*| = |C^*|$ as in Definition 28). Thus $v_1 = \hat{c}_i \in X_{c_i}$ and $v_{2t-1} \in B$. Since $v_2 v_{2t-1} \in E$, $v_{2t-1} = \hat{b}_i \in X_{b_i}$, thus implying that no pair in $\{a_i, b_i, c_i\}$ is fuzzy. But then $v_{2t} = \hat{c}_k \in X_{c_k}$ with $k \neq i$ and $v_{2t} v_{2t-1} \in E$, a contradiction.

Thus $z_0 \notin H$ and a vertex $\hat{c}_i \in H$ otherwise H would be partitioned into two cliques. Set $\hat{c}_i = v_0$ for some $i \in \{1, \dots, n\}$ where n is defined as above. Then $v_1 \in X_{a_i} \cup X_{b_i}$. Without loss of generality, let $v_1 = \hat{a}_i$. Since $\hat{a}_i$ is anticomplete to $B \setminus X_{b_i}$, $H \setminus \{v_0, v_1, v_2\}$ does not contain any node in $B \setminus X_{b_i}$.

If $v_2 \in X_{c_i}$ then the pair $\{a_i, c_i\}$ is fuzzy and so, $X_{b_i} = \emptyset$. Thus $V(H) \subseteq A \cup C$, a contradiction. As a consequence, since $v_2 \in N(\hat{c}_i) \setminus N(\hat{a}_i)$, $v_2 = \hat{b}_j \in X_{b_j}$ with $j \neq i$. Furthermore, $v_{2t} \notin X_{a_i}$ since otherwise $v_{2t} v_2$ would not be adjacent. Thus $v_{2t} = \hat{b}_i \in X_{b_i}$, i.e., no pair in $\{a_i, b_i, c_i\}$ is fuzzy. As $\hat{b}_i$ is anticomplete to $A \setminus X_{a_i}$, $H \setminus \{v_{2t-1}, v_{2t}, v_0, v_1\}$ does not contain any node in $A \setminus X_{a_i}$. Since $v_{2t-1} \in N(\hat{c}_i) \setminus N(\hat{b}_i)$, $v_{2t-1} = \hat{a}_k \in X_{a_k}$ with $k \neq i$.

Thus $H \setminus \{v_{2t-1}, v_{2t}, v_0, v_1, v_2\}$ is contained in C and consists of at most one node, contradicting the hypothesis that H has length at least 7. □

It is easy to see that a closed fuzzy Z_2-strip G that is a thickening with $F = \emptyset$ is a nice three-cliqued graph and, with a little effort, it can also be proved that the thickening performed on any admissible set of fuzzy pairs does not increase the stability number of closed fuzzy Z_2-strips. Therefore, by Theorems 26 and 27, the stable set polytopes of a closed fuzzy Z_2-strip and its contraction along the super simplicial edge $a_0 b_0$ are described by: nonnegativity and clique-neighbourhood inequalities. In the remainder of this section, we provide details on the structure of these inequalities.

Theorem 30. *Let $(G, a_0 b_0)$ be a closed fuzzy Z_2-strip. Then every nontrivial inequality (β, β_0) that is facet defining for $STAB(G)$ $(STAB(G/a_0 b_0))$ is a rank or a lifted 5-wheel inequality. Moreover, if $\beta_{a_0} = \beta_{b_0} > 0$ ($\beta_{z_0} > 0$, respectively), then (β, β_0) is a rank inequality.*

Proof. Since a closed fuzzy Z_2-strip is a nice three-cliqued graph, it follows, by Theorem 26 (Theorem 27), that every facet defining inequality for $STAB(G)$ ($STAB(G/a_0 b_0)$, respectively) is either a nonnegativity or a clique-neighbourhood inequality.

Since $A \cap B = \emptyset$, the edge a_0b_0 is super simplicial in G and so, by Theorem 26, every facet defining clique-neighbourhood inequality of $STAB(G)$ with coefficients $\{0,1,2\}$ has zero coefficients on a_0 and b_0. Therefore, the supporting graph of every clique-neighbourhood inequality of $STAB(G)$ that is not a rank inequality is a subgraph G' of $G \setminus \{a_0, b_0\}$ and so also of G/a_0b_0.

By Corollary 8, every nontrivial facet defining inequality of $STAB(G/a_0b_0)$ that is not a rank inequality is a clique-neighbourhood inequality generated by a nonempty clique $K \subset C$. Since z_0 is C-anticomplete it follows that z_0 has coefficient zero in every clique-neighbourhood inequality of $STAB(G/a_0b_0)$ with coefficients $\{0,1,2\}$.

To complete the proof we need to show that every clique-neighbourhood inequality (β, β_0) that is facet defining for $STAB(G)$ (and for $STAB(G/a_0b_0)$) and is not a rank inequality, is a lifted 5-wheel inequality. By Lemma 29, G and G/a_0b_0 do not contain any $(2t+1)$-antiwheel with $t \geqslant 3$. Thus, by Lemma 9, it suffices to show that $\overline{G_\beta}[\tilde{N}(K)]$ is connected, where K is the clique that generates (β, β_0).

Let $\hat{c}_k \in K$ and let $(\hat{a}_i, \hat{b}_j, \hat{c}_j, \hat{a}_j, \hat{b}_r)$ be a 5-hole contained in $\overline{G_\beta}[\tilde{N}(K)]$ (it exists since G is claw-free, it does not contain a $(2t+1)$-antiwheel with $t \geqslant 3$, and $\overline{G_\beta}[\tilde{N}(K)]$ is not bipartite) with $i \neq j \neq r$ and i, j, r different from k. Note that r might coincide with i in case G is fuzzy and $\{a_i, b_i\}$ is a fuzzy pair. Now, in $\overline{G}$ each node $\hat{a}_q \in A \cap \tilde{N}_{G_\beta}(K)$, $q \neq i, j$ is adjacent to $\hat{b}_j$ and each node $\hat{b}_p \in B \cap \tilde{N}_{G_\beta}(K)$, $p \neq j, r$ is adjacent to $\hat{a}_j$. Moreover each node $\hat{c}_t \in C \cap \tilde{N}_{G_\beta}(K)$, $t \neq j$, is adjacent in $\overline{G}$ to at least one node in $(A \cup B) \cap \tilde{N}_{G_\beta}(K)$, otherwise $\hat{c}_t$ is isolated in $\overline{G_\beta}[\tilde{N}(K)]$ and Theorem 6 would be contradicted. Hence, $\overline{G_\beta}[\tilde{N}(K)]$ is connected and the thesis follows. □

Theorem 30 provides us the following useful information: when performing the 2-clique-bond composition of a closed fuzzy Z_2-strip G with another graph, clique-neighbourhood inequalities of $STAB(G)$ and $STAB(G/a_0b_0)$ with coefficients $\{0,1,2\}$ are never involved in even-odd combinations of inequalities. This because the only facet defining inequalities of $STAB(G)$ that are even are rank inequalities and the only facet defining inequalities of $STAB(G/a_0b_0)$ that are odd are rank inequalities as well.

The closed fuzzy Z_2-strips are not the only three-cliqued graphs involved in the decomposition theorem of claw-free graphs. In particular two other types of three-cliqued graphs are needed to construct fuzzy antihat graphs: closed fuzzy Z_3-strips and closed fuzzy Z_4-strips. These three-cliqued graphs are not nice in general and for each of them we need specific proofs to yield the linear descriptions of their stable set polytope. This will be discussed in the next two sections.

7. Fuzzy Z_3-strips

In [3], fuzzy Z_3-strips are defined as follows:

Definition 31. Let H be a graph and let $(h_1, h_2, h_3, h_4, h_5)$ be a path in H such that h_1 and h_5 both have degree one in H and every other edge of H is incident with one of

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h_2, h_3, h_4. The graph H' obtained from the line graph of H by performing a thickening on the pair $\{h_2h_3, h_3h_4\}$ or by deleting the edge $\{h_2h_3, h_3h_4\}$ is a fuzzy Z_3-strip with simplicial nodes $\{h_1h_2, h_4h_5\}$.

An equivalent definition can be given directly without producing H' as a thickening of the line graph of an original graph H.

Definition 32. Let G^* be a three-cliqued strip $(A^*, B^*, C^*, a_0, b_0)$ where the following conditions hold:

1) $A^* = \{z_1, a_1, a_2, \ldots, a_n\}$, $B^* = \{z_2, b_1, b_2, \ldots, b_n\}$, and $C^* = \{c_1, c_2, \ldots, c_n\}$ are three pairwise disjoint cliques;
2) for $1 \leqslant i, j \leqslant n$, a_ib_j, a_ic_j, $b_ic_j \in E(G^*)$ if and only if $i = j$;
3) z_1 is $(A^* \cup \{a_0\} \cup C^*)$-complete and $(B^* \cup \{b_0\})$-anticomplete; z_2 is $(B^* \cup \{b_0\} \cup C^*)$-complete and $(A^* \cup \{a_0\})$-anticomplete; $z_1z_2 \in E(G^*)$.

A fuzzy Z_3-strip (A, B, C, a_0, b_0) is obtained from G^* by deleting a (possibly empty) set of nodes $Y \subseteq A^* \cup B^* \cup C^* \setminus \{z_1, z_2\}$ such that $A = A^* \setminus Y$, $B = B^* \setminus Y$, and $C = C^* \setminus Y$ and by performing a thickening on a set F containing the pair $\{z_1, z_2\}$ and possibly the following pairs:

- $\{a_i, c_i\}$, with $b_i \in Y$;
- $\{b_i, c_i\}$, with $a_i \in Y$;
- $\{a_i, b_i\}$, with either $c_i \in Y$ or $C = \{c_i\}$.

It can be verified that the fuzzy Z_3-strip as defined in Definition 32 is equivalent to the thickening of a Z_3-strip (H, h_1h_2, h_4h_5) as defined in Definition 31 provided that the thickening of Chudnovsky and Seymour is modified as in Definition 21.

In the following, we say that a (closed/contracted closed) fuzzy Z_3-strip G contains an *ab-pair* $\{a_i, b_i\}$ if $a_i, b_i \in V(G)$ and $c_i \in Y$, an *ac-pair* $\{a_i, c_i\}$ if $a_i, c_i \in V(G)$ and $b_i \in Y$, a *bc-pair* $\{b_i, c_i\}$ if $b_i, c_i \in V(G)$ and $a_i \in Y$. Moreover, if $a_i, b_i, c_i \in V(G)$ for some index i, we say that G contains a *complete triple*, while, if $c_i \in V(G)$ and $a_i, b_i \in Y$, we say that G contains a *c-singleton*. According to Definition 20, each of the above pairs is fuzzy. In the following, when we say that G contains, say, an *ab*-pair $\{a_i, b_i\}$ for some i, we mean that G contains the thickening (X_{a_i}, X_{b_i}) of that pair (if any). The same apply to *bc*-pairs, *ac*-pairs, and the complete triple $\{a_i, b_i, c_i\}$ when $C = \{c_i\}$ according to Definition 32.

In Fig. 4 we depict two examples of Z_3-strip: (a) shows a Z_3-strip with one complete triple, one *ab*-pair and a *c*-singleton; (b) shows a Z_3-strip with two complete triples. The dashed bold edges represent fuzzy pairs, while the grey areas emphasize the three cliques A, B, and C.

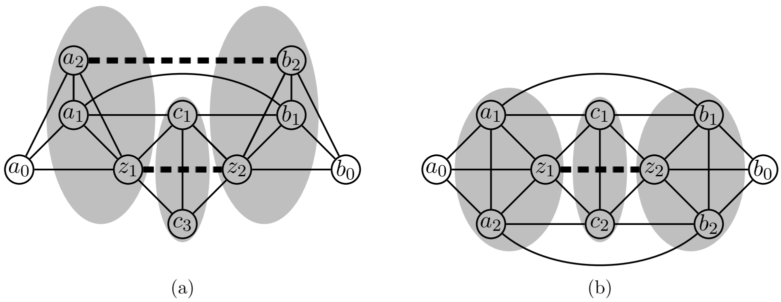

Fig. 4. Z_3-strips.

In the following we study the facet defining inequalities of $STAB(G)$ and $STAB(G/a_0b_0)$ where G is a closed fuzzy Z_3-strip with a super simplicial edge a_0b_0. A number of preliminary observations can be made to specify which subgraphs of a closed fuzzy Z_3-strip can support a nontrivial facet defining inequality.

Observation 33. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_3-strip and let Y be as in Definition 32. If Y contains a pair of nodes $\{b_i, c_i\}$ (or $\{a_i, c_i\}$), then the node a_i (b_i respectively) is simplicial in G, as it belongs only to the clique $A \cup \{a_0\}$ ($B \cup \{b_0\}$ respectively). By ii) of Corollary 14, the associated clique inequality is the only nontrivial facet defining inequality for $STAB(G)$ with nonzero coefficient on a_i (b_i). Analogous result holds for G/a_0b_0 with the cliques $A \cup \{z_0\}$ and $B \cup \{z_0\}$.*

Based on the previous observation, we restrict our attention to nontrivial facet defining inequalities (β, β_0) with the following property: if $a_i \in V(G_\beta)$ then $V(G_\beta) \cap \{b_i, c_i\} \neq \emptyset$ and if $b_i \in V(G_\beta)$ then $V(G_\beta) \cap \{a_i, c_i\} \neq \emptyset$.

Observation 34. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_3-strip. If c_i and c_k are c-singletons, then c_i and c_k are twins in G and in G/a_0b_0.*

As twins always have the same coefficient in every facet defining inequality, hereafter we consider only nontrivial facet defining inequalities whose supporting graph contains at most one c-singleton.

Lemma 35. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_3-strip. If G does not contain any complete triple, then G and G/a_0b_0 are fuzzy line graphs.*

Proof. Let $H = (A, B, C, x_0, y_0)$ denote the fuzzy Z_3-strip obtained from $G - a_0b_0$ by renaming the nodes a_0 and b_0 as x_0 and y_0, respectively.

Consider now the fuzzy Z_3-strip $H' = (A', B', C, x_1, x_2)$ obtained from H by removing the nodes of A and B that form ab-pairs and adding two nodes x_1 and x_2 that are twins of x_0 and y_0, respectively.

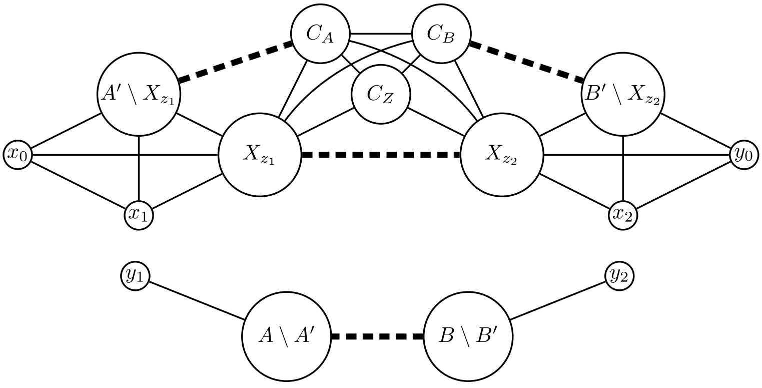

Fig. 5. The two strips H' and H'' of the proof of Lemma 35.

We show that (H', x_1, x_2) is a fuzzy linear interval strip. We partition the set C into three sets: C_A, C_B and C_Z, where C_A and C_B contains exactly the nodes of C that are adjacent to some nodes in $A' \setminus X_{z_1}$ and $B' \setminus X_{z_2}$, respectively, and $C_Z = C \setminus (C_A \cup C_B)$ (see Fig. 5).

Then consider the mapping ϕ from $V(H')$ to the points $l_0, l_1, \ldots, l_{10}$ of a line L (ordered from left to right) such that: $\phi^{-1}(l_0) = x_1$, $\phi^{-1}(l_1) = x_0$, $\phi^{-1}(l_2) = A' \setminus X_{z_1}$, $\phi^{-1}(l_3) = X_{z_1}$, $\phi^{-1}(l_4) = C_A$, $\phi^{-1}(l_5) = C_Z$, $\phi^{-1}(l_6) = C_B$, $\phi^{-1}(l_7) = X_{z_2}$, $\phi^{-1}(l_8) = B' \setminus X_{z_2}$, $\phi^{-1}(l_9) = y_0$, $\phi^{-1}(l_{10}) = x_2$, and the intervals: $I_1 = [l_0, l_3 + \epsilon]$, $I_2 = [l_2, l_4]$, $I_3 = [l_3, l_7]$, $I_4 = [l_6, l_8]$, $I_5 = [l_7 - \epsilon, l_{10}]$, where ϵ is an opportunely small value and the intervals I_2, I_3, I_4 are fuzzy (according to Definition 10).

To obtain H it suffices to perform a strip composition between (H', x_1, x_2) and a fuzzy linear interval strip (H'', y_1, y_2) consisting of a unique fuzzy interval with endpoints $A \setminus A'$ and $B \setminus B'$ depicted in Fig. 5.

According to Observation 1, G and G/a_0b_0 are obtained as strip compositions of (H, x_0, y_0) with the 4-path $\{x'_0, a_0, b_0, y'_0\}$ and the 3-path $\{x'_0, z_0, y'_0\}$, respectively. Thus, G and G/a_0b_0 are fuzzy line. □

The next lemmas specify the structure of the supporting graph of nontrivial facet defining inequalities of $STAB(G/a_0b_0)$.

Lemma 36. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_3-strip. Let (β, β_0) be a nontrivial facet defining inequality of $STAB(G/a_0b_0)$ that is not a rank inequality. If G_β does not contain twins, then one of the following cases occurs:*

1. *$G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ consists of two complete triples, no pairs, and at most one c-singleton;*
2. *$G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ consists of one complete triple, one pair, and at most one c-singleton;*

3. *$G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ consists of one complete triple, no pairs, and at most one c-singleton;*
4. *$G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ consists of one complete triple, one ac-pair, one bc-pair, no ab-pair, and at most one c-singleton.*

Proof. Let $A_\beta = A \cap V(G_\beta)$ and $B_\beta = B \cap V(G_\beta)$. By Lemma 3, there exists a stable set S missing the clique $A_\beta \cup \{z_0\}$. Thus every node $a_i \in A_\beta \setminus X_{z_1}$ must be adjacent to at least one node in $S \cap \{b_i, c_i\}$ since otherwise $S \cup \{a_i\}$ would violate (β, β_0). As $B \cup C$ contains at most two nonadjacent nodes, it follows that $|V(G_\beta) \cap (A_\beta \setminus X_{z_1})| \leqslant 2$.

By using a tight stable set S missing the clique $B_\beta \cup \{z_0\}$, it can be proved that $|V(G_\beta) \cap (B_\beta \setminus X_{z_2})| \leqslant 2$. If G_β does not contain any complete triple, then, by Lemma 35 and Theorem 11, (β, β_0) is a rank inequality, a contradiction. Hence, G_β contains at least one complete triple and, by Observation 34, G_β contains at most one c-singleton. Finally, by Observation 33, it follows that all feasible configurations are those listed in the thesis. □

Lemma 37. *Let $G = (A, B, C, a_0 b_0)$ be a closed fuzzy Z_3-strip. Then every nontrivial inequality (β, β_0) that is facet defining for STAB$(G/a_0 b_0)$ is either a rank or a lifted 5-wheel inequality. Moreover, if $\beta_{z_0} > 0$, then (β, β_0) is a rank inequality.*

Proof. We may assume that (β, β_0) does not contain twins and every homogeneous pair in G_β is canonical. According to Lemma 36 we distinguish four cases.

Case 1: *$G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ consists of two complete triples $\{a_i, b_i, c_i\}$, $i = 1, 2$, no pairs, and at most one c-singleton c_3.*

If $c_3 \notin V(G_\beta)$, then consider the mapping $\phi : V(G_\beta) \to V(G'_\beta)$ defined as follows:

- $V(G_\beta) = \{z_0, a_1, c_1, b_1, a_2, c_2, b_2, z_1, z_2\}$;
- $V(G'_\beta) = \{z'_0, a'_1, c'_2, b'_1, a'_2, c'_1, b'_2, a'_3, b'_3\}$;
- $\phi(a_i) = a'_i$, $\phi(b_i) = b'_i$, for $i = 1, 2$, $\phi(z_0) = z'_0$, $\phi(z_1) = a'_3$, and $\phi(z_2) = b'_3$, $\phi(c_1) = c'_2$, $\phi(c_2) = c'_1$.

Note that, since $c'_3 \notin V(G'_\beta)$, the pair $\{a'_3, b'_3\}$ is fuzzy and a thickening of this pair can be always performed in G'_β as well as on the pair $\{z_1, z_2\}$ in G_β. Thus G_β is isomorphic to a closed fuzzy Z_2-strip G'_β and, by Theorem 30, we are done, as (β, β_0) is either a rank or a lifted 5-wheel inequality; moreover, in the latter case, $\beta_{z_0} = 0$.

Suppose now that $c_3 \in V(G_\beta)$ and that (β, β_0) is not a rank inequality. First observe that $V(G_\beta) \cap X_{z_i} \neq \emptyset$, $i = 1, 2$, since otherwise c_3 would be simplicial in G_β and (β, β_0) would be a clique inequality by (ii) of Corollary 14, contradicting the hypothesis. Then, since $\alpha(G_\beta \setminus \{c_3\}) = 2$, we consider the rank inequality $(\gamma, 2)$ defined as $\sum_{v \in V'} x_v \leqslant 2$ where $V' = V(G_\beta) \setminus \{c_3\}$. By Lemma 3, there exists a tight stable set S for (β, β_0) that is not tight for $(\gamma, 2)$; then $S = \{c_3, z_0\}$, thus implying $\beta_{z_0} > \beta_u$ for any $u \in \{a_1, a_2, b_1, b_2\}$. Now let S' be a tight stable set of (β, β_0) missing the clique $A \cup \{z_0\}$; $S' \cap \{b_1, b_2\} = \emptyset$

otherwise $S' \setminus \{b_1, b_2\} \cup \{z_0\}$ would violate (β, β_0). Hence, $z_2^1 \in S'$ and $S' \cup \{a_1\}$ and $S' \cup \{a_2\}$ are stable sets that violate (β, β_0). A contradiction.

Case 2: $G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ *consists of one complete triple* $\{a_1, b_1, c_1\}$*, one pair, and at most one c-singleton* c_3.

Suppose that (β, β_0) is not a rank inequality. Then we distinguish two nonsymmetric subcases according with the type of pair contained in G_β.

Subcase 2a): G_β contains an ab-pair $\{a_2, b_2\}$.

Let (X_{a_2}, X_{b_2}) be a canonical homogeneous pair of cliques contained in G_β with $X_{a_2} = \{a_2^1, \dots, a_2^n\}$ and $X_{b_2} = \{b_2^1, \dots, b_2^m\}$. Then $a_2^n b_2^1 \in E(G_\beta)$, since otherwise no stable set tight for (β, β_0) would miss the clique inequality on $A \cup \{z_0\}$. Symmetrically, $a_2^1 b_2^m \in E(G_\beta)$. Thus $n = m$ by Lemma 24. Let (X_{z_1}, X_{z_2}) be a canonical homogeneous pair of cliques contained in G_β and let S' be a tight stable set missing the clique $A \cup \{z_0\}$. Then $S' = \{b_2^1, c_1\}$ and so, $\beta_{c_1} \geqslant \beta_{z_1^1}$ and $\beta_{c_1} \geqslant \beta_{c_3} + \beta_{a_1}$.

Consider the set $U = \{a_1, b_1, c_1, z_0, b_2^1\} \cup X_{z_1} \cup X_{z_2} \cup X_{a_2}$ and observe that $\alpha(G[U]) = 2$. As a consequence, the rank inequality $(\gamma, 2)$, whose support is $G[U]$, is valid for $STAB(G/a_0b_0)$. As (β, β_0) is not a rank inequality, there exists a stable set S that is tight for (β, β_0) and not for $(\gamma, 2)$, by Lemma 3. Then $S = \{c_3, z_0\}$ (if $m = n = 1$) or $S \in \{\{c_3, z_0\}, \{c_3, a_2^i, b_2^j\}, \{c_3, a_1, b_2^j\}, \{z_1^1, b_2^j\}\}$ for some $i, j > 1$ with $a_2^i b_2^j \notin E(G)$ (if $m = n > 1$). If $S \in \{\{c_3, z_0\}, \{c_3, a_2^i, b_2^j\}\}$, then $S \setminus \{c_3\} \cup \{c_1\}$ violates (β, β_0), contradiction. If $S \in \{\{c_3, a_1, b_2^j\}, \{z_1^1, b_2^j\}\}$, then S can be augmented by first replacing $\{c_3, a_1\}$ or $\{z_1^1\}$ with c_1 and then adding the node a_2^n, so violating (β, β_0), contradiction.

Subcase 2b): G_β contains an ac-pair $\{a_2, c_2\}$.

Let (X_{a_2}, X_{c_2}) be a canonical homogeneous pair of cliques contained in G_β with $X_{a_2} = \{a_2^1, \dots, a_2^p\}$ and $X_{c_2} = \{c_2^1, \dots, c_2^q\}$. Consider a stable set S that is tight for (β, β_0) and misses the clique $A \cup \{z_0\}$. Then $S = \{c_2^1, b_1\}$ and, consequently, $\beta_{b_1} \geqslant \beta_{z_0}, \beta_{a_1}$ and $\beta_{c_2^1} \geqslant \beta_{c_3} + \beta_{a_2^1}$. Moreover, $c_2^1 a_2^p \in E(G)$ and therefore, if $c_2^j \in V(G_\beta)$ for some $1 < j \leqslant q$, then there exists $a_2^i \in V(G_\beta)$ with $1 \leqslant i \leqslant p$ such that $a_2^i c_2^j \notin E(G)$ (otherwise c_2^1 and c_2^j would be twins).

Consider the set $U = \{z_0, a_1, b_1, c_1, c_2^1\} \cup X_{z_1} \cup X_{z_2} \cup X_{a_2}$ and observe that $\alpha(G[U]) = 2$. As a consequence, the rank inequality $(\gamma, 2)$, whose support is $G[U]$, is valid for $STAB(G/a_0b_0)$. As (β, β_0) is not a rank inequality, there exists a stable set S' that is tight for (β, β_0) and not for $(\gamma, 2)$, by Lemma 3. Therefore, either $S' \in \{\{c_2^j, a_1\}, \{c_2^j, z_0\}\}$ (if $c_3 \notin V(G_\beta)$) or $S' \in \{\{c_2^j, a_1\}, \{c_2^j, z_0\}, \{c_3, z_0\}, \{c_3, a_1\}\}$ (if $c_3 \in V(G_\beta)$), for some $1 < j \leqslant q$.

But then, either $S' \setminus \{a_1, z_0\} \cup \{b_1, a_2^i\}$ with $1 \leqslant i \leqslant p$ or $S' \setminus \{c_3\} \cup \{c_2^1\}$ (if $c_3 \in S'$) violates (β, β_0), a contradiction.

Case 3: $G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ *consists of one complete triple* $\{a_1, b_1, c_1\}$*, no pairs, and at most one c-singleton* c_3.

In this case, we can use the same mapping ϕ of Case 1 to show that G_β is isomorphic to a Z_2-strip G' and, by Theorem 30, the thesis follows.

Case 4: $G_\beta \setminus (X_{z_1} \cup X_{z_2} \cup \{z_0\})$ *consists of one complete triple* $\{a_1, b_1, c_1\}$*, one ac-pair* $\{a_2, c_2\}$*, one bc-pair* $\{b_3, c_3\}$*, no ab-pair, and at most one c-singleton.*

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Since G_β is not a clique, there exists a stable set S missing the clique $A \cup \{z_0\}$; then $c_2 \in S$, otherwise $S \cup \{a_2\}$ violates (β, β_0), and $b_1 \in S$, otherwise $S \cup \{a_1\}$ violates (β, β_0). As $S \setminus \{b_1\} \cup \{a_1, b_3\}$ is a stable set, $\beta_{b_1} > \beta_{a_1}$. Symmetrically, let S' be a tight stable set missing the clique $B \cup \{z_0\}$, then $c_3 \in S'$, otherwise $S' \cup \{b_3\}$ violates (β, β_0), and $a_1 \in S'$, otherwise $S' \cup \{b_1\}$ violates (β, β_0). As $S' \setminus \{a_1\} \cup \{b_1, a_2\}$ is a stable set, $\beta_{a_1} > \beta_{b_1}$. A contradiction. It is not difficult to check that this proof still works if thickening operations are performed on the pairs $\{a_2, c_2\}$ and $\{b_3, c_3\}$. □

We can now state the final result for closed fuzzy Z_3-strips.

Theorem 38. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_3-strip. Then every nontrivial inequality (β, β_0) that is facet defining for STAB(G) (STAB(G/a_0b_0)) is a rank or a lifted 5-wheel inequality. Moreover, if $\beta_{a_0} = \beta_{b_0} > 0$ ($\beta_{z_0} > 0$, respectively), then (β, β_0) is a rank inequality.*

Proof. As usual, we assume that G_β does not contain twins. By Lemma 35, we may assume that G_β contains at least one complete triple, say $\{a_1, b_1, c_1\}$, since otherwise G_β is fuzzy line and so, it supports a rank inequality by Theorem 11. The proof consists of three cases.

Case 1: *(β, β_0) is facet defining for STAB(G) and $\beta_{a_0} = \beta_{b_0} > 0$.*

Let S be a tight stable set for (β, β_0) missing the clique $A \cup \{a_0\}$ (it must exists by Lemma 3). Then $b_0 \in S$, otherwise $S \cup \{a_0\}$ violates (β, β_0). Then $c_1 \in S$ otherwise $S \cup \{a_1\}$ would violate (β, β_0). If G_β contains a triple $\{a_2, b_2, c_2\}$, or an ab-pair $\{a_3b_3\}$, or an ac-pair $\{a_4, c_4\}$ then $S \cup \{a_2\}$, or $S \cup \{a_3\}$, or $S \cup \{a_4\}$, respectively, violates (β, β_0). A contradiction.

Symmetric arguments prove that G_β contains no bc-pair. Since, by Observation 34, G_β contains at most one c-singleton, the same mapping used to prove Case 1 of Lemma 37 (with simply $\phi(z_0) = z'_0$ replaced by $\phi(a_0) = a'_0$ and $\phi(b_0) = b'_0$) can be used to show that G_β is isomorphic to a closed fuzzy Z_2-strip. Thus the thesis follows by Theorem 30.

Case 2: *(β, β_0) is facet defining for STAB(G/a_0b_0).*

The thesis follows from Lemma 37.

Case 3: *(β, β_0) is facet defining for STAB(G) and $\beta_{a_0} = \beta_{b_0} = 0$.*

The inequality (β, β_0) is also facet defining for STAB$((G/a_0b_0) \setminus \{z_0\})$. Hence, the inequality (β', β_0) obtained from (β, β_0) by lifting the node z_0 is facet defining for STAB(G/a_0b_0). By Lemma 37, (β', β_0) is either a lifted 5-wheel inequality and $\beta_{z_0} = 0$, or a rank inequality and $\beta_{z_0} \in \{0, 1\}$. Hence (β, β_0) satisfies the thesis. □

8. Fuzzy Z_4-strips

The last three-cliqued strip to be considered in the decomposition of claw-free graphs with large stability number was named Z_4-strip in [3] and defined as follows:

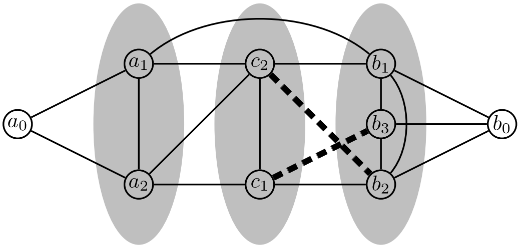

Fig. 6. A Z_4-strip (A, B, C, a_0, b_0). The dashed bold edges represent fuzzy pairs, while the grey areas emphasize the three cliques A, B, and C.

Definition 39. A three-cliqued strip (A, B, C, a_0, b_0) is a *Z_4-strip* if adjacencies are as follows:

1) $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_2\}$ are three pairwise disjoint cliques;
2) $\{a_2, c_1, c_2\}, \{a_1, b_1, c_2\}$ are cliques; b_2c_1, b_2c_2, and b_3c_1 are edges.

A Z_4-strip is *fuzzy* if and only if a thickening has been performed on at least one of the following pairs: $\{b_2, c_2\}$ and $\{b_3, c_1\}$ (see Fig. 6).

As usual, we denote by z_0 the node of G/a_0b_0 obtained by contracting a_0b_0.

Lemma 40. *Let $G = (A, B, C, a_0b_0)$ be a closed fuzzy Z_4-strip. If G does not contain an induced 5-wheel, then G is fuzzy line. The same holds for G/a_0b_0.*

Proof. The 5-wheels in G or G/a_0b_0 are all of type $(c_2^i : a_1, a_2, c_1^h, b_2^j, b_1)$ where $i, j, h \geqslant 1$ and $c_2^i b_2^j \in E(G)$. So we consider all the cases when G and G/a_0b_0 contain none of the above 5-wheels. Let us denote by $H = (A, B, C, x_0, y_0)$ the fuzzy Z_4-strip obtained from $G - a_0b_0$ by renaming the nodes a_0 and b_0 as x_0 and y_0, respectively.

According to Observation 1, G and G/a_0b_0 are obtained as strip compositions of (H, x_0, y_0) with the 4-path $\{x_0', a_0, b_0, y_0'\}$ and the 3-path $\{x_0', z_0, y_0'\}$, respectively. Moreover, as a_0b_0 is super simplicial, G (G/a_0b_0) contains a 5-wheel if and only if H contains a 5-wheel. Hence, it is sufficient to prove that if H does not contain a 5-wheel, then H is fuzzy line.

If H does not contain a_1 then H is a fuzzy linear interval strip. Indeed a map from $V(H)$ to the points l_0, l_1, l_2, l_3, l_4 of a line L (ordered from left to right) is the following: $\phi(x_0) = l_0$, $\phi(a_2) = l_1$, $\phi(X_{c_1}) = \phi(X_{c_2}) = l_2$, $\phi(b_1) = \phi(X_{b_2}) = \phi(X_{b_3}) = l_3$, $\phi(y_0) = l_4$, and the intervals are: $I_1 = [l_0, l_1 + \epsilon]$, $I_2 = [l_1, l_2 + \epsilon]$, $I_3 = [l_2, l_3]$, $I_4 = [l_3 - \epsilon, l_4]$, where ϵ is a suitably small value and the interval I_3 is fuzzy (according to Definition 10).

Similar maps can be found when H does not contain any node in X_{c_1} or H does not contain b_1.

If H does not contain a_2 then it is obtained as a strip composition of the following strips: (H_1, x_1, x_2) with node set $\{x_0, x_1, a_1, x_2, y_0\} \cup X_{c_2} \cup X_{b_2} \cup X_{c_1} \cup X_{b_3}$ depicted in Fig. 7 (a) and the strip consisting of the 3-node path $\{y_1, b_1, y_2\}$.

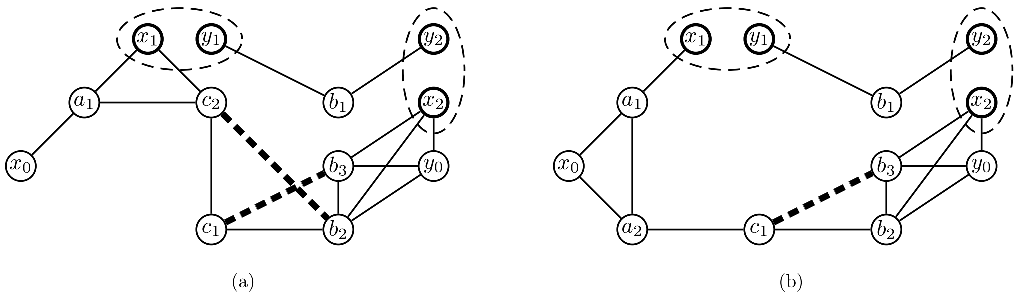

Fig. 7. Fuzzy Z_4-strips without node a_2 (a) or node c_2 (b) are compositions of fuzzy linear interval strips.

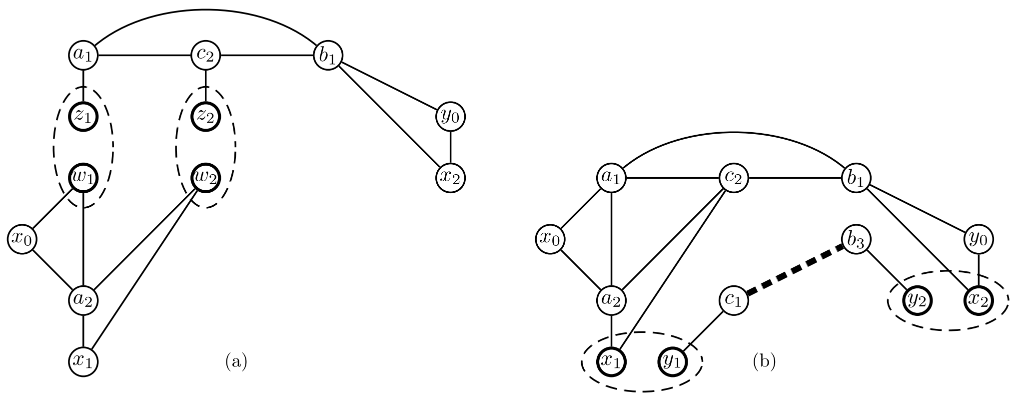

Fig. 8. Fuzzy Z_4-strips without node b_2 can be obtained by two subsequent strip compositions (a) and (b).

If H does not contain any node $x \in X_{c_2}$ then the strip composition is depicted in Fig. 7 (b).

If H does not contain any node $x \in X_{b_2}$ then H has the decomposition depicted in Fig. 8.

Finally, consider the case where $c_2^i b_2^j \notin E$ for any $c_2^i \in X_{c_2}$ and $b_2^j \in X_{b_2}$. Then a decomposition similar to the one depicted in Fig. 8 holds with b_2 added and adjacent to c_1, b_3 and y_2. This completes the proof. □

Theorem 41. *Let $G = (A, B, C, a_0 b_0)$ be a closed fuzzy Z_4-strip. Then every nontrivial inequality (β, β_0) that is facet defining for $STAB(G)$ ($STAB(G/a_0 b_0)$) is a rank or a lifted 5-wheel inequality. Moreover, if $\beta_{a_0} = \beta_{b_0} > 0$ ($\beta_{z_0} > 0$, respectively), then (β, β_0) is a rank inequality.*

Proof. Suppose conversely that (β, β_0) is neither a rank nor a lifted 5-wheel inequality. As usual, we assume that G_β does not contain twins and every homogeneous pair in G_β is canonical.

If G_β does not contain a 5-wheel then, by Lemma 40 and Theorem 11, (β, β_0) is a rank inequality, a contradiction. So, we may assume, without loss of generality, that $a_1, a_2, c_1^1, c_2^1, b_1, b_2^1 \in V(G_\beta)$.

Observe that if $X_{b_3} \cap V(G_\beta) = \emptyset$ then G_β is isomorphic to a closed fuzzy Z_2-strip (it suffices to rename a_2 as a_3) and the thesis follows from Theorem 30.

Hence, $X_{b_3} \cap V(G_\beta) \neq \emptyset$, i.e., $|X_{b_3} \cap V(G_\beta)| = m \geqslant 1$.

First observe that $b_3^m c_1^1 \in E$ otherwise b_3^m would be simplicial in G_β, contradicting ii) of Corollary 14. Now, when $m = 1$, G_β is still isomorphic to a closed fuzzy Z_2-strip (it suffices to rename a_2 as a_3 and put b_3^1 into X_{b_2}) and the thesis follows from Theorem 30. So, we assume that $m > 1$.

To complete the proof we consider three cases for (β, β_0) each of which yields a contradiction:

Case 1: (β, β_0) *is facet defining for* $STAB(G)$ *and* $\beta_{a_0} = \beta_{b_0} > 0$.

Let S be a tight stable set of (β, β_0) missing the clique $B \cup \{b_0\}$. Clearly $a_0 \in S$ otherwise $S \cup \{b_0\}$ violates (β, β_0). Then $S \cap C \neq \emptyset$. If $S \cap X_{c_2} \neq \emptyset$ then $S \cup \{b_3^1\}$ violates (β, β_0), a contradiction. If $S \cap X_{c_1} \neq \emptyset$ then $S \cup \{b_1\}$ violates (β, β_0), a contradiction.

Case 2: (β, β_0) *is facet defining for* $STAB(G/a_0 b_0)$.

By Lemma 24, the nodes of X_{c_1} and X_{b_3} can be ordered so that nodes with smaller index have more neighbours and, by Proposition 4, we have that: $\beta_{c_1^1} \geqslant \beta_{c_1^2} \geqslant \cdots \geqslant \beta_{c_1^n}$, and $\beta_{b_3^1} \geqslant \beta_{b_3^2} \geqslant \cdots \geqslant \beta_{b_3^m}$, where $|n - m| \leqslant 1$. An analogous assumption can be made for X_{c_2} and X_{b_2}.

Consider the 5-hole $H = (a_1, a_2, c_1^1, b_2^1, b_1)$ and the inequality $(\gamma, 2)$ supported by H plus the nodes $(X_{b_2} \setminus \{b_2^1\}) \cup X_{b_3} \cup X_{c_2} \cup \{z_0\}$ lifted according to the following order:

$$b_2^2, b_2^3, \ldots, b_2^p, \quad b_3^1, b_3^2, \ldots, b_3^m, \quad c_2^1, c_2^2, \ldots, c_2^q, \quad z_0 \tag{4}$$

where $p = |X_{b_2}|$ and $q = |X_{c_2}|$. All the above nodes receive lifting coefficient 1 according to formula (1). Indeed, when lifting nodes $u \in (X_{b_2} \setminus \{b_2^1\}) \cup X_{b_3}$, observe that $\{a_1\}$ is a maximum stable set of $G_\gamma \setminus N[u]$; when lifting nodes $v \in X_{c_2}$, $\{b_3^1\}$ is a maximum stable set of $G_\gamma \setminus N[v]$; and when lifting the node z_0, $\{c_1^1\}$ is a maximum stable set of $G_\gamma \setminus N[z_0]$. Lastly, the nodes $u \in X_{c_1} \setminus \{c_1^1\}$ are lifted with zero coefficient because $\{a_1, b_3^m\}$ is a maximum stable set of $G_\gamma \setminus N[u]$.

Let S' be a stable set that is tight for (β, β_0) and is not tight for $(\gamma, 2)$. S' is $\{c_1^j, b_1\}$ or $\{c_1^j, z_0\}$ for some $j > 1$ (because $\{c_1^j, a_1\}$ is augmentable with b_3^m). Since both sets $S' \setminus \{b_1\} \cup \{a_1, b_3^m\}$ and $S' \setminus \{z_0\} \cup \{a_1, b_3^m\}$ are stable, $\beta_{b_1} > \beta_{a_1}$ or $\beta_{z_0} > \beta_{a_1}$.

Let S'' be a tight stable set of (β, β_0) missing $B \cup \{z_0\}$. If $a_2 \in S''$ then $S'' \cup \{b\}$ violates (β, β_0) for any $b \in B$, a contradiction. If $a_1 \notin S''$ then $c_2 \in S''$ and $S'' \cup \{b_3^1\}$ violates (β, β_0), contradiction. Then $a_1 \in S''$ and $S'' = \{a_1, c_1^1\}$. Since $S'' \setminus \{a_1\} \cup \{b_1\}$ and $S'' \setminus \{a_1\} \cup \{z_0\}$ are stable sets, $\beta_{a_1} \geqslant \beta_{b_1}$ and $\beta_{a_1} \geqslant \beta_{z_0}$, a contradiction.

Case 3: (β, β_0) *is facet defining for* $STAB(G)$ *and* $\beta_{a_0} = \beta_{b_0} = 0$.

The inequality (β, β_0) is also facet defining for $STAB((G/a_0 b_0) \setminus \{z_0\})$. Hence, the inequality (β', β_0) obtained from (β, β_0) by lifting the node z_0 is facet defining for $STAB(G/a_0 b_0)$. By Case 2, (β', β_0) is either a lifted 5-wheel inequality and $\beta_{z_0} = 0$, or a rank inequality and $\beta_{z_0} \in \{0, 1\}$. Hence (β, β_0) satisfies the thesis. $\square$

9. The stable set polytope of fuzzy antihat graphs

The structure theorem for claw-free graphs of Chudnovsky and Seymour [3] states that a claw-free graph without 1-join and with stability number at least four is either a striped graph or a fuzzy circular interval graph.

The defining linear system of the stable set polytope of a graph G containing a clique-cutset is the union of the linear systems defining the stable set polytopes of the two graphs composing G (this follows from Theorem 13). Since in claw-free graphs a 1-join gives rise to a clique-cutset, the study of the linear description of the stable set polytope of claw-free graphs can be restricted to claw-free graphs that do not admit 1-joins. Thus, a complete linear description of the stable set polytope of claw-free graphs is available as soon as we know an explicit linear description of $STAB(G)$ when G is striped, or fuzzy circular interval or G has stability number less than or equal to 3. In this section, we study the facial structure of the stable set polytope of a large subclass of striped graphs: the *fuzzy antihat graphs*.

To maintain the analogy with the notation in [3], we denote by $\mathcal{Z}_i$ the set of closed fuzzy Z_i-strips for $i = 2, 3, 4$. Moreover, a_0b_0 will always indicate the super simplicial edge of a closed fuzzy Z_i-strip according to Definitions 28, 32, and 39.

Definition 42. A *fuzzy antihat graph* is a graph obtained from a fuzzy line graph H by iteratively performing 2-clique-bond compositions of closed fuzzy strips T_i belonging to $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$ along pairs (u_i, v_i) and (a_0^i, b_0^i) such that: $\Gamma_H = \{e_i = u_iv_i,\ i = 1, \dots, k\}$ is a set of pairwise non-incident super simplicial edges of H and $f_i = a_0^ib_0^i$ is the super simplicial edge of T_i for $i = 1, \dots, k$.

We say that a graph is $\mathcal{W}$*-perfect* if its stable set polytope is described by: nonnegativity, rank, and lifted 5-wheel inequalities. The next theorem shows that the 2-clique-bond composition preserves the $\mathcal{W}$-perfectness provided that the closed strips used in the composition belong to $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$.

Theorem 43. *Let G be a graph obtained as the 2-clique-bond composition of a claw-free graph H and a closed fuzzy strip Z belonging to $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$, along pairs (u, v) and (a_0, b_0) such that $f = uv$ is a super simplicial edge of H and $e = a_0b_0$ a super simplicial edge of Z.*

If H and H/f are $\mathcal{W}$-perfect, then G is $\mathcal{W}$-perfect.

Proof. Let z_e (z_f) denote the node resulting from the contraction of the edge e (f, respectively). By Theorem 18, $STAB(G)$ is described by the following inequalities:

i) nonnegativity inequalities;
ii) clique inequalities;
iii) facet defining inequalities of $STAB(H)$ with zero coefficients on the endnodes of f;

iv) facet defining inequalities of $STAB(Z)$ with zero coefficients on the endnodes of e;
v) even-odd combinations of facet defining inequalities of $STAB(H)$ and $STAB(Z/e)$;
vi) even-odd combinations of facet defining inequalities of $STAB(H/f)$ and $STAB(Z)$.

By hypothesis, $STAB(H)$ and $STAB(H/f)$ are described by nonnegativity, rank, and lifted 5-wheel inequalities. Since f is super simplicial and H is claw-free, it follows from Corollary 8 that the only inequalities of $STAB(H)$ and $STAB(H/f)$ having nonzero coefficients on the endnodes of f and on z_f (those involved into even-odd combinations) are rank inequalities.

By Theorems 30, 38, 41, $STAB(Z)$ and $STAB(Z/e)$ are described only by nonnegativity, rank, and lifted 5-wheel inequalities. Moreover, only rank inequalities have nonzero coefficients on both endnodes of e in $STAB(Z)$ and on the node z_e in $STAB(Z/e)$. This implies, by Lemma 19, that the even-odd combinations v) and vi) resulting from the 2-clique-bond composition of H and Z are rank inequalities and the theorem follows. □

We can now state the main result of this paper:

Theorem 44. *Let G be a fuzzy antihat graph. Then $STAB(G)$ is described by the following inequalities:*

- *nonnegativity inequalities,*
- *rank inequalities,*
- *lifted 5-wheel inequalities.*

Proof. According to Definition 42, let H_k be a fuzzy antihat graph, i.e., a graph obtained from a fuzzy line graph H by k applications of the 2-clique-bond composition with graphs $T_i \in \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$, $i = 1, \ldots, k$ along pairs (a_i, b_i) and (u_i, v_i) such that: $\Gamma_H = \{e_i = a_i b_i, i = 1, \ldots, k\}$ is a set of pairwise non-incident super simplicial edges of H and $f_i = u_i v_i$ is a super simplicial edge of T_i for $i = 1, \ldots, k$.

First observe that the class of fuzzy line graphs is closed under contraction of super simplicial edges; indeed, H/e is obtained by composing the fuzzy line strip $(H - e, v_1, v_2)$ with the strip (P, a_0, b_0) where P is the path (a_0, z_0, b_0). Thus, H/F is fuzzy line for any set $F \subseteq \Gamma_H$.

The proof is by induction on k.

If $k = 1$ then H_1 is $\mathcal{W}$-perfect by Theorem 43. Then we assume that $k > 1$ and H_k is the 2-clique-bond composition of H_{k-1} and T_k along (a_k, b_k) and (u_k, v_k).

By Theorem 43, in order to prove that H_k is $\mathcal{W}$-perfect it suffices to show that H_{k-1} and H_{k-1}/e_k are $\mathcal{W}$-perfect. Since H_{k-1} is $\mathcal{W}$-perfect by inductive hypothesis, it remains to show that H_{k-1}/e_k is $\mathcal{W}$-perfect.

As the edges $\{e_1, e_2, \ldots, e_k\}$ are pairwise non-incident, the edges $\{e_1, e_2, \ldots, e_{k-1}\}$ are super simplicial in H/e_k. This implies that the graph H_{k-1}/e_k can be obtained from the graph H/e_k by $k-1$ iterated 2-clique-bond compositions of T_1, T_2, ..., T_{k-1} along the pairs $(a_1, b_1), \ldots, (a_{k-1}, b_{k-1})$.

Since H/e_k is fuzzy line, it follows that H_{k-1}/e_k is a fuzzy antihat graph obtained with $k-1$ iterations of the 2-clique-bond composition and, by induction, it is $\mathcal{W}$-perfect. Thus the theorem follows. $\square$

The above theorem together with the characterization of rank facet defining inequalities in [13] provides the minimal defining linear system for the stable set polytope of fuzzy antihat graphs. Since the polytope $STAB(G)$ is full dimensional, this linear system is unique.

As explained at the beginning of this section, a complete linear description of the stable set polytope of claw-free graphs will be available as soon as we know the explicit linear description of $STAB(G)$ when G is striped, or fuzzy circular interval or G has stability number less than or equal to 3. Now the only strip that is missed to complete the construction of striped graphs is the so-called Z_5-strip. This strip differs from the other Z_i-strips, $i = 1, 2, 3, 4$, because it is not three-cliqued. Moreover, the Z_5-strip gives rise to a class of more complicated facet defining inequalities for $STAB(G)$ that have coefficients $\{0, 1, 2\}$ and that are different from the lifted 5-wheel inequalities (see [9,10] for details). We consider this case in a subsequent paper [12] where we complete the study of the stable set polytope of striped graphs.

If G is fuzzy circular interval, a linear description of $STAB(G)$ has been provided in [7]. Therefore, to have a complete linear description of $STAB(G)$ when G is claw-free, it remains to consider the case $\alpha(G) = 3$. This case seems to be difficult because the defining linear system of $STAB(G)$, when G has stability number three, contains inequalities with arbitrarily many different coefficients [26,18].

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